ACCURATE SEQUENTIAL TESTS ON THE MEAN OF AN EXPONENTIAL DISTRIBUTION

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0. Summary. In this paper, methods introduced earlier by the author [1] are used to obtain simple, accurate formulas for the decision boundaries for sequential probability ratio tests for simple hypotheses and alternatives on the mean θ of the exponential distribution $\theta^{-1}\exp(-u/\theta)$. Examples are provided to indicate the accuracy and the degree of complexity of the results. It is hoped that the results given here will find applications in life testing and statistical studies of radioactive decay.

1. Some integral equations. Consider a sequential probability ratio test for a simple hypothesis θ_1 and alternative $\theta_2 > \theta_1$ on the mean of the exponential distribution

$$g(u;\theta) = \theta^{-1}\exp(-u/\theta), \qquad u > 0. \tag{1}$$

The substitutions $\xi = \theta^{-1}$, $Q(\xi) = \log(-\xi)$, $v = u$ and $f(v;\xi) = (-\xi)\exp(\xi v)$ identify the present problem with a more general one studied in Sections 4 and 6 of [1]. This identification will not be used here because it introduces needless complication of notation in the simple problem at hand. No confusion should arise from similarities or differences between the notations used in [1] and those used here.

Define the parameters

$$r = \theta_2/\theta_1, \qquad h = \log r, \qquad m = \frac{\theta}{\theta_1} - \frac{\theta}{\theta_2}.$$

The logarithm of the probability ratio for the test takes the form

$$z = \log\frac{g(u;\theta_2)}{g(u;\theta_1)} = \frac{mu}{\theta} - h,$$

and its p.d.f. is

$$\begin{aligned} f(z;m) &= m^{-1}\exp[-(z+h)/m], & z &\geqq -h, \\ &= 0, & z &< -h. \end{aligned} \tag{2}$$

As in [1], Part II, let $-b$ and a be the decision boundaries on the cumulative sums $x_N = \sum_{i=0}^{N} z_i$ of z and let the starting point $x_0 = z_0$ of the test be chosen arbitrarily in the open interval $(-b, a)$. When θ is the true value of the mean of

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(1), the probability $P_1(x_0\,;\theta)$ of deciding in favor of the hypothesis θ_1 and the expected duration $M_1(x_0\,;\theta)$ of the test satisfy the integral equations

$$P_1(x;\theta) = \int_{-\infty}^{-b} f(y-x;m)\,dy + \int_{-b}^{a} P_1(y;\theta)f(y-x;m)dy, \tag{3}$$

$$M_1(x;\theta) = 1 + \int_{-b}^{a} M_1(y;\theta)f(y-x;m)\,dy \tag{4}$$

on the interval $(-b, a)$. The kernel $f(y-x;m)$ is obtained from (2). The probability of deciding in favor of the alternative hypothesis θ_2 is given by $P_2(x;\theta) = 1 - P_1(x;\theta)$.

The integral equations (3) and (4) can be solved exactly by a simple device that will be indicated in Section 3, below. The results are too unwieldy for practical use in the determination of decision boundaries for preassigned error risks. Approximate solutions for the integral equations will also be obtained. These are relatively easy to use and are demonstrated to be of sufficient accuracy to be considered essentially exact for practical purposes.

It will be convenient to transform the integral equations slightly by introducing the quantities

$$H = h/m, \qquad A = a/m, \qquad B = b/m, \qquad s = x/m, \qquad t = y/m. \tag{5}$$

Account being taken of the discontinuity of the kernel, the equations (3) and (4) take the forms

$$\begin{aligned} P_1(ms;\theta) &= 1 - e^{s+B-H} + \int_{-B}^{A} P_1(mt;\theta)e^{-t+s-H}\,dt, && -B < s \leqq -B+H, \\ &= \int_{s-H}^{A} P_1(mt;\theta)e^{-t+s-H}\,dt, && -B+H \leqq s < A, \end{aligned} \tag{6}$$

and

$$\begin{aligned} M_1(ms,\theta) &= 1 + \int_{-B}^{A} M_1(mt;\theta)e^{-t+s-H}\,dt, && -B < s \leqq -B+H, \\ &= 1 + \int_{s-H}^{A} M_1(mt;\theta)e^{-t+s-H}\,dt, && -B+H \leqq s < A. \end{aligned} \tag{7}$$

2. Approximate solutions of (6). Let n be any positive integer or zero with the restriction that $-B + nH \leqq A$ and let $\lambda = \lambda(\theta)$ be the non-zero solution of the equation

$$1 + \lambda = e^{\lambda H}.$$

Define the functions $G_n(s)$ and $\psi_n(s)$ by

$$G_n(s) \equiv \sum_{j=0}^{n} \frac{(-1)^j}{j!} e^{-jH}(s + B - jH)^j,$$

$$\psi_n(s) \equiv \frac{e^{-\lambda(s-H)} - e^{-\lambda A}}{\delta_n - e^{-\lambda A}},$$

where δ_n is a constant to be determined. Let $\phi_n(s; \theta)$ denote a function defined on the interval $(-B, A)$ by

$$\text{(8)} \quad \phi_n(s; \theta) \equiv \begin{cases} 1 - C_n e^{-H+s+B} G_{k-1}(s), & -B + (k-1)H \leqq s \leqq -B + kH, \\ & k = 1, 2, \cdots, n, \\ \psi_n(s), \quad -B + nH \leqq s < A, & \end{cases}$$

where C_n is a constant to be determined. When $n = 0$, the first form on the right in (8) is to be deleted. Let $\Phi_n(s; \theta)$ denote the iterate of the function (8) under the operator on the right in equation (6).

It will be shown that if C_n and δ_n are any pair of constants related by the equation

$$\text{(9)} \qquad C_n e^{nH} G_n\{-B + (n+1)H\} = 1 - \psi_n\{-B + (n+1)H\},$$

then $\phi_n(s; \theta)$ and its iterate $\Phi_n(s; \theta)$ are identical on the subintervals $-B < s \leqq -B + nH$ and $-B + (n+1)H \leqq s < A$ of $(-B, A)$. The identity does not persist to the subinterval $-B + nH \leqq s \leqq -B + (n+1)H$, but it will be shown that pairs of values $\delta_n(\theta, U)$, $C_n(\theta, U)$ and $\delta_n(\theta, L)$, $C_n(\theta, L)$ exist for δ_n and C_n such that (9) is satisfied and the resulting functions $\phi_n(s; \theta, U)$ and $\phi_n(s; \theta, L)$ and their iterates $\Phi_n(s; \theta, U)$ and $\Phi_n(s; \theta, L)$ have the properties

$$\text{(10)} \qquad \Phi_n(s; \theta, U) \leqq \phi_n(s; \theta, U), \qquad \Phi_n(s; \theta, L) \geqq \phi_n(s; \theta, L), \\ -B + nH \leqq s \leqq -B + (n+1)H.$$

It follows from Theorem 4 of [1] that $\phi_n(s; \theta, U)$ and $\phi_n(s; \theta, L)$ are respectively *upper and lower bounds for the function* $P_1(ms; \theta)$ *over the entire interval* $(-B, A)$.

Specifically, the stated values of δ_n are defined by the following: Let

$$\text{(11)} \qquad Q_n(x; \theta) \equiv \frac{G_n\{-B + (n+1)H\} e^{(1+\lambda)x} - G_n\{-B + (n+1)H - x\}}{G_n\{-B + (n+1)H\} e^{x} - G_n\{-B + (n+1)H - x\}}, \\ 0 \leqq x \leqq H,$$

then

$$\text{(12)} \qquad \begin{aligned} \delta_n(\theta, U) e^{-\lambda(B-nH)} &= \min Q_n(x; \theta), \qquad & 0 \leqq x \leqq H, \\ \delta_n(\theta, L) e^{-\lambda(B-nH)} &= \max Q_n(x; \theta), \qquad & 0 \leqq x \leqq H. \end{aligned}$$

The values $C_n(\theta, U)$ and $C_n(\theta, L)$ are then defined by using (12) in (9).

In general, the extrema required in (12) must be determined numerically. This is inconvenient. The value of δ_n specified by

$$\text{(13)} \qquad \delta_n = \delta_n(\theta, C) = Q_n(H; \theta) e^{\lambda(B-nH)}$$

is relatively easy to calculate. It and its companion value $C_n(\theta; C)$, defined by (9) and (13), define an approximate solution $\phi_n(s; \theta, C)$ for the equation (6) that is continuous on the whole interval $(-B, A)$ and which lies between the corresponding upper and lower bounds, at least on the important subinterval $-B + nH \leqq s < A$.

The proofs of the facts stated above are a matter of laborious detail that will be sketched next. The uninterested reader should proceed at once to the next section. The proofs of (10) will be divided into cases.

CASE (i): $-B < s \leqq -B + H$. In this case the iterate of $\phi_n(s; \theta)$ is given by

$$\Phi_n(s; \theta) = 1 - e^{s+B-H} + \sum_{k=1}^{n} \int_{-B+(k-1)H}^{-B+kH} e^{-t+s-H}\{1 - C_n e^{t+B-H} G_{k-1}(t)\}\, dt + \int_{-B+nH}^{A} e^{-t+s-H}\psi_n(t)\, dt$$

$$= 1 - e^{s+B-H}\Big\{1 - \sum_{k=1}^{n} [e^{-(k-1)H} - e^{-kH}] - C_n \sum_{k=1}^{n} [G_k[-B + (k+1)H] - G_k[-B + (k+2)H]] - (\delta_n - e^{-\lambda A})^{-1}\left[\frac{e^{\lambda(B+H)-(1+\lambda)nH}}{1+\lambda} - e^{-\lambda A - nH}\right]\Big\}.$$

This can be simplified by the use of $1 + \lambda = e^{\lambda H}$, the identities

$$1 - \sum_{k=1}^{n} [e^{-(k-1)H} - e^{-kH}] = e^{-nH},$$

$$\sum_{k=1}^{n} \{G_k[-B + (k+1)H] - G_k[-B + (k+2)H\} = G_n[-B + (n+1)H] - 1,$$

and the relation (9) to the form $\Phi_n(s; \theta) = 1 - C_n e^{s+B-H} = \phi_n(s; \theta)$.

CASE (ii): $-B + (M-1)H \leqq s \leqq -B + MH$, $1 < M \leqq n$. In this case the iterate is given by

$$\Phi_n(s; \theta) = \int_{s-H}^{-B+(M-1)H} e^{-t+s-H}\{1 - C_n e^{t+B-H} G_{M-2}(t)\}\, dt + \sum_{k=M}^{n} \int_{-B+(k-1)H}^{-B+kH} e^{-t+s-H}\{1 - C_n e^{t+B-H} G_{k-1}(t)\}\, dt + \int_{-B+nH}^{A} e^{-t+s-H}\, \psi_n(t)\, dt.$$

Devices similar to those used in Case (i) reduce this to the form $\phi_n(s; \theta) = 1 - C_n e^{s+B-H} G_{M-1}(s)$ appropriate to this interval.

CASE (iii): $-B + (n+1)H \leqq s < A$. In this case

$$\Phi_n(s; \theta) = \int_{s-H}^{A} \psi_n(t) e^{-t+s-H}\, dt = \psi_n(s) \tag{14}$$

is almost immediate. The only reduction needed follows from $1 + \lambda = \exp(\lambda H)$.

CASE (iv): $-B + nH \leqq s \leqq -B + (n + 1)H$. In this case the iterate takes the form

$$\Phi_n(s; \theta) = \int_{s-H}^{-B+nH} e^{-t+s-H}\{1 - C_n e^{t+B-H} G_{n-1}(t)\}\, dt + \int_{-B+nH}^{A} e^{-t+s-H}\psi_n(t)\, dt.$$

Add and subtract the integral over $(s - H, -B + nH)$ of the quantity $\psi_n(t)$ $\exp(-t + s - H)$ and use the formal identity (14) to reduce this to the form $\Phi_n(s; \theta) = \psi_n(s) + \epsilon_n(s; \theta)$,

$$\epsilon_n(s; \theta) \equiv \{1 - \psi_n(s)\} - \{1 - \psi_n[-B + (n + 1)H]\}e^{s+B-(n+1)H} + C_n e^{s+B-H} \{G_n[-B + (n + 1)H] - G_n(s)\}.$$

The relation (9) may be used to eliminate the constant C_n from $\epsilon_n(s; \theta)$ to obtain

$$\epsilon_n(s; \theta) = \{1 - \psi_n(s)\} - \{1 - \psi_n[-B + (n + 1)H]\}e^{s+B-(n+1)H} \cdot \frac{G_n(s)}{G_n[-B + (n + 1)H]}, \quad -B + nH \leqq s \leqq -B + (n + 1)H. \tag{15}$$

The inequalities (10) result from the requirements $\epsilon_n(s; \theta) \leqq 0$ and $\epsilon_n(s; \theta) \geqq 0$, respectively, enforced over the interval of definition of $\epsilon_n(s; \theta)$. The definitions (12) are easily derived by setting $s = -B + (n + 1)H - x$ and assuming that $\delta_n \geqq \exp(-\lambda A)$. This last assumption is always justifiable in practical cases.

3. Exact solutions of (6). Exact solutions for the integral equation (6) may be found by a modification of the above-described technique. Omit the final form $\psi_n(s)$ in the definition of $\phi_n(s; \theta)$ and choose n large enough that

$$-B + nH \geqq A > -B + (n - 1)H.$$

For example, if, for some integer L, $A + B = LH$, choose $n = L$. A relation comparable to (9) is found:

$$C_n e^{nH} G_n[-B + (n + 1)H] = 1, \qquad A + B = LH.$$

This determines C_n. If $A + B = (L + \nu)H$ where L is an integer and $0 < \nu < 1$, a more complicated relation is found for the determination of C_n.

These exact results are almost useless for practical determinations of decision boundaries to effect desired risk probabilities. The formulas are so nearly indeterminate that the writer obtained absurd results from them using modest computing facilities. In comparison, it will be shown in later sections that quite accurate determinations of decision boundaries may be made easily by use of the approximations $\varphi_n(s; \theta, C)$.

The method of derivation of the function (8) may be of interest. From well-known theory, the integral equation (6) has a unique, continuous solution on $-B < s < A$. From the first form of the equation it is obvious that on the subinterval $-B < s \leqq -B + H$, $P_1(ms; \theta) \equiv 1 - C \exp(s + B - H)$ for some choice of C. Differentiation of the second form of the integral equation leads to the differential-difference equation

$$\frac{d}{ds} P_1(ms; \theta) - P_1(ms; \theta) = -P_1[m(s - H); \theta],$$

valid over $-B + H \leqq s < A$. The change of form of $P_1(ms; \theta)$ from any interval $-B + (k - 1)H \leqq s \leqq -B + kH$ to the next is easily determined from this equation and the continuity requirement. Finally, the form (8) is simply an expedient combination of the form for the exact solution and the function $\psi_n(s)$, which was studied in [1].

For sequential tests on the mean occurrence time of a Poisson process with a continuous time parameter, Dvoretzky, Kiefer, and Wolfowitz [3] found a differential-difference equation similar to that above and exact formulas for the operating characteristics of the tests that were of the same structure as the exact solution indicated above for the integral equation (6). Their discussion can be interpreted in a manner to apply to sequential tests on the mean of an exponential population. Anscombe and Page [2] show how this is done and indicate another derivation of the quoted results of Dvoretzky, Kiefer, and Wolfowitz. These papers do not consider the problem of obtaining useful approximate results.

4. Remarks on the approximation of $P_1(0; \theta_2)$. As in Wald [6], it is usual to start a sequential probability ratio test of θ_1 versus θ_2 at $x_0 = z_0 = 0$. The design problem consists in the determination of the boundaries a and $-b$ to achieve preassigned probabilities

$$\alpha = P_2(0; \theta_1), \qquad \beta = P_1(0; \theta_2),$$

of the first and second kinds of error. Easy success in this problem will depend on two things: (i) a choice of n in (8) small enough that the starting point $s_0 = 0$ lies in the subinterval $-B + nH \leqq s < A$ of validity of the simple form $\psi_n(s)$ of $\phi_n(s; \theta)$, and (ii) a choice of n large enough that the bounds $\phi_n(s_0; \theta, U)$ and $\phi_n(s_0; \theta, L)$ are close enough together to give the accuracy desired in the test.

Explicit calculation of the values of the constants defined in (12) must be done numerically if $n > 0$. Sample calculations performed by the writer indicate that for a given set of values of a, b and s_0, the difference $\phi_n(s_0; \theta, U) - \phi_n(s_0; \theta, L)$ decreases with increasing n or with decreasing r. The computational difficulty in obtaining needed values of δ_n and C_n increases rapidly with n. It appears then that a good rule is to use the smallest value of n which will provide the accuracy desired. As a rough guide, the writer's experience has been that the choice $n = 2$ will usually give bounds for $P_1(0; \theta_2)$ that differ by less than one per cent; for the choice $n = 3$, the bounds usually differ by something less than one tenth of one per cent. Both of these estimates of accuracy are based on values of r between 1.0 and 2. The choices $n = 0, 1$ are quite poor unless r is near unity.

After a value for n has been chosen, either by the rough suggestions given above or by actual computation of the series of bounds for $P_1(0; \theta_2)$, either the upper or the lower bound or any value between them may be used as an approximate formula. Since the extrema required in (12) must be calculated numerically, the continuous approximation given by use of (13) seems simpler to use. With n chosen so that the simple form $\psi_n(s)$ of $\phi_n(s; \theta_2)$ is to be used at $s = 0$, it is clear that the value given by the continuous approximation will lie between the corresponding upper and lower bounds.

The approximation $\phi_2(s; \theta_2, C)$ is simple to calculate and is quite accurate. It is suggested here as a practical compromise approximation to be used in most designs. A brief table of data to facilitate the study of $\phi_2(s; \theta_2, C)$ and the corresponding bounds is given in Table 1.

TABLE 1

Data for the computation of $P_1(0; \theta_2)$

$\theta = \theta_2$,	$m = m_2 = r - 1$,	$\lambda = \lambda_2 = r - 1$,	$H = H_2 = (\log r)/(r - 1)$	
r	H_2	$\exp(-H_2)$	$Q_2(H_2; \theta_2)/r^2$	$\log [Q_2(H_2; \theta_2)/r^2]$
1.01	.995039	.369709	1.01343	.013341
1.05	.975804	.376889	1.06758	.065393
1.10	.953102	.385543	1.13626	.127741
1.15	.931747	.393865	1.20602	.187324
1.20	.911605	.401879	1.27682	.244376
1.25	.892576	.409600	1.34863	.299090
1.30	.874543	.417053	1.42142	.351658
1.40	.841180	.431202	1.56985	.450982
1.50	.810930	.444444	1.72191	.543433
1.75	.746155	.474187	2.11714	.750023
2.00	.693147	.500000	2.53198	.929003

5. Approximation of $P_2(0; \theta_1)$. Approximate formulas and bounds for $P_2(x; \theta_1)$ are to be found from the identity $P_2(x; \theta_1) = 1 - P_1(x; \theta_1)$ and the results given in Section 2 for P_1. Clearly, $P_2(0; \theta_1) \cong 1 - \phi_n(0; \theta_1, C)$ and $1 - \phi_n(0; \theta_1, U) \leqq P_2(0; \theta_1) \leqq 1 - \phi_n(0; \theta_1, L)$.

For the case $\theta = \theta_1$, one finds that $m = m_1 = 1 - (1/r)$, $\lambda = \lambda_1 = (1/r) - 1$, and $H = H_1 = rH_2$. It is easy to show that $Q_n(rx; \theta_1) \equiv 1/Q_n(x; \theta_2)$. From this it follows that

$$\delta_n(\theta_1, U)e^{-\lambda_1(B-nH_1)} = 1/\delta_n(\theta_2, L)e^{-\lambda_2(B-nH_2)},$$

$$\delta_n(\theta_1, L)e^{-\lambda_1(B-nH_1)} = 1/\delta_n(\theta_2, U)e^{-\lambda_2(B-nH_2)}.$$

These relations give the values of δ_n and C_n needed for bounds on $P_2(x; \theta_1)$ in terms of those used for bounds on $P_1(x; \theta_2)$. Clearly, the same reciprocal relationship may be used to obtain $\delta_n(\theta_1; C)$ and the corresponding value $C_n(\theta_1; C)$ for a continuous approximation to $P_2(x; \theta_1)$.

6. Approximate decision boundaries. Page [5] shows a simple method for an improvement of Wald's approximate formulas for setting the decision boundaries and estimating the expected sample size of a sequential test. He illustrates his method for a normal population. Epstein and Sobel [4] give improvements over Wald's formulas for the specific case of a semicontinuous sequential decision procedure on the mean of an exponential population. The latter authors study also the accuracy of their results by means of upper and lower bounds on the operating characteristic and expected sample size for their specific test setup. This line of attack will be continued in the sections that follow here by using the continuous approximations for $P_1(0;\theta_2)$ and $P_2(0;\theta_1)$ obtained above to derive a series of simple formulas of increasing accuracy for setting the decision boundaries a and $-b$ to achieve preassigned risk probabilities α and β. The corresponding upper and lower bounds for $P_1(0;\theta_2)$ and $P_2(0;\theta_1)$ will be used to establish the accuracy of the formulas for a and $-b$.

Assume that, for a chosen n, the boundaries will be such that $-b + nh < 0 < a$. One then has the equations

$$\beta = \frac{e^{\lambda_2 H_2} - e^{-\lambda_2 A_2}}{\delta_n(\theta_2\,;C) - e^{-\lambda_2 A_2}}, \qquad \alpha = 1 - \frac{e^{\lambda_1 H_1} - e^{-\lambda_1 A_1}}{\delta_n(\theta_1\,;C) - e^{-\lambda_1 A_1}} \tag{16}$$

to be solved for a and b. By (13) and the remarks in Section 5,

$$\delta_n(\theta_2\,;C) = e^{\lambda_2(B_2 - nH_2)} Q_n(H_2\,;\theta_2) = \frac{e^b Q_n(H_2\,;\theta_2)}{r^n},$$

$$\delta_n(\theta_1\,;C) = e^{\lambda_1(B_1 - nH_1)} \,/\, Q_n(H_2\,;\theta_2) = \frac{r^n e^{-b}}{Q_n(H_2\,;\theta_2)},$$

where

$$Q_n(H_2\,;\theta_2) = \frac{G_n\{-B_2 + (n+1)H_2\}e^{(1+\lambda_2)H_2} - G_n\{-B_2 + nH_2\}}{G_n\{-B_2 + (n+1)H_2\}e^{H_2} - G_n\{-B_2 + nH_2\}}$$

and

$$e^{\lambda_i H_i} = 1 + \lambda_i, \qquad \lambda_i A_i = (-1)^i a, \qquad \lambda_i B_i = (-1)^i b, \qquad i = 1, 2.$$

The solution of (16) for a and $-b$ is readily found to be

$$\begin{aligned} a &= \log\frac{1-\beta}{\alpha} - \log r, \\ -b &= \log\frac{\beta}{1-\alpha} + \log Q_n(H_2, \theta_2) - (n+1)\log r. \end{aligned} \tag{17}$$

The assumptions made in deriving (17) are easily checked in any special case. Wald's results [6] were the first terms on the right in (17). It is remarkable that so simple a modification of his results should suffice for the accuracy that will be indicated in Section 8.

It might be noted that the results (16) and (17) could have been stated in terms of a general starting point x_0 for the test if $-b + nh < x_0 < a$. The effect would have been to replace a and $-b$ by $a - x_0$ and $-b - x_0$, respectively.

A judgment of the accuracy of (17) in any given example is easily obtained by computing bounds upon the true values of $P_1(0; \theta_2)$ and $P_2(0; \theta_1)$ generated by the use of (17). One finds

$$(18)\quad \begin{aligned} \frac{r^{n+1}e^{a} - r^{n}}{e^{a+b}K_1 - r^{n}} &\leqq P_1(0; \theta_2) \leqq \frac{r^{n+1}e^{a} - r^{n}}{e^{a+b}K_2 - r^{n}}, \\ \frac{e^{b}K_1 - r^{n+1}}{re^{a+b}K_1 - r^{n+1}} &\leqq P_2(0; \theta_1) \leqq \frac{e^{b}K_2 - r^{n+1}}{re^{a+b}K_2 - r^{n+1}}, \end{aligned}$$

where

$$K_1 = \max_{(0,H_2)} Q_n(x; \theta_2), \qquad K_2 = \min_{(0,H_2)} Q_n(x; \theta_2).$$

These bounds are to be evaluated by use of the results

$$(19)\quad e^{a} = \frac{1 - \beta}{\alpha r}, \qquad e^{a+b} = \frac{(1 - \alpha)(1 - \beta)r^{n}}{\alpha\beta Q_n(H_2\,; \theta_2)}$$

of (17) and a graph of $Q_n(x; \theta_2)$.

7. The ASN. It is convenient to transform the integral equation (7) as follows. The formal identity

$$1 + A - s = 1 - H + \int_{s-H}^{\infty} (1 + A - t)e^{-t+s-H}\, dt$$

may be written in the form

$$\begin{aligned} 1 + A - s &= 1 - H + \int_{s-H}^{-B} (1 + A - t)e^{-t+s-H}\, dt \\ &\qquad + \int_{-B}^{A} (1 + A - t)e^{-t+s-H}\, dt, \quad -B < s \leqq -B + H, \\ &= 1 - H + \int_{s-H}^{A} (1 + A - t)e^{-t+s-H}\, dt, \qquad -B + H \leqq s < A. \end{aligned}$$

From this and the equations (6) and (7), it is seen that the function

$$(20)\quad R(ms; \theta) \equiv (H - 1)M_1(ms; \theta) + 1 + A - s - (A + B)P_1(ms; \theta)$$

satisfies the integral equation

$$(21)\quad \begin{aligned} R(ms; \theta) &= H - B - s + \int_{-B}^{A} R(mt; \theta)e^{-t+s-H}\, dt, \quad -B < s \leqq -B + H, \\ &= \int_{s-H}^{A} R(mt; \theta)e^{-t+s-H}\, dt, \qquad -B + H \leqq s < A. \end{aligned}$$

Equation (21) for R is quite similar to equation (6) for P_1 and may be treated in much the same way. This will be indicated below. First, easily obtained bounds for R will be given. They are probably good enough for most practical purposes.

It is trivial to show that

$$1 - e^{s+B-H} \leqq H - B - s \leqq \frac{H(1 - e^{s+B-H})}{1 - e^{-H}}, \qquad -B \leqq s \leqq -B + H.$$

It follows at once from Part (ii) of Theorem 5 in [1] that

$$P_1(ms; \theta) \leqq R(ms; \theta) \leqq \frac{HP_1(ms; \theta)}{1 - e^{-H}}, \qquad -B < s < A.$$

Bounds on the expected sample size are then readily obtained by use of (20).

To obtain more accurate results, proceed in the manner of Sections 2 and 3. It is evident that on the subinterval $-B < s \leqq -B + H$, $R(ms; \theta) = H - B - s + D \exp(s - H + B)$, where D is some constant. Over the remaining subinterval of $(-B, A)$, $R(ms; \theta)$ satisfies the same differential-difference equation as does $P_1(ms; \theta)$. An exact continuous solution to (21) may be obtained. Approximate solutions and bounds for the solution to (21) may be found by the technique indicated in Section 2. Since accuracy in the determination of the ASN is not of basic importance in the design of sequential experiments, the calculations just indicated will be left to the reader who is interested.

8. Examples. The following examples are based on the values $\alpha = \beta = 0.05$ and upon two choices of r, $r = 1.1$ and $r = 1.5$. They will serve to show the order of accuracy to be expected and the computation needed in the use of the decision boundary formulas (17). Six-digit accuracy appears to be useful in the computations.

EXAMPLE 1. $r = 1.5$ *and* $n = 2$. For this case one needs $Q_2(x; \theta_2)$, $0 \leqq x \leqq H_2 = 0.810930$. From (11), it is easy to obtain

$$Q_2(x; \theta_2) = \frac{e^{rx} - g_2(x)}{e^x - g_2(x)},$$

$$g_2(x) = 1 + \frac{xe^{-H_2}(1 - H_2 e^{-H_2}) + \frac{1}{2}x^2 e^{-2H_2}}{1 - 2H_2 e^{-H_2} + \frac{1}{2}H_2^2 e^{-2H_2}},$$

$$= 1 + 0.826045x + 0.287005x^2.$$

This yields the results: $Q_2(H_2 ; \theta_2) = 3.874306$ and $\max Q_2(x; \theta_2) \cong 3.877046$, $\min Q_2(x; \theta_2) \cong 3.855812$. Formulas (17) then give $a = 2.53898$ and $b = 2.80647$. By the inequalities (18), the true values of $P_1(0; \theta_2)$ and $P_2(0; \theta_1)$ satisfy $0.04996 \leqq P_1(0; \theta_2) \leqq 0.05024$ and $0.0499874 \leqq P_2(0; \theta_1) \leqq 0.0500017$.

As a comparison, the choice $n = 1$ gives $a = 2.53898$ and $b = 2.78504$ and the bounds on $P_1(0; \theta_2)$ are 0.05000 and 0.05166. For the choice $n = 3$, the bounds on $P_1(0; \theta_2)$ would be 0.049975 and 0.0500099.

It is interesting to note that the choice of decision boundaries from Wald [6] would be $a = b = 2.94444$. For these values, the bounds (18) give $0.04428 \leqq P_1(0; \theta_2) \leqq 0.04452$ and $0.03269 \leqq P_2(0; \theta_1) \leqq 0.03353$.

EXAMPLE 2. $r = 1.1$ *and* $n = 2$. For this case $H_2 = 0.953102$ and $Q_2(H_2 ; \theta_2) =$

1.374880, max $Q_2(x;\, \theta_2) \cong 1.375097$ and min $Q_2(x;\, \theta_2) \cong 1.373430$. These yield the decision boundaries $a = 2.84913$ and $b = 2.91201$ and the bounds

$$0.049992 \leqq P_1(0;\, \theta_2) \leqq 0.050052,\; 0.049997 \leqq P_2(0;\, \theta_1) \leqq 0.0500004.$$

In this example, the choice $n = 1$ might be satisfactory. It yields $a = 2.84913$, $b = 2.90703$ with the bounds $0.5000 \leqq P_1(0;\, \theta_2) \leqq 0.05043$.

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