A WAITING LINE PROCESS OF MARKOV TYPE[1]

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Summary. Waiting-line or queuing processes of the Markov type are studied, the incoming traffic being of Poisson type and having negative-exponential holding time. The parameters are allowed to depend on time. The problem of finding an exact solution for the probability distribution of the waiting-line length as a function of time is reduced to the solution of an integral equation of the Volterra type. When the ratio of the parameters for the incoming and outgoing traffic is constant, this equation can be solved explicitly and the required distribution obtained. Using this solution, the behavior of the process for large values of t is studied, particularly for the unstable case with traffic intensity $\geqq 1$.

Statement of the problem. We shall consider a Markov process $\mathbf{n}(t)$ taking values in the discrete space of nonnegative integers $0, 1, 2, \cdots$, for which there exist nonnegative continuous functions $\lambda(t)$ and $\mu(t)$ satisfying

(i) for each $n_0 = 0, 1, 2, \cdots$, $\Pr\{\mathbf{n}(t + \Delta t) - \mathbf{n}(t) = 1 \mid \mathbf{n}(t) = n_0\}$
$$= \lambda(t)\Delta t + o(\Delta t),$$

(ii) for each $n_0 = 1, 2, 3, \cdots$, $\Pr\{\mathbf{n}(t + \Delta t) - \mathbf{n}(t) = -1 \mid \mathbf{n}(t) = n_0\}$
$$= \mu(t)\Delta t + o(\Delta t),$$

(iii) $\Pr\{\mid \mathbf{n}(t + \Delta t) - \mathbf{n}(t) \mid > 1\} = o(\Delta t)$.

Intuitively, this states that the probability of an "arrival" to the "waiting-line" during the time interval $(t, t + \Delta t)$ is $\lambda(t)\Delta t + o(\Delta t)$, and the probability of a "departure" during this interval is $\mu(t)\Delta t + o(\Delta t)$. Thus the system differs from a process which is simply the difference of two independent Poisson processes ("arrivals to" and "departures from" the waiting line) only in that $\mathbf{n}(t)$ is restricted to nonnegative values.

Letting

$$P_{\nu,n} = P_{\nu,n}(t) = \Pr\{\mathbf{n}(t) = n \mid \mathbf{n}(0) = \nu\} \quad (n, \nu = 0, 1, 2, \cdots),$$

the basic "forward" set of Kolmogorov equations for the system becomes

$$\frac{d}{dt} P_{\nu,n} = -(\lambda(t) + \mu(t))P_{\nu,n} + \lambda(t)P_{\nu,n-1} + \mu(t)P_{\nu,n+1} \qquad (n > 0), \tag{1}$$

$$\frac{d}{dt} P_{\nu,0} = -\lambda(t)P_{\nu,0} + \mu(t)P_{\nu,1} \tag{2}$$

(see, for example, [4], p. 377).

Received September 24, 1954.

[1] The research reported here was supported in part by the Office of Naval Research, Contract N6 ONR 232-1.

Our basic problem is the solution of this system of differential equations under the initial conditions

$$P_{\nu,n}(0) = \delta_{\nu,n} \qquad (n, \nu = 0, 1, 2, \cdots). \tag{3}$$

Reduction to an integral equation. By the results of [5] one is assured of the existence of a unique, nonnegative, continuously differentiable system of solutions satisfying $\sum_{n=0}^{\infty} P_{\nu,n} \equiv 1$.

On rescaling the time axis by introducing the new variable

$$\tau = \int_0^t \mu(s)\, ds$$

in place of t, it is seen that equations (1) and (2) transform into a new system in which μ is identically equal to 1. This rescaling is possible provided $\mu(t)$ has only discrete zeros. This assumption, while in no way essential to the results of this paper, will be made in order to simplify the details.

Under these conditions, the system (1) through (3) becomes

$$\frac{d}{d\tau} P_{\nu,n} = -[\rho(\tau) + 1]P_{\nu,n} + \rho(\tau)P_{\nu,n-1} + P_{\nu,n+1} \qquad (n > 0), \tag{4}$$

$$\frac{d}{d\tau} P_{\nu,0} = -\rho(\tau)P_{\nu,0} + P_{\nu,1}, \tag{5}$$

$$P_{\nu,n}(0) = \delta_{\nu,n} \qquad (n, \nu = 0, 1, 2, \cdots), \tag{6}$$

where

$$\rho(\tau) = \frac{\lambda(t)}{\mu(t)};$$

this ratio represents, in the terminology of telephone waiting-line theory, the *instantaneous relative traffic intensity* of the process.

By analogy, the quantity

$$R(\tau) = \frac{\int_0^t \lambda(s)\, ds}{\int_0^t \mu(s)\, ds} = \frac{1}{\tau}\int_0^\tau \rho(\sigma)\, d\sigma$$

might be termed the *smoothed relative traffic intensity* of the process.

The system (4) through (6) can be simplified by introducing a new system of dependent variables:

$$Q_{\nu,n}(\tau) = e^{\tau[1+R(\tau)]}P_{\nu,n} \qquad (\nu, n = 0, 1, 2, \cdots).$$

In terms of these variables the system (4) through (6) becomes

$$\frac{d}{d\tau} Q_{\nu,n}(\tau) = \rho(\tau)Q_{\nu,n-1}(\tau) + Q_{\nu,n+1}(\tau) \qquad (n > 0), \tag{7}$$

$$\frac{d}{d\tau} Q_{\nu,0}(\tau) = Q_{\nu,0}(\tau) + Q_{\nu,1}(\tau), \tag{8}$$

$$Q_{\nu,n}(0) = \delta_{\nu,n} \qquad (\nu, n = 0, 1, 2, \cdots). \tag{9}$$

In order to reduce this difference-differential equation (7) to a partial differential equation, we introduce the generating function

$$Q_\nu(z, \tau) = \sum_{n=0}^{\infty} Q_{\nu,n}(\tau) \frac{(z-\tau)^n}{n!} \qquad (\nu = 0, 1, 2, \cdots).$$

This function will be analytic in z and continuously differentiable in τ. Differentiation with respect to z and τ gives, using (7), the following hyperbolic partial differential equation for $Q_\nu(z, \tau)$:

$$\frac{\partial^2 Q_\nu}{\partial\tau\, \partial z} = \rho(\tau) Q_\nu. \tag{10}$$

The solution of such an equation ("the Telegrapher's equation") in general requires two boundary conditions. These are given by (8) and (9) which transform into

$$\left. \frac{\partial Q_\nu(z, \tau)}{\partial \tau} \right|_{z=\tau} = Q_\nu(\tau, \tau), \tag{11}$$

and

$$Q_\nu(z, 0) = \frac{z^\nu}{\nu!}. \tag{12}$$

In the method of solution to be used here, we first solve for $Q_\nu(z, \tau)$ in terms of the (unknown) function

$$f_\nu(\tau) = \frac{\partial Q_\nu(0, \tau)}{\partial \tau} \tag{13}$$

using the classical Riemann method, with boundary conditions (12) and (13) (see, for example, [3], p. 316). The condition (11) is then used to derive an integral equation for $f_\nu(\tau)$.

The Riemann function associated with (10) is easily seen to be

$$I_0[2\{[R(\tau)\tau - R(\sigma)\sigma](z - \zeta)\}^{1/2}],$$

where $I_n(u)$ denotes the modified Bessel function $i^{-n} J_n(iu)$. Application of standard methods and integration formulas for Bessel functions ([11], p. 373) gives the solution

$$Q_\nu(z, \tau) = A_\nu(0, \tau, z) + \int_0^\tau A_0(\sigma, \tau, z) f_\nu(\sigma)\, d\sigma, \tag{14}$$

where

$$A_n(\sigma, \tau, z) = z^{n/2}[R(\tau)\tau - R(\sigma)\sigma]^{-n/2} I_n[2\{[R(\tau)\tau - R(\sigma)\sigma]z\}^{1/2}] \tag{15}$$

$$(n = 0, \pm 1, \pm 2, \cdots).$$

Noting that

$$\frac{\partial}{\partial z} A_n(\sigma, \tau, z) = A_{n-1}(\sigma, \tau, z),$$

(16)
$$\frac{\partial}{\partial \tau} A_n(\sigma, \tau, z) = \rho(\tau) A_{n+1}(\sigma, \tau, z),$$

$$A_n(0, 0, 0) = \delta_{0,n},$$

$$A_{-n}(\tau, \tau, \tau) = 0 \qquad \text{for } n > 0, A_0(\tau, \tau, \tau) = 1,$$

one finds

(17)
$$P_{\nu,n} = e^{-\tau[1+R(\tau)]} \left. \frac{\partial^n Q(z, \tau)}{\partial z^n} \right|_{z=\tau}$$

$$= e^{-\tau[1+R(\tau)]} \left[A_{\nu-n}(0, \tau, \tau) + \int_0^\tau A_{-n}(\sigma, \tau, \tau) f_\nu(\sigma)\, d\sigma \right].$$

On substituting this into (11) using (16), one obtains the following Volterra-type integral equation for $f_\nu(\tau)$:

(18)
$$f_\nu(\tau) = B_\nu(0, \tau) + \int_0^\tau B_0(\sigma, \tau) f_\nu(\sigma)\, d\sigma,$$

where

$$B_n(\sigma, \tau) = A_n(\sigma, \tau, \tau) - \rho(\tau) A_{n+1}(\sigma, \tau, \tau).$$

Consequently, (17) gives the solution to our problem, provided a solution $f_\nu(\tau)$ to (18) can be found. In the important special case of $\rho(\tau) =$ *constant*, (18) can be solved explicitly. In other cases it provides information as to the limiting behavior of $f_\nu(\tau)$.

The case of constant traffic intensity. Let us now assume the relative traffic intensity to be constant, $\rho(\tau) \equiv \rho$. Under these conditions $R(\tau) \equiv \rho$, as well.

Note that the three conditions: $\rho(\tau) \equiv$ *constant*, $R(\tau) \equiv$ *constant*, and $\rho(\tau) \equiv R(\tau)$ are all equivalent.

Several methods are available for obtaining the explicit solution of (18). The one used here is possibly the simplest if nót the most elegant.

Let us now assume that $f_\nu(\tau)$ is representable by a power series

(19)
$$f_\nu(\tau) = \sum_{k=0}^{\infty} a_{\nu,k} \tau^k,$$

convergent for all values of τ. This can be proved directly using (7); however, this is not required. If a solution of (18) can be found in the form of such a power series, then the uniqueness property for the solutions of such an integral equation assures us that this series must be $f_\nu(\tau)$.

Using Sonine's integral formula (see, for example, [11], p. 373), we have the following formula

$$\int_0^\tau A_{-n}(\sigma, \tau, \tau)[R(\sigma)\sigma]^k \rho(\sigma)\, d\sigma = k! A_{-n-k-1}(0, \tau, \tau),$$

for $n \geqq 0$, $k \geqq 0$. In the case $R(\tau) = \rho(\tau) = \rho$, this becomes

$$\int_0^\tau A_{-n}(\sigma, \tau, \tau)\sigma^k\, d\sigma = \frac{k!}{\rho^{k+1}} A_{-n-k-1}(0, \tau, \tau).$$

Since $I_{-n}(u) = I_n(u)$, $A_{-n}(0, \tau, \tau) = \rho^n A_n(0, \tau, \tau)$, this is also equivalent to

$$\int_0^\tau A_{-n}(\sigma, \tau, \tau)\, \sigma^k\, d\sigma = k! \rho^n A_{n+k+1}(0, \tau, \tau). \tag{20}$$

Consequently, on substituting (19) into (14) and integrating, one finds

$$Q_\nu(z, \tau) = A_\nu(0, \tau, z) + \sum_{k=0}^{\infty} a_{\nu,k}\, k!\, A_{k+1}(0, \tau, z).$$

Substituting this into (11) using (16), one obtains the identity

$$\begin{aligned} A_\nu(0, \tau, \tau) &- \rho A_{\nu+1}(0, \tau, \tau) \\ &= a_{\nu,0} A_0(0, \tau, \tau) + \sum_{k=1}^{\infty} (a_{\nu,k} k! - a_{\nu,k-1}(k-1)!) A_k(0, \tau, \tau). \end{aligned} \tag{21}$$

On equating coefficients of $A_k(0, \tau, \tau)$ $(k = 0, 1, 2, \cdots)$, one finds the following recurrence relations for the $a_{\nu,k}$:

$$a_{\nu,0} = 0 \qquad \text{for} \qquad \nu > 0,$$

$$a_{\nu,k} = \frac{1}{k} a_{\nu,k-1}, \qquad \text{for } 0 < k < \nu \quad \text{or} \quad \nu + 1 < k,$$

$$a_{\nu,\nu} = \frac{1}{\nu} a_{\nu,\nu-1} + \frac{1}{\nu!},$$

$$a_{\nu,\nu+1} = \frac{1}{\nu+1} a_{\nu,\nu} - \frac{\rho}{(\nu+1)!}.$$

The solution of this system is found to be

$$a_{\nu,k} = 0, \qquad k < \nu,$$

$$a_{\nu,\nu} = \frac{1}{\nu!},$$

$$a_{\nu,k} = \frac{(1-\rho)}{k!}, \qquad k > \nu,$$

whence

$$f_\nu(\tau) = \frac{\tau^\nu}{\nu!} + (1-\rho) \sum_{k=\nu+1}^{\infty} \frac{\tau^k}{k!} = (1-\rho)\left(e^\tau - \sum_{k=0}^{\nu-1} \frac{\tau^k}{k!}\right) + \frac{\rho\tau^\nu}{\nu!}. \tag{22}$$

In the special case $\nu = 0$,

$$f_0(\tau) = (1 - \rho)e^{\tau} + \rho.$$

Substituting (22) into (17) and integrating term-by-term using (20), one finds the final solution

$$P_{\nu,n} = e^{-\tau(1+\rho)} \Big[A_{\nu-n}(0, \tau, \tau) + \rho^n A_{n+\nu+1}(0, \tau, \tau) + (1 - \rho)\rho^n \sum_{k=n+\nu+2}^{\infty} A_k(0, \tau, \tau) \Big]. \tag{23}$$

Using a table of Bessel functions, $P_{\nu,n}$ can be tabulated for various values of τ and ρ from this formula. Some tables are available for the case $\nu = 0$ (see [2]). For other values of ν, $P_{\nu,n}$ can be found from the formula

$$P_{\nu,n} = \rho^{-\nu} P_{0,\nu+n} + e^{-\tau(1+\rho)}[A_{\nu-n}(0, \tau, \tau) - \rho^n A_{\nu+n}(0, \tau, \tau)],$$

which is easily derived from (23).

Formulas essentially equivalent to (23) have been derived by Ledermann and Reuter [9] and by Bailey [1] for the case of constant λ and μ, using somewhat different methods.

Limiting formulas for mean and variance. All the well-known limiting results for the case $\rho =$ *constant* can be derived directly from (23). When $\rho < 1$, the probability distribution of $\mathbf{n}(t)$ approaches a geometric equilibrium distribution with common ratio ρ as $\tau \to \infty$, independent of ν. When $\rho \geqq 1$, no such limiting distribution exists. (See [7] and [8] for precise statements of the results in this and in more general cases.)

Let us temporarily drop the restriction that ρ be constant, and proceed to develop formulas for the mean $M_\nu(\tau)$ and the standard deviation $\sigma_\nu(\tau)$ of the distribution. By definition,

$$M_\nu(\tau) = \sum_{n=0}^{\infty} nP_{\nu,n},$$

$$\sigma_\nu^2(\tau) = \sum_{n=0}^{\infty} [n - M_\nu(\tau)]^2 P_{\nu,n}.$$

Assuming term-by-term differentiation to be justified (which can easily be proved), these series may be differentiated using (4), together with the fact that $\Sigma P_{\nu,n} = 1$, to give

$$\frac{d}{d\tau} M_\nu(\tau) = \rho(\tau) - 1 + P_{\nu,0},$$

$$\frac{d}{d\tau} \sigma_\nu^2(\tau) = 2[\rho(\tau) + (\rho(\tau) - 1)M_\nu(\tau)] - [1 + 2M_\nu(\tau)] \frac{dM_\nu(\tau)}{d\tau}$$

whence

$$(24)\quad M_\nu(\tau) = [R(\tau) - 1]\tau + \int_0^\tau P_{\nu,0}(\sigma)\, d\sigma + \nu,$$

$$(25)\quad \sigma_\nu^2(\tau) = 2\int_0^\tau [\rho(\sigma) + [\rho(\sigma) - 1]M_\nu(\sigma)]\, d\sigma - M_\nu(\tau) - [M_\nu(\tau)]^2 + \nu + \nu^2.$$

If $\rho(\tau)$ is bounded for all $\tau \geqq 0$, then one sees from (24) and (25) that both $M_\nu(\tau)$ and $\sigma_\nu(\tau) = O(\tau)$ as $\tau \to \infty$.

In the case of constant $\rho \geqq 1$, (24) and (25) will be used to determine more explicitly the limiting behavior of $M_\nu(\tau)$ and $\sigma_\nu(\tau)$ as $\tau \to \infty$. (It is here assumed that $\tau = \int_0^t \mu(s)\, ds \to \infty$ as $t \to \infty$.)

Let us first assume that $\rho \equiv$ *constant* and that $\rho > 1$. Using standard integration formulas for Bessel functions ([11], p. 386), one finds that

$$\int_0^\infty e^{-\tau(1+\rho)} A_n(0, \tau, \tau)\, d\tau = \frac{1}{\rho^n(\rho - 1)}.$$

When this formula is used to integrate (23) term-by-term, one obtains

$$\begin{aligned}\int_0^\infty P_{\nu,0}(\sigma)\, d\sigma &= \frac{1}{\rho^\nu(\rho - 1)} + \frac{1}{\rho^{\nu+1}(\rho - 1)} + (1 - \rho) \sum_{k=\nu+2}^{\infty} \frac{1}{\rho^k(\rho - 1)} \\ &= \frac{1}{\rho^\nu(\rho - 1)}.\end{aligned}$$

Consequently,

$$\int_0^\tau P_{\nu,0}(\sigma)\, d\sigma = \frac{1}{\rho^\nu(\rho - 1)} + o(1),$$

and, substituting this result into (24),

$$(26)\qquad M_\nu(\tau) = (\rho - 1)\tau + \frac{1}{\rho^\nu(\rho - 1)} + \nu + o(1).$$

If (26) is substituted into (25), then it is easily shown that

$$(27)\qquad \sigma_\nu(\tau) = \sqrt{\tau(\rho + 1)} + o(\sqrt{\tau}).$$

Let us now assume $\rho = 1$. In this case, from (23),

$$\begin{aligned}P_{\nu,0}(\tau) &= e^{-2\tau}[I_\nu(2\tau) + I_{\nu+1}(2\tau)] \\ &= \frac{1}{\sqrt{\pi\tau}}\left[1 + O\left(\frac{1}{\tau}\right)\right],\end{aligned}$$

using an asymptotic formula for $I_n(z)$. Consequently,

$$(28)\qquad M_\nu(\tau) = \int_0^\tau P_{\nu,0}(\sigma)\, d\sigma + \nu = 2\sqrt{\tau/\pi} + O(1),$$

and

$$\sigma_\nu(\tau) = \sqrt{2\tau(1 - 2/\pi)} + O(1). \tag{29}$$

Note that neither (26) nor (27) reduces to (28) or (29) when ρ is set equal to 1.

Waiting times. If $\mathbf{T}(\tau)$ is a random variable representing the time required to complete the servicing of an individual arriving at time τ, then

$$\mathbf{T}(\tau) = \mathbf{S}[\mathbf{n}(\tau)]$$

where $\mathbf{S}[n]$ is a random variable independent of $\mathbf{n}(\tau)$ and represents the time required for $n + 1$ transitions in a Poisson process having parameter 1; i.e., $\mathbf{S}[n]$ is the sum of $n + 1$ independent random variables each having the probability density function $e^{-\tau}$, $0 < \tau < \infty$, and will thus have a Gamma distribution. Khintchine [6] and Volberg [10] have derived asymptotic formulas for the distribution of $\mathbf{T}(\tau)$ as $\tau \to \infty$ for the case of constant ρ. Using the above formula, one sees that the probability density function for $\mathbf{T}(\tau)$ is

$$\varphi(s; \tau) = e^{-s} \sum_{n=0}^{\infty} P_{\nu,n} \frac{s^n}{n!} = e^{-[s+\tau(1+\rho)]} Q_\nu(s + \tau, \tau),$$

for $s > 0$. By using this result, these asymptotic formulas may be derived from the results of this paper.

Note. I am indebted to the referee for the references to the papers of Volberg [10] and Ledermann and Reuter [9].

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