THE WAGR SEQUENTIAL t-TEST REACHES A DECISION WITH PROBABILITY ONE[1]

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0. Summary. The WAGR test[3] is a sequential procedure for testing the null hypothesis that the proportion of a normal population greater than a given constant is p_0 (given) against the alternative that it is p_1 (given). These are equivalent (after a translation) to hypotheses specifying the value of μ/σ, where μ and σ^2 are the mean and the variance of the normal population under test. We prove that, with probability one, a decision is reached when the WAGR test is applied. This fact is of importance in its own right; it also has indirect interest because, unless it were true, the standard Wald inequalities on probabilities of error at the two hypothesis points could not be applied.

1. Introduction. The WAGR sequential test for one-sided proportion defective may formally be described as follows: Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed normal random variables with mean μ and variance σ^2. Let U be a given number. Let

$$u_n = \sum_{i=1}^{n} (U - X_i) \bigg/ \left(\sum_{i=1}^{n} (U - X_i)^2 \right)^{1/2}.$$

Let α and β be given numbers, between 0 and 1, such that $\alpha + \beta < 1$. Define for reference the inequality

$$\ln \frac{\beta}{1-\alpha} \leqq l_n(u_n) \leqq \ln \frac{1-\beta}{\alpha}, \tag{1.1}$$

where

$$l_n(u_n) = \tfrac{1}{2}(n - u_n^2)(K_0^2 - K_1^2) + \ln \frac{Hh_{n-1}(-u_n K_1)}{Hh_{n-1}(-u_n K_0)}, \tag{1.2}$$

$$Hh_\nu(x) = \int_0^\infty \frac{z^\nu}{\nu!} \exp\left[-\tfrac{1}{2}(z + x)^2\right] dz, \tag{1.3}$$

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[3] The name WAGR stems from the initials of those individuals who suggested and developed the test in question: A. Wald, K. Arnold, Goldberg ([4] p. 83n), and Rushton [8]. The name of G. A. Barnard [1] should be added to this list, and in fact a particular case of the WAGR test has been called Barnard's test in at least one exposition [3]. The test may be called a (one-sided) sequential t-test, although perhaps this name should be reserved for the special case in which $p_0 = \frac{1}{2}$ (i.e., $\mu = 0$ under the null hypothesis).

and $K_1 < K_0$ are given. K_i $(i = 0, 1)$ is the unit-normal deviate exceeded with given probability p_i. It is no loss of generality to take $p_0 < p_1$, and consequently $K_1 < K_0$.

Proceed by observing X_1, X_2, and computing u_2 and $l_2(u_2)$. If (1.1) is broken at the left, accept $H_0: (U - \mu)/\sigma = K_0$; if (1.1) is broken at the right, accept $H_1: (U - \mu)/\sigma = K_1$. If (1.1) is satisfied, observe X_3, compute u_3 and $l_3(u_3)$, and look at (1.1) with $n = 3$. If (1.1) is broken, take the appropriate decision; otherwise, continue taking observations, each time looking at (1.1), until (1.1) is broken. When (1.1) is broken, stop taking observations. The rationale behind the above procedure is discussed in [8], [1], and in Chapter 1 of [4].

It is important to know whether or not the WAGR test procedure as described above will lead to a decision at finite n for almost every sample sequence, that is with probability one, whatever $(U - \mu)/\sigma = K$ in fact happens to be. Were this not the case, the WAGR test might not be a satisfactory statistical procedure, and doubt would be thrown on the accuracy of the Wald approximation to the probabilities of Type I and Type II errors. (This approximation says that the probability of Type I error is approximately α and that the probability of Type II error is approximately β; its derivation depends on the knowledge that almost every sample sequence leads to a decision at finite n.)

The proof to be given is rather direct. An alternative proof could no doubt be obtained by the method used by Barnard [1], in which the WAGR test is thought of as a limiting procedure in a sequence of tests each of which is a weighted sequential probability ratio test in the sense of Wald ([9], Section 4.2.2), weighted by a distribution on population variance. Such a proof would depend on knowing that the weighted sequential probability ratio tests mentioned in the last sentence themselves reach decisions with probability one. Although Wald ([9], Section 3.2) states that under general circumstances such weighted procedures do reach a decision with probability one, so far as we know no exact statement or proof of this has appeared in the literature.

It is also possible that a proof might be had along the lines expounded by Nandi [7], but we are not able to follow Nandi's arguments.

In order to prove that almost every sequence leads to decision at finite n, it is useful to rewrite $l_n(u_n)$ in terms of the variable $v_n = u_n/\sqrt{n}$. We shall feel free to drop the subscript n when convenient. Then, letting $\nu = n - 1$, $l_n(u_n)$, the center of (1.1), becomes, in terms of v_n,

$$
\begin{aligned}
l_n &= \frac{\nu + 1}{2}(1 - v_n^2)(K_0^2 - K_1^2) + \ln \frac{Hh_\nu(-\sqrt{\nu + 1}\, v_n K_1)}{Hh_\nu(-\sqrt{\nu + 1}\, v_n K_0)} \\
&= \frac{\nu + 1}{2}(K_0^2 - K_1^2) + \ln \frac{\int_0^\infty z^\nu \exp\left(-\tfrac{1}{2}z^2 + z\sqrt{\nu + 1}\, v_n K_1\right) dz}{\int_0^\infty z^\nu \exp\left(-\tfrac{1}{2}z^2 + z\sqrt{\nu + 1}\, v_n K_0\right) dz}.
\end{aligned} \tag{1.4}
$$

It is known that for n (or ν) fixed, l_n as a function of v_n is strictly monotone decreasing (see [5], and note that our v_n is u of [5]). Also $v_n^2 \leq 1$.

Hence (1.1) is equivalent to

$$A_n \geqq v_n \geqq R_n\,, \tag{1.5}$$

where A_n and R_n are acceptance (accept H_0) and rejection (accept H_1) numbers for v_n, functions of n, K_1, and K_0.

The first problem is to show that A_n and R_n have the same limit as $n \to \infty$. From this it will be possible to show that the probability of decision is unity.

Note that A_n and R_n are the solutions respectively of

$$l_n(A_n) = \ln \frac{\beta}{1-\alpha}, \qquad l_n(R_n) = \ln \frac{1-\beta}{\alpha}. \tag{1.6}$$

Hence if we can show that there exists a number L such that $\lim_{n\to\infty} l_n(v) = \infty(-\infty)$ as v is $<(>)L$, we may conclude that $\lim_{n\to\infty} A_n = \lim_{n\to\infty} R_n = L$, the desired result.

In order to show this it is essential to obtain asymptotic formulas for the integrals appearing in (1.4). This is our first step.

2. Asymptotic formulas for the integrals of (1.4). This section is devoted to the statement and proof of the following Lemma:

LEMMA.

$$\int_0^\infty z^\nu \exp\left(-\tfrac{1}{2}z^2 + z\sqrt{\nu+1}Kv\right) dz \backsim \sqrt{2\pi}(\bar{z}/e)^\nu \exp\left(\tfrac{1}{2}\bar{z}^2\right)$$
$$\times \{1 + (\sqrt{K^2v^2+4} - Kv)^2/4\}^{-1/2}$$

as $\nu \to \infty$. *K and v are fixed and*

$$\bar{z} = \tfrac{1}{2}\sqrt{\nu+1}Kv + \sqrt{\tfrac{1}{4}(\nu+1)K^2v^2 + \nu}.$$

The right side of the above asymptotic relation may be replaced by

$$\sqrt{2\pi}(\bar{z}/e)^\nu \exp\left(\tfrac{1}{2}\bar{z}^2\right)\sqrt{\bar{z}^2/(\bar{z}^2+\nu)}.$$

Since K and v always appear together in this section as a product, we may as well set $Kv = w$. The integral of interest then is

$$\int_0^\infty z^\nu \exp\left(-\tfrac{1}{2}z^2 + z\sqrt{\nu+1}\,w\right) dz. \tag{2.1}$$

The maximum of the integrand occurs at (differentiate)

$$\bar{z} = \tfrac{1}{2}\sqrt{\nu+1}\,w + \sqrt{\tfrac{1}{4}(\nu+1)w^2 + \nu} \tag{2.2}$$

satisfying

$$\bar{z}^2 - \bar{z}\sqrt{\nu+1}\,w - \nu = 0 \tag{2.3}$$

and suggesting the change of variable $y = z - \bar{z}$. Thus we may write

$$(2.1) = \int_{-\bar{z}}^\infty (y+\bar{z})^\nu \exp\left[-\tfrac{1}{2}(y+\bar{z})^2 + (y+\bar{z})\sqrt{\nu+1}\,w\right] dy \tag{2.4}$$

or, by virtue of (2.3),

$$(2.5)\quad (2.1) = \bar{z}^{\nu} \exp\left(-\tfrac{1}{2}\bar{z}^2 + \bar{z}\sqrt{\nu+1}\, w\right) \int_{-\bar{z}}^{\infty} \left(1 + \frac{y}{\bar{z}}\right)^{\nu} \exp\left(-\tfrac{1}{2}y^2 - y\frac{\nu}{\bar{z}}\right) dy.$$

The factors preceding the integral equal, again using (2.3),

$$\bar{z}^{\nu} \exp\left(-\tfrac{1}{2}\bar{z}^2 + \bar{z}^2 - \nu\right) = (\bar{z}/e)^{\nu} \exp\left(\bar{z}^2/2\right).$$

It will suffice then to show that the integral in (2.5) has a limit, as $\nu \to \infty$, equal to the square root of the quantity in curly brackets in the statement of the lemma.

Note that

$$\bar{z} = \frac{\sqrt{\nu+1}}{2}\left[w + \sqrt{w^2 + 4\frac{\nu}{\nu+1}}\right]$$

and hence that $\bar{z} \to \infty$ as $\sqrt{\nu}$. Now rewrite the integral of (2.5) in the form

$$(2.6)\quad \int_{-\infty}^{\infty} \varphi_{\bar{z}}(y) \exp\left\{-\tfrac{1}{2}y^2 + \nu\left[\ln\left(1 + \frac{y}{\bar{z}}\right) - \frac{y}{\bar{z}}\right]\right\} dy,$$

where $\varphi_{\bar{z}}(y) = 0$ or 1 as $y <$ or $\geqq -\bar{z}$. It is readily checked that when $-\bar{z} \leqq y < \infty$

$$\ln\left(1 + \frac{y}{\bar{z}}\right) - \frac{y}{\bar{z}} \leqq 0.$$

Hence, for all y,

$$0 \leqq \varphi_{\bar{z}}(y) \exp\left\{-\tfrac{1}{2}y^2 + \nu\left[\ln\left(1 + \frac{y}{\bar{z}}\right) - \frac{y}{\bar{z}}\right]\right\} \leqq \exp\left(-\tfrac{1}{2}y^2\right),$$

and $\exp\left(-\tfrac{1}{2}y^2\right)$ is integrable. Thus, by Lebesgue's theorem, the limit of (2.6) as $\nu \to \infty$ is the integral of the pointwise limit. But for each y

$$\lim_{\nu\to\infty} \varphi_{\bar{z}}(y) \exp\left\{-\tfrac{1}{2}y^2 + \nu\left[\ln\left(1 + \frac{y}{\bar{z}}\right) - \frac{y}{\bar{z}}\right]\right\} = \exp\left[-\tfrac{1}{2}y^2\left(1 + \lim\frac{\nu}{\bar{z}^2}\right)\right]$$
$$= \exp\left\{-\tfrac{1}{2}y^2\left[1 + \left(\sqrt{w^2+4} - w\right)^2/4\right]\right\}.$$

Hence the limit of (2.6) is the square root of

$$\frac{2\pi}{1 + \left(\sqrt{w^2+4} - w\right)^2/4}.$$

This gives the first asymptotic form of the lemma. The second is obtained by noting that

$$\lim\left(\bar{z}/\sqrt{\nu}\right) = 2/\left(\sqrt{w^2+4} - w\right),$$

so that (2.6) is asymptotically equivalent to the square root of

$$\frac{2\pi}{1 + 4\nu/(4\bar{z}^2)} = 2\pi\frac{\bar{z}^2}{\bar{z}^2 + \nu}.$$

For $Kv = 0$, the above lemma provides an asymptotic expression for the Gamma function that is equivalent to Stirling's formula. Similar methods have been applied to obtain inequalities on the Gamma integral. See, for example: J. R. Wilton, "A note on Stirling's theorem," *Mathematical Notes* (Edinburgh Mathematical Society), No. 28 (1933), xii–xiii.

3. The limit of l_n. From the lemma we see that for large ν and fixed v, (1.4) behaves like

$$\frac{\nu+1}{2}(K_0^2 - K_1^2) + \nu \ln\left(\frac{\bar{z}_1}{\bar{z}_0}\right) + \tfrac{1}{2}(\bar{z}_1^2 - \bar{z}_0^2) + \frac{1}{2}\ln \frac{1 + \dfrac{4}{[K_0 v + \sqrt{K_0^2 v^2 + 4}]^2}}{1 + \dfrac{4}{[K_1 v + \sqrt{K_1^2 v^2 + 4}]^2}}, \tag{3.1}$$

where for $i = 0, 1$

$$\bar{z}_i = \tfrac{1}{2}\sqrt{\nu + 1}K_i v + \sqrt{\tfrac{1}{4}(\nu + 1)K_i^2 v^2 + \nu}. \tag{3.2}$$

For fixed v, the last term of (3.1) is a constant and the first three terms are of order ν.

Hence, as the following quantity is positive or negative,

$$\frac{1}{2}(K_0^2 - K_1^2) + \ln \frac{K_1 v + \sqrt{K_1^2 v^2 + 4}}{K_0 v + \sqrt{K_0^2 v^2 + 4}} + \tfrac{1}{4}v^2(K_1^2 - K_0^2) + \frac{v}{4}\{K_1\sqrt{K_1^2 v^2 + 4} - K_0\sqrt{K_0^2 v^2 + 4}\} \tag{3.3}$$

l_n will have the limit $+\infty$ or $-\infty$. (In obtaining (3.3), (2.3) is used.)

Next we wish to show that there exists a number L ($|L| < 1$) such that (3.3) is positive ($l_n \to \infty$) for $v < L$ and (3.3) is negative ($l_n \to -\infty$) for $v > L$. Such an L will obviously be unique. The derivative of (3.3) with respect to v is

$$\frac{1}{2}([K_1\sqrt{K_1^2 v^2 + 4} - K_0\sqrt{K_0^2 v^2 + 4}] + \frac{v}{2}(K_1^2 - K_0^2). \tag{3.4}$$

We shall show that (3.4) is negative, that (3.3) is positive at $v = -1$, and that (3.3) is negative at $v = 1$; these three facts suffice to show the existence of L with the stated properties.

First, (3.4) is, for any fixed v, equal to $f(K_1) - f(K_0)$, where

$$f(K) = \tfrac{1}{2}[K\sqrt{K^2 v^2 + 4} + Kv^2]. \tag{3.5}$$

Since

$$\frac{d}{dK}f(K) = [\sqrt{K^2v^2 + 4} + Kv]^2/\sqrt{K^2v^2 + 4} > 0, \tag{3.6}$$

and $K_1 < K_0$ by hypothesis, (3.4) is negative for all v.

Second, (3.3) evaluated for $v = -1$ is equal to $g(K_1) - g(K_0)$, where

$$g(K) = -\tfrac{1}{4} K^2 + \ln [-K + \sqrt{K^2 + 4}] - \tfrac{1}{4} K\sqrt{K^2 + 4}. \tag{3.7}$$

Since

$$\frac{d}{dK} g(K) = -\tfrac{1}{2}[K + \sqrt{K^2 + 4}] < 0, \tag{3.8}$$

and $K_1 < K_0$, (3.3) at $v = -1$ is positive.

Third, (3.3) evaluated at $v = 1$ is equal to $h(K_1) - h(K_0)$, where

$$h(K) = -\tfrac{1}{4} K^2 + \ln [K + \sqrt{K^2 + 4}] + \tfrac{1}{4} K\sqrt{K^2 + 4}. \tag{3.9}$$

Since

$$\frac{d}{dK} h(K) = \tfrac{1}{2}[K + \sqrt{K^2 + 4}] > 0, \tag{3.10}$$

and $K_1 < K_0$, (3.3) at $v = 1$ is negative.

This completes the proof that L exists and lies between -1 and 1. Although we do not need the information here, the sign of L may readily be found by evaluating (3.3) at $v = 0$. For there, (3.3) is just $(K_0^2 - K_1^2)/2$. Hence

$$\begin{aligned} &\text{if } K_1 + K_0 > 0, && 0 < L < 1; \\ &\text{if } K_1 + K_0 = 0, && L = 0; \\ &\text{if } K_1 + K_0 < 0, && -1 < L < 0. \end{aligned}$$

One might expect L always to lie between $K_0/\sqrt{1 + K_0^2}$ and $K_1/\sqrt{1 + K_1^2}$, since these are the stochastic limits of v_n under H_0 and H_1, respectively. However, this does not seem to be the case in general.

4. Proof that the probability of decision is one. It is now easy to prove that the WAGR test reaches a decision with probability one for every value of K save one. An immediate method is to note that $v_n = u_n/\sqrt{n}$ converges almost everywhere to $K/\sqrt{1 + K^2}$. Hence for every value of K except one, v_n will, for large enough n (depending on K) have crossed a decision limit. Were the probability of no decision positive, this would contradict the strong convergence of v_n. The one exception is the case for which $K/\sqrt{1 + K^2} = L$.

An alternative argument follows that of Cox [2]. We note that

$$\frac{u_n - \sqrt{n}K/\sqrt{1 + K^2}}{\sqrt{(1 + \tfrac{1}{2}K^2)/(1 + K^2)^3}} \tag{4.1}$$

converges in distribution to the unit-normal distribution. This follows from the delta method theorem applied to u_n in terms of sample mean and variance for the $(U - X_i)$'s.

Hence the probability that v_n lies in the interval $[R_n, A_n]$, which is the same as the probability that u_n lies in the interval $[\sqrt{n}R_n, \sqrt{n}A_n]$, becomes arbitrarily close, by uniformity of convergence, to

$$(4.2)\quad \Phi\left(\frac{\sqrt{n}A_n - \sqrt{n}K/\sqrt{1+K^2}}{\sqrt{(1+\frac{1}{2}K^2)/(1+K^2)^3}}\right) - \Phi\left(\frac{\sqrt{n}R_n - \sqrt{n}K/\sqrt{1+K^2}}{\sqrt{(1+\frac{1}{2}K^2)/(1+K^2)^3}}\right).$$

Now, provided L is *not* $K\sqrt{1+K^2}$, (4.2) $\to 0$, for both terms together approach either one or zero. But the probability that decision is not reached by n is $\leqq$ the probability that $R_n \leqq v_n \leqq A_n$, and this has the limit zero.

5. Special argument if K is such that the common limit of A_n and R_n is $K/\sqrt{1+K^2}$. The above arguments fail if L is $K/\sqrt{1+K^2}$, and it then becomes necessary to look at the speed of convergence.

If we differentiate (1.4) with respect to v_n, we obtain

$$(5.1)\quad K_1\sqrt{\nu+1}\,\frac{\int_0^\infty z^{\nu+1}\exp\left(-\frac{1}{2}z^2 + z\sqrt{\nu+1}\,v_nK_1\right)dz}{\int_0^\infty z^{\nu}\exp\left(-\frac{1}{2}z^2 + z\sqrt{\nu+1}\,v_nK_1\right)dz}$$

minus a similar quantity with K_1 replaced by K_0. Following the same argument as that in Section 3, but with $\tilde{z}$ replaced by $\bar{z}$,

$$(5.2)\quad \bar{z} = \tfrac{1}{2}\sqrt{\nu+1}Kv_n + \sqrt{\tfrac{1}{4}(\nu+1)K^2v_n^2 + (\nu+1)},$$

where

$$(5.3)\quad \bar{z}^2 - \bar{z}\sqrt{\nu+1}Kv_n - (\nu+1) = 0,$$

we find that (5.1) for large ν behaves like

$$(5.4)\quad K_1\sqrt{\nu+1}\,\bar{z}_1\left(\frac{\tilde{z}_1}{\bar{z}_1}\right)^{\nu}\exp\left[\tfrac{1}{2}(\tilde{z}_1^2 - \bar{z}_1^2) - 1\right],$$

where $\bar{z}_1$ is $\bar{z}$ for $K = K_1$. Now, letting $w_1 = v_nK_1$,

$$(5.5)\quad \lim\left(\frac{\tilde{z}_1}{\bar{z}_1}\right)^{\nu} = \exp\left[2(w_1\sqrt{w_1^2+4} + w_1^2 + 4)^{-1}\right]$$

and

$$(5.6)\quad \lim\,(\tilde{z}_1^2 - \bar{z}_1^2) = 1 + \frac{w_1}{\sqrt{w_1^2+4}}.$$

It follows that the derivative of (1.4) with respect to v_n, divided by $\nu + 1$, has the limit

$$(5.7)\quad \tfrac{1}{2}K_1[w_1 + \sqrt{w_1^2+4}] - \tfrac{1}{2}K_0[w_0 + \sqrt{w_0^2+4}],$$

where $w_0 = v_nK_0$. Call this limit $\Delta(v_n)$; we shall assume that $\Delta(L) < 0$ and demonstrate this later.

By the law of the mean

$$(5.8)\quad \frac{\ln\left(\frac{1-\beta}{\beta}\,\frac{1-\alpha}{\alpha}\right)}{(A_n - R_n)n} = \frac{1}{n}\times[\text{Derivative of (1.4) at } v_n = \theta],$$

where θ lies between A_n and R_n. Now $\Delta(v_n)$ is negative in some interval about $v_n = L$, by the assumption of the last paragraph and the continuity of (5.7). Hence the right side of (5.8) has the negative limit $\Delta(L)$ as $n \to \infty$. Hence $\lim n(A_n - R_n) =$ constant and $\lim v_n(A_n - R_n) = 0$. Thus the argument at the end of Section 4 applies.

It remains to show that $\Delta(L) < 0$. This follows directly from the observation that the partial derivative with respect to K of $K[Kv + \sqrt{K^2v^2 + 4}]$ is positive.

6. Further comments. The question of whether or not a decision is reached with probability one may be asked, not only about the WAGR test itself, but also about each of the several approximations to it that have been suggested.

For example, Wallis ([4], Chapter 1) suggested a sequential procedure that approximates the WAGR test in a manner described by Kruskal [6]. It may be shown relatively easily that the Wallis procedure leads to a decision with probability one. Other approximations have been suggested by Kruskal [6], but for them the probabilities of decision have not been investigated. Rushton [8] suggests three approximations to the WAGR test, but we know of no theoretical work on their properties.

It may be noted that the asymptotic expressions of this paper suggest still another approximation to the WAGR test, namely the use of (3.1) as an approximation to l_n.

Although we have shown that A_n and R_n converge to the same limit as $n \to \infty$, nothing has been said about monotonicity of approach. One feels that A_n should approach L from above and R_n from below monotonely. However, consideration of the crossing point of l_n and l_{n+1} suggests that this cannot be true for all α's and β's. It may very likely be true, for given α and β, when n is sufficiently large; and again it may be true for all n if α and β are restricted. These seem to be open questions.

We feel that the conclusions of this paper should become a special case of some much more general results. Perhaps when they do, questions such as those of the above paragraph will become more easily handled.

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