DISTRIBUTION OF THE SUM IN RANDOM SAMPLES FROM A DISCRETE POPULATION

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1. Introduction and summary. In [1], the author proposed a "rank sum" criterion for testing whether or not a sample was drawn from a population having a completely specified continuous distribution. This criterion under the null hypothesis is distributed as the sum in a random sample from a discrete uniform population; it is otherwise distributed as the sum from a more general discrete population. In this paper we give a method of finding numerically the distribution of such random variables. Tables are given for the distribution of the sums from certain selected discrete uniform distributions. Normal approximations are investigated and applications are briefly discussed.

2. Distribution of the sample sum. Let $X_1, X_2, \cdots, X_m$ be a random sample drawn from a population having the discrete density

$$f(x; p) = p_x, \qquad x = 1, 2, \cdots, k, \tag{2.1}$$

where

$$p = (p_1, p_2, \cdots, p_k), \qquad p_1 + p_2 + \cdots + p_k = 1. \tag{2.2}$$

Let S be the sum of the sample values $X_1, X_2, \cdots, X_m$; i.e.,

$$S = \sum_{i=1}^{m} X_i . \tag{2.3}$$

We shall be interested in finding the distribution of the statistic S.

It is well known that when $k = 2$, the distribution of S is binomial. For $k > 2$, the distribution of S in general does not assume a simple form. However, as it is easily seen, the pdf of S, denoted by $g(s; p, m)$, is given by the coefficient of t^s in the power expansion of the generating function:

$$D(t; p, m) = \left(\sum_{i=1}^{k} p_i t^i\right)^m, \tag{2.4}$$

and the cdf $G(s; p, m) = \sum_{y=m}^{s} g(y; p, m)$ is given by the sum of coefficients of t^y in (2.4) for $y = m, m + 1, \cdots, s$.

Furthermore, for $m = 2, 3, \cdots$, we have

$$D(t; p, m) = D(t; p, m - 1)D(t; p, 1). \tag{2.5}$$

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It then follows that, for $m = 2, 3, \cdots$, we have

$$(2.6)\quad \begin{aligned} g(s; p, m) &= \sum_{i=1}^{k} p_i g(s - i; p, m - 1), \\ & \qquad\qquad s = m, m + 1, \cdots, km, \\ G(s; p, m) &= \sum_{i=1}^{k} p_i G(s - i; p, m - 1), \end{aligned}$$

where

$$(2.7)\quad \begin{aligned} g(s; p, 1) &= \begin{cases} p_s, & s = 1, 2, \cdots, k, \\ 0, & \text{otherwise}, \end{cases} \\ G(s; p, 1) &= \begin{cases} 0, & s < 1, \\ \sum_{i=1}^{s} p_i, & s = 1, 2, \cdots, k, \\ 1, & s \geqq k. \end{cases} \end{aligned}$$

The above formulae provide a quick method of finding numerically the values of $g(s; p, m)$ and $G(s; p, m)$, once $p = (p_1, p_2, \cdots, p_k)$ is known. In the special case where $p_1 = p_2 = \cdots = p_k = 1/k$, the formulae (2.6) and (2.7) become

$$(2.6a)\quad \begin{aligned} g(s; 1/k, m) &= \frac{1}{k} \sum_{i=1}^{k} g(s - i; 1/k, m - 1), \\ & \qquad\qquad s = m, m + 1, \cdots, km, \\ G(s;1/k, m) &= \frac{1}{k} \sum_{i=1}^{k} G(s - i; 1/k, m - 1), \end{aligned}$$

$$(2.7a)\quad \begin{aligned} g(s; 1/k, 1) &= \begin{cases} 1/k, & s = 1, 2, \cdots, k, \\ 0, & \text{otherwise}, \end{cases} \\ G(s; 1/k, 1) &= \begin{cases} 0, & s < 1, \\ s/k, & s = 1, 2, \cdots, k, \\ 1, & s \geqq k. \end{cases} \end{aligned}$$

Here, we have used $g(s; 1/k, m)$ and $G(s; 1/k, m)$ as short notations for $g(s; 1/k, \cdots, 1/k, m)$ and $G(s; 1/k, \cdots, 1/k, m)$ respectively.

The functions $g(s; 1/k, m)$ and $G(s; 1/k, m)$ are useful, and they are uniquely determined for any given k and m. The values of $G(s; 1/k, m)$ for $k = 3, 4, 5, 6$ and $m = 1, 2, \cdots, 20$ are tabulated in Tables 1, 2, 3, and 4 with $s = m + r$. Since $g(s; 1/k, m)$ is symmetrical (see [1]), only half of the values of the cdf $G(s; 1/k, m)$ are given. The remaining half can be obtained from the following identity:

$$(2.8)\quad G(km - r; 1/k, m) = 1 - G(m + r - 1; 1/k, m), \qquad r = 0, 1, \cdots, (k - 1)m.$$

TABLE 1

Values of $G(m + r; 1/3, m)$

r	m = 1	2	3	4	5	6	7
0	.3333333	.1111111	.0370370	.0123457	.0041152	.0013717	$.0^{3}45725$
1	.6666667	.3333333	.1481482	.0617284	.0246914	.0096022	.0036580
2		.6666667	.3703704	.1851852	.0864198	.0384088	.0164609
3			.6296296	.3827160	.2098766	.1069959	.0516690
4				.6172840	.3950617	.2304527	.1252858
5					.6049383	.4032922	.2469136
6						.5967078	.4101509
7							.5898491

r	m = 8	9	10	11	12	13	14
0	$.0^{3}15242$	$.0^{4}50805$	$.0^{4}16935$	$.0^{5}56450$	$.0^{5}18817$	$.0^{6}62723$	$.0^{6}20908$
1	.0013717	$.0^{3}50805$	$.0^{3}18629$	$.0^{4}67740$	$.0^{4}24462$	$.0^{5}87812$	$.0^{5}31361$
2	.0068587	.0027943	.0011177	$.0^{3}44031$	$.0^{3}17123$	$.0^{4}65859$	$.0^{4}25089$
3	.0239293	.0107199	.0046741	.0019927	$.0^{3}83358$	$.0^{3}34309$	$.0^{3}13924$
4	.0644719	.0317533	.0150892	.0069603	.0031311	.0013786	$.0^{3}59586$
5	.1412894	.0765635	.0396789	.0198140	.0095890	.0045179	.0020799
6	.2607834	.1555149	.0879439	.0475707	.0247817	.0125006	.0061324
7	.4156379	.2725702	.1682162	.0986130	.0553326	.0299011	.0156399
8	.5843621	.4202611	.2827821	.1796474	.1086104	.0629082	.0351033
9		.5797389	.4241901	.2917295	.1899966	.1179799	.0702630
10			.5758099	.4275940	.2996570	.1994213	.1267698
11				.5724060	.4305765	.3067434	.2080482
12					.5694235	.4332190	.3131279
13						.5667810	.4355811
14							.5644189

r	m = 15	16	17	18	19	20	r*
0	$.0^{7}69692$	$.0^{7}23231$	$.0^{8}77435$	$.0^{8}25812$	$.0^{9}86039$	$.0^{9}28680$	
1	$.0^{5}11151$	$.0^{6}39492$	$.0^{6}13938$	$.0^{7}49042$	$.0^{7}17208$	$.0^{8}60227$	
2	$.0^{5}94781$	$.0^{5}35543$	$.0^{5}13241$	$.0^{6}49042$	$.0^{6}18068$	$.0^{7}66250$	
3	$.0^{4}55823$	$.0^{4}22139$	$.0^{5}86960$	$.0^{5}33865$	$.0^{5}13087$	$.0^{6}50218$	
4	$.0^{3}25340$	$.0^{3}10623$	$.0^{4}43976$	$.0^{4}17999$	$.0^{5}72918$	$.0^{5}29271$	
5	$.0^{3}93833$	$.0^{3}41585$	$.0^{3}18141$	$.0^{4}78026$	$.0^{4}33137$	$.0^{4}13913$	
6	.0029360	.0013759	$.0^{3}63267$	$.0^{3}28602$	$.0^{3}12735$	$.0^{4}55926$	
7	.0079507	.0039417	.0019112	$.0^{3}90841$	$.0^{3}42415$	$.0^{3}19488$	
8	.0189585	.0099484	.0050887	.0025442	.0012462	$.0^{3}59923$	
9	.0403354	.0224149	.0121017	.0063672	.0032733	.0016479	9.268
10	.0773787	.0455575	.0259736	.0143880	.0077664	.0040953	10.345
11	.1350270	.0842470	.0507398	.0296050	.0167867	.0092755	11.403
12	.2159820	.1427959	.0908668	.0558601	.0332844	.0192792	12.446
13	.3189191	.2233093	.1501174	.0972414	.0609022	.0369911	13.476
14	.4377093	.3242034	.2301029	.1570291	.1033768	.0658545	14.496
15	.5622907	.4396397	.3290508	.2364237	.1635647	.1092812	15.508
16		.5603603	.4414011	.3335183	.2423237	.1697551	16.512
17			.5585989	.4430169	.3376530	.2478471	17.512
18				.5569831	.4445061	.3414943	18.509
19					.5554939	.4458843	19.503
20						.5541157	20.497

TABLE 2

Values of $G(m + r; 1/4, m)$

r	m = 1	2	3	4	5	6	7
0	.2500000	.0625000	.0156250	.0039063	$.0^{3}97656$	$.0^{3}24414$	$.0^{4}61035$
1	.5000000	.1875000	.0625000	.0195313	.0058594	.0017090	$.0^{3}48829$
2		.3750000	.1562500	.0585938	.0205078	.0068359	.0021973
3		.6250000	.3125000	.1367188	.0546875	.0205078	.0073242
4			.5000000	.2578125	.1181641	.0498047	.0197144
5				.4140625	.2167969	.1025391	.0449219
6				.5859375	.3486328	.1845703	.0893555
7					.5000000	.2958985	.1582031
8						.4291992	.2530518
9						.5708008	.3701172
10							.5000000

r	m = 8	9	10	11	12	13	14
0	$.0^{4}15259$	$.0^{5}38147$	$.0^{6}95367$	$.0^{6}23842$	$.0^{7}59605$	$.0^{7}14901$	$.0^{8}37253$
1	$.0^{3}13733$	$.0^{4}38147$	$.0^{4}10490$	$.0^{5}28610$	$.0^{6}77486$	$.0^{6}20862$	$.0^{7}55879$
2	$.0^{3}68665$	$.0^{3}20981$	$.0^{4}62943$	$.0^{4}18597$	$.0^{5}54240$	$.0^{5}15646$	$.0^{6}44703$
3	.0025177	$.0^{3}83923$	$.0^{3}27275$	$.0^{4}86784$	$.0^{4}27120$	$.0^{5}83447$	$.0^{5}25332$
4	.0074310	.0026932	$.0^{3}94509$	$.0^{3}32282$	$.0^{3}10777$	$.0^{4}35271$	$.0^{4}11347$
5	.0185394	.0072937	.0027590	.0010099	$.0^{3}35954$	$.0^{3}12496$	$.0^{4}42535$
6	.0403290	.0172043	.0070076	.0027461	.0010414	$.0^{3}38396$	$.0^{3}13813$
7	.0780487	.0360870	.0158196	.0066328	.0026779	.0010467	$.0^{3}39771$
8	.1363831	.0683250	.0322275	.0144534	.0062106	.0025724	.0010320
9	.2176819	.1181106	.0599318	.0287466	.0131447	.0057687	.0024429
10	.3203430	.1881142	.1026592	.0526595	.0256231	.0119141	.0053254
11	.4382629	.2781677	.1631794	.0894995	.0463398	.0228295	.0107712
12	.5617371	.3845062	.2422247	.1419988	.0782261	.0408334	.0203364
13		.5000000	.3376970	.2114401	.1238995	.0685221	.0360248
14			.4445419	.2969108	.1849623	.1083569	.0601355
15			.5554581	.3949804	.2613325	.1621051	.0949544
16				.5000000	.3508328	.2302568	.1423102
17					.4492277	.3115888	.2030769
18					.5507723	.4030413	.2767480
19						.5000000	.3612217
20							.4528972
21							.5471028

r	m = 15	16	17	18	19	20	r^*
0	$.0^{9}93132$	$.0^{9}23283$	$.0^{10}58208$	$.0^{10}14552$	$.0^{11}36380$	$.0^{12}90950$	
1	$.0^{7}14901$	$.0^{8}39581$	$.0^{8}10477$	$.0^{9}27649$	$.0^{10}72760$	$.0^{10}19099$	
2	$.0^{6}12666$	$.0^{7}35623$	$.0^{8}99525$	$.0^{8}27649$	$.0^{9}76398$	$.0^{9}21009$	
3	$.0^{6}75996$	$.0^{6}22561$	$.0^{7}66356$	$.0^{7}19354$	$.0^{8}56025$	$.0^{8}16107$	

TABLE 2.—*Continued*

r	m = 15	16	17	18	19	20	r^*
4	$.0^5 35958$	$.0^5 11243$	$.0^6 34738$	$.0^6 10619$	$.0^7 32145$	$.0^8 96461$	
5	$.0^4 14216$	$.0^5 46745$	$.0^5 15150$	$.0^6 48468$	$.0^6 15325$	$.0^7 47940$	
6	$.0^4 48637$	$.0^4 16802$	$.0^5 57067$	$.0^5 19089$	$.0^6 62977$	$.0^6 20519$	
7	$.0^3 14743$	$.0^4 53470$	$.0^4 19018$	$.0^5 66467$	$.0^5 22866$	$.0^6 77545$	
8	$.0^3 40259$	$.0^3 15322$	$.0^4 57042$	$.0^4 20820$	$.0^5 74651$	$.0^5 26337$	
9	.0010027	$.0^3 40034$	$.0^3 15596$	$.0^4 59431$	$.0^4 22202$	$.0^5 81458$	
10	.0022995	$.0^3 96305$	$.0^3 39252$	$.0^3 15613$	$.0^4 60758$	$.0^4 23178$	
11	.0048929	.0021494	$.0^3 91651$	$.0^3 38051$	$.0^3 15422$	$.0^4 61162$	
12	.0097190	.0044785	.0019978	$.0^3 86570$	$.0^3 36544$	$.0^3 15066$	
13	.0181145	.0087565	.0040869	.0018484	$.0^3 81269$	$.0^3 34828$	
14	.0318170	.0161358	.0078801	.0037203	.0017037	$.0^3 75902$	
15	.0528628	.0281283	.0143748	.0070849	.0033798	.0015654	15.227
16	.0833562	.0465376	.0248895	.0128078	.0063654	.0030654	16.296
17	.1251192	.0732888	.0410226	.0220418	.0114137	.0057157	17.354
18	.1792724	.1101526	.0645268	.0362035	.0195345	.0101733	18.401
19	.2458392	.1583968	.0970940	.0568833	.0319841	.0173244	19.438
20	.3234860	.2184292	.1400669	.0856776	.0502015	.0282834	20.467
21	.4094924	.2895225	.1941253	.1239532	.0756794	.0443499	21.489
22	.5000000	.3697044	.2590132	.1725748	.1097722	.0669093	22.504
23		.4558715	.3333819	.2316468	.1534631	.0972791	23.514
24		.5441285	.4148067	.3003318	.2071267	.1365104	24.519
25			.5000000	.3768005	.2703385	.1851751	25.521
26				.4583455	.3417811	.2431774	26.519
27				.5416545	.4192831	.2096323	27.516
28					.5000000	.3828507	28.510
29						.4604453	29.503
30						.5395547	30.497

It perhaps should be pointed out that the density function of the sample mean, $\bar{X}$, is given by

$$h(u; p, m) = g(mu; p, m), \qquad u = 1, (m + 1)/m, \cdots, k. \tag{2.9}$$

3. Normal approximations. Since S is the sum of observations in a random sample, then, by the central limit theorem, it is asymptotically normally distributed. Let us denote by $N(y; \mu, \sigma)$ the normal distribution with mean μ and standard deviation σ. Then, for sufficiently large m and suitable s, the value $\alpha = G(s; p, m)$ can be approximated by $N(z; 0, 1)$ with

$$z = \left(s - m \sum_{i=1}^{k} ip_i\right) \Big/ \sqrt{m\left[\sum_{i=1}^{k} i^2 p_i - \left(\sum_{i=1}^{k} ip_i\right)^2\right]}. \tag{3.1}$$

On the other hand, when a suitable size probability α is given, the formula (3.1) can also be used to obtain an approximate value of s or of $r = s - m$. For example, consider the case $p_1 = p_2 = \cdots = p_k = 1/k$. When m is large, the approximate value of an integer r is given by

TABLE 3

Values of G (m + r; 1/5, m)

r	m = 1	2	3	4	5	6	7
0	.2000000	.0400000	.0080000	.0016000	$.0^{3}32000$	$.0^{4}64000$	$.0^{4}12800$
1	.4000000	.1200000	.0320000	.0080000	.0019200	$.0^{3}44800$	$.0^{3}10240$
2	.6000000	.2400000	.0800000	.0240000	.0067200	.0017920	$.0^{3}46080$
3		.4000000	.1600000	.0560000	.0179200	.0053760	.0015360
4		.6000000	.2800000	.1120000	.0403200	.0134400	.0042240
5			.4240000	.1952000	.0790400	.0291840	.0100480
6			.5760000	.3040000	.1382400	.0564480	.0212480
7				.4320000	.2198400	.0990720	.0407040
8				.5680000	.3222400	.1599360	.0716160
9					.4390400	.2396800	.1168640
10					.5609600	.3360640	.1782400
11						.4439680	.2557440
12						.5560320	.3471360
13							.4479360
14							.5520640

r	m = 8	9	10	11	12	13	14
0	$.0^{5}25600$	$.0^{6}51200$	$.0^{6}10240$	$.0^{7}20480$	$.0^{8}40960$	$.0^{9}81920$	$.0^{9}16384$
1	$.0^{4}23040$	$.0^{5}51200$	$.0^{5}11264$	$.0^{6}24576$	$.0^{7}53248$	$.0^{7}11469$	$.0^{8}24576$
2	$.0^{3}11520$	$.0^{4}28160$	$.0^{5}67584$	$.0^{5}15974$	$.0^{6}37274$	$.0^{7}86016$	$.0^{7}19661$
3	$.0^{3}42240$	$.0^{3}11264$	$.0^{4}29286$	$.0^{5}74547$	$.0^{5}18637$	$.0^{6}45875$	$.0^{6}11141$
4	.0012672	$.0^{3}36608$	$.0^{3}10250$	$.0^{4}27955$	$.0^{5}74547$	$.0^{5}19497$	$.0^{6}50135$
5	.0032742	.0010204	$.0^{3}30648$	$.0^{4}89231$	$.0^{4}25297$	$.0^{5}70083$	$.0^{5}19028$
6	.0075034	.0025165	$.0^{3}80876$	$.0^{3}25076$	$.0^{4}75399$	$.0^{4}22078$	$.0^{5}63160$
7	.0155520	.0056038	.0019239	$.0^{3}63418$	$.0^{3}20192$	$.0^{4}62386$	$.0^{4}18776$
8	.0295680	.0114330	.0041880	.0014659	$.0^{3}49361$	$.0^{3}16074$	$.0^{4}50831$
9	.0520960	.0215987	.0084345	.0031323	.0011145	$.0^{3}38214$	$.0^{3}12687$
10	.0857344	.0380908	.0158486	.0062407	.0023448	$.0^{3}84604$	$.0^{3}29468$
11	.1326336	.0631168	.0279686	.0116727	.0046292	.0017568	$.0^{3}64162$
12	.1939200	.0987904	.0466059	.0206091	.0086242	.0034412	.0013174
13	.2691840	.1467136	.0736621	.0345039	.0152318	.0063889	.0025630
14	.3562240	.2075392	.1108502	.0549871	.0256027	.0112865	.0047439
15	.4511488	.2806221	.1593564	.0836886	.0410923	.0190360	.0083819
16	.5488512	.3638656	.2195062	.1219961	.0631570	.0307416	.0141789
17		.4538368	.2905155	.1707781	.0931908	.0476549	.0230216
18		.5461632	.3704054	.2301267	.1323153	.0710716	.0359581
19			.4561244	.2991816	.1811542	.1021819	.0541372
20			.5438756	.3760854	.2396336	.1418902	.0787080
21				.4581031	.3068550	.1906298	.1106857
22				.5418969	.3810787	.2482074	.1507962
23					.4598363	.3137116	.1993242
24					.5401637	.3855134	.2559905
25						.4613710	.3198866
26						.5386290	.3894865
27							.4627423
28							.5372577

TABLE 3—*Continued*

r	m						
	15	16	17	18	19	20	r^*
0	$.0^{10}32768$	$.0^{11}65536$	$.0^{11}13107$	$.0^{12}26214$	$.0^{13}52429$	$.0^{13}10486$	
1	$.0^{9}52429$	$.0^{9}11141$	$.0^{10}23593$	$.0^{11}49807$	$.0^{11}10486$	$.0^{12}22021$	
2	$.0^{8}44565$	$.0^{8}10027$	$.0^{9}22413$	$.0^{10}49807$	$.0^{10}11010$	$.0^{11}24222$	
3	$.0^{7}26739$	$.0^{8}63504$	$.0^{8}14942$	$.0^{9}34865$	$.0^{10}80740$	$.0^{10}18570$	
4	$.0^{6}12701$	$.0^{7}31752$	$.0^{8}78447$	$.0^{8}19176$	$.0^{9}46426$	$.0^{9}11142$	
5	$.0^{6}50754$	$.0^{6}13325$	$.0^{7}34494$	$.0^{8}88162$	$.0^{8}22274$	$.0^{9}55690$	
6	$.0^{5}17703$	$.0^{6}48720$	$.0^{6}13191$	$.0^{7}35194$	$.0^{8}92652$	$.0^{8}24097$	
7	$.0^{5}55215$	$.0^{5}15902$	$.0^{6}44984$	$.0^{6}12512$	$.0^{7}34279$	$.0^{8}92632$	
8	$.0^{4}15666$	$.0^{5}47180$	$.0^{5}13922$	$.0^{6}40327$	$.0^{6}11486$	$.0^{7}32219$	
9	$.0^{4}40939$	$.0^{4}12880$	$.0^{5}39617$	$.0^{5}11941$	$.0^{6}35330$	$.0^{6}10279$	
10	$.0^{4}99494$	$.0^{4}32678$	$.0^{4}10471$	$.0^{5}32814$	$.0^{5}10078$	$.0^{6}30391$	
11	$.0^{3}22655$	$.0^{4}77634$	$.0^{4}25901$	$.0^{5}84351$	$.0^{5}26878$	$.0^{6}83961$	
12	$.0^{3}48628$	$.0^{3}17379$	$.0^{4}60340$	$.0^{4}20413$	$.0^{5}67454$	$.0^{5}21818$	
13	$.0^{3}98871$	$.0^{3}36840$	$.0^{3}13308$	$.0^{4}46750$	$.0^{4}16015$	$.0^{5}53618$	
14	.0019121	$.0^{3}74263$	$.0^{3}27903$	$.0^{3}10176$	$.0^{4}36128$	$.0^{4}12517$	
15	.0035296	.0014286	$.0^{3}55822$	$.0^{3}21131$	$.0^{4}77734$	$.0^{4}27862$	
16	.0062370	.0026307	.0010688	$.0^{3}41990$	$.0^{3}16003$	$.0^{4}59330$	
17	.0105778	.0046490	.0019639	$.0^{3}80061$	$.0^{3}31607$	$.0^{3}12119$	
18	.0172569	.0079027	.0034708	.0014681	$.0^{3}60035$	$.0^{3}23806$	
19	.0271355	.0129474	.0059117	.0025947	.0010989	$.0^{3}45062$	
20	.0412008	.0204816	.0097223	.0044275	.0019422	$.0^{3}82351$	
21	.0605021	.0313346	.0154631	.0073063	.0033195	.0014554	21.172
22	.0860570	.0464305	.0238194	.0116774	.0054948	.0024911	22.240
23	.1187303	.0667251	.0355838	.0181000	.0088212	.0041353	23.298
24	.1591009	.0931182	.0516180	.0272413	.0137505	.0066656	24.348
25	.2073366	.1263454	.0727908	.0398550	.0208360	.0104444	25.390
26	.2630968	.1668643	.0998967	.0567417	.0307231	.0159251	26.426
27	.3254860	.2147501	.1335606	.0786900	.0441256	.0236513	27.455
28	.3930727	.2696186	.1741393	.1064011	.0617858	.0342442	28.478
29	.4639773	.3305939	.2216345	.1404044	.0844184	.0483778	29.496
30	.5360227	.3963311	.2756316	.1809725	.1126419	.0667390	30.510
31		.4650972	.3352782	.2280488	.1469034	.0899750	31.519
32		.5349028	.3993087	.2811985	.1874051	.1186309	32.525
33			.4661188	.3395944	.2340437	.1530825	33.528
34			.5338812	.4020437	.2863716	.1934731	34.528
35				.4670556	.3435882	.2396624	35.526
36				.5329444	.4045673	.2911952	36.522
37					.4679189	.3472979	37.517
38					.5320811	.4069054	38.510
39						.4687176	39.504
40						.5312824	40.496

$$r^* = m(k - 1)/2 + z_\alpha\sqrt{m(k^2 - 1)/12}, \tag{3.2}$$

where z_α is so determined that $N(z_\alpha\,;\, 0,\, 1) = \alpha$, assuming, of course, that the value $\alpha = G(m + r;\, 1/k,\, m)$ is some moderate size. Thus, in the last column of each of the Tables 1, 2, 3, and 4, r^* is calculated for $m = 20$ and for each integer

TABLE 4

Values of G(m + r; 1/6, m)

r	m = 1	2	3	4	5	6	7
0	.1666667	.0277778	.0046296	$.0^{3}77161$	$.0^{3}12860$	$.0^{4}21434$	$.0^{5}35723$
1	.3333333	.0833333	.0185185	.0038580	$.0^{3}77161$	$.0^{3}15004$	$.0^{4}28578$
2	.5000000	.1666667	.0462963	.0115741	.0027006	$.0^{3}60014$	$.0^{3}12860$
3		.2777778	.0925926	.0270062	.0072017	.0018004	$.0^{3}42867$
4		.4166667	.1620371	.0540124	.0162037	.0045010	.0011788
5		.5833333	.2592593	.0972222	.0324074	.0099023	.0028292
6			.3750000	.1589506	.0587706	.0196759	.0061050
7			.5000000	.2391975	.0979939	.0358796	.0120599
8				.3356482	.1520062	.0607639	.0220872
9				.4436728	.2214507	.0964720	.0378658
10				.5563272	.3051698	.1446331	.0612211
11					.3996914	.2058470	.0938786
12					.5000000	.2793853	.1371635
13						.3631044	.1917010
14						.4535751	.2571695
15						.5464249	.3321616
16							.4142054
17							.5000000

r	m = 8	9	10	11	12	13	14
0	$.0^{6}59537$	$.0^{7}99229$	$.0^{7}16538$	$.0^{8}27564$	$.0^{9}45939$	$.0^{10}76566$	$.0^{10}12761$
1	$.0^{5}53584$	$.0^{6}99229$	$.0^{6}18192$	$.0^{7}33076$	$.0^{8}59721$	$.0^{8}10719$	$.0^{9}19141$
2	$.0^{4}26792$	$.0^{5}54576$	$.0^{5}10915$	$.0^{6}21500$	$.0^{7}41805$	$.0^{8}80394$	$.0^{8}15313$
3	$.0^{4}98237$	$.0^{4}21830$	$.0^{5}47299$	$.0^{5}10033$	$.0^{6}20902$	$.0^{7}42877$	$.0^{8}86774$
4	$.0^{3}29471$	$.0^{4}70949$	$.0^{4}16555$	$.0^{5}37624$	$.0^{6}83610$	$.0^{6}18223$	$.0^{7}39048$
5	$.0^{3}76625$	$.0^{3}19866$	$.0^{4}49664$	$.0^{4}12040$	$.0^{5}28427$	$.0^{6}65602$	$.0^{6}14838$
6	.0017832	$.0^{3}49575$	$.0^{3}13227$	$.0^{4}34082$	$.0^{5}85227$	$.0^{5}20764$	$.0^{6}49444$
7	.0037884	.0011263	$.0^{3}31982$	$.0^{4}87355$	$.0^{4}23076$	$.0^{5}59214$	$.0^{5}14812$
8	.0074481	.0023631	$.0^{3}71276$	$.0^{3}20597$	$.0^{4}57368$	$.0^{4}15476$	$.0^{5}40591$
9	.0136877	.0046280	.0014805	$.0^{3}45192$	$.0^{3}13252$	$.0^{4}37528$	$.0^{4}10307$
10	.0236947	.0085280	.0028900	$.0^{3}93083$	$.0^{3}28703$	$.0^{4}85227$	$.0^{4}24481$
11	.0388696	.0148786	.0053366	.0018120	$.0^{3}58702$	$.0^{3}18259$	$.0^{4}54803$
12	.0607127	.0247002	.0093707	.0033517	.0011400	$.0^{3}37117$	$.0^{3}11632$
13	.0906529	.0391776	.0157126	.0059172	.0021116	$.0^{3}71925$	$.0^{3}23521$
14	.1298333	.0595751	.0252479	.0100064	.0037450	.0013339	$.0^{3}45494$
15	.1788826	.0871076	.0389945	.0162587	.0063795	.0023750	$.0^{3}84452$
16	.2377133	.1227774	.0580361	.0254498	.0104660	.0040715	.0015089
17	.3054002	.1671991	.0834228	.0384641	.0165747	.0067361	.0026012
18	.3801720	.2204424	.1160465	.0562434	.0253899	.0107778	.0043356
19	.4595282	.2819216	.1565039	.0797086	.0376885	.0167073	.0070003
20	.5404718	.3503613	.2049682	.1096620	.0542978	.0251327	.0109667
21		.4238522	.2610923	.1466783	.0760344	.0367419	.0166945
22		.5000000	.3239628	.1909994	.1036260	.0522685	.0247274
23			.3921209	.2424491	.1376235	.0724434	.0356786
24			.4636536	.3003836	.1783135	.0979306	.0502041
25			.5363464	.3636907	.2256439	.1292565	.0689623
26				.4308425	.2791739	.1667359	.0925628
27				.5000000	.3380609	.2104070	.1215070
28					.4010872	.2599838	.1561262
29					.4667306	.3148350	.1965248
30					.5332694	.3739943	.2425354
31						.4362058	.2936936
32						.5000000	.3492377
33							.4081355
34							.4691392
35							.5308608

TABLE 4—*Continued*

r	m						
	15	16	17	18	19	20	r^*
0	$.0^{11}21268$	$.0^{12}35447$	$.0^{13}59078$	$.0^{14}98464$	$.0^{14}16411$	$.0^{15}27351$	
1	$.0^{10}34029$	$.0^{11}60260$	$.0^{11}10634$	$.0^{12}18708$	$.0^{13}32821$	$.0^{14}57437$	
2	$.0^{9}28925$	$.0^{10}54234$	$.0^{10}10102$	$.0^{11}18708$	$.0^{12}34462$	$.0^{13}63181$	
3	$.0^{8}17355$	$.0^{9}34348$	$.0^{10}67349$	$.0^{10}13096$	$.0^{11}25272$	$.0^{12}48439$	
4	$.0^{8}82436$	$.0^{8}17174$	$.0^{9}35358$	$.0^{10}72026$	$.0^{10}14532$	$.0^{11}29063$	
5	$.0^{7}32974$	$.0^{8}72131$	$.0^{8}15558$	$.0^{9}33132$	$.0^{10}69752$	$.0^{10}14532$	
6	$.0^{6}11538$	$.0^{7}26443$	$.0^{8}59628$	$.0^{8}13251$	$.0^{9}29060$	$.0^{10}62965$	
7	$.0^{6}36221$	$.0^{7}86805$	$.0^{7}20429$	$.0^{8}47298$	$.0^{8}10789$	$.0^{9}24277$	
8	$.0^{5}10385$	$.0^{6}25984$	$.0^{7}63726$	$.0^{7}15349$	$.0^{8}36367$	$.0^{9}84884$	
9	$.0^{5}27548$	$.0^{6}71868$	$.0^{6}18345$	$.0^{7}45913$	$.0^{7}11287$	$.0^{8}27295$	
10	$.0^{5}68284$	$.0^{5}18554$	$.0^{6}49241$	$.0^{6}12792$	$.0^{7}32595$	$.0^{8}81595$	
11	$.0^{4}15938$	$.0^{5}45062$	$.0^{5}12422$	$.0^{6}33469$	$.0^{7}88322$	$.0^{7}22868$	
12	$.0^{4}35242$	$.0^{4}10361$	$.0^{5}29646$	$.0^{6}82779$	$.0^{6}22607$	$.0^{7}60498$	
13	$.0^{4}74196$	$.0^{4}22666$	$.0^{5}67278$	$.0^{5}19457$	$.0^{6}54956$	$.0^{6}15191$	
14	$.0^{3}14934$	$.0^{4}47383$	$.0^{4}14582$	$.0^{5}43653$	$.0^{5}12746$	$.0^{6}36373$	
15	$.0^{3}28838$	$.0^{4}94987$	$.0^{4}30293$	$.0^{5}93836$	$.0^{5}28308$	$.0^{6}83366$	
16	$.0^{3}53578$	$.0^{3}18315$	$.0^{4}60508$	$.0^{4}19386$	$.0^{5}60406$	$.0^{5}18350$	
17	$.0^{3}96017$	$.0^{3}34052$	$.0^{3}11651$	$.0^{4}38598$	$.0^{4}12418$	$.0^{5}38899$	
18	.0016634	$.0^{3}61188$	$.0^{3}21676$	$.0^{4}74231$	$.0^{4}24652$	$.0^{5}79608$	
19	.0027909	.0010647	$.0^{3}39043$	$.0^{3}13818$	$.0^{4}47357$	$.0^{4}15762$	
20	.0045429	.0017969	$.0^{3}68202$	$.0^{3}24942$	$.0^{4}88200$	$.0^{4}30250$	
21	.0071845	.0029463	.0011572	$.0^{3}43724$	$.0^{3}15951$	$.0^{4}56363$	
22	.0110543	.0046994	.0019099	$.0^{3}74548$	$.0^{3}28052$	$.0^{3}10211$	
23	.0165672	.0073005	.0030699	.0012377	$.0^{3}48038$	$.0^{3}18010$	
24	.0242119	.0110586	.0048111	.0020034	$.0^{3}80191$	$.0^{3}30965$	
25	.0345389	.0163500	.0073586	.0031648	.0013064	$.0^{3}51948$	
26	.0481383	.0236159	.0109951	.0048836	.0020787	$.0^{3}85123$	
27	.0656070	.0333529	.0160629	.0073679	.0032338	.0013636	27.111
28	.0875068	.0460950	.0229622	.0108766	.0049224	.0021373	28.177
29	.1143145	.0623862	.0321431	.0157221	.0073364	.0032799	29.238
30	.1463697	.0827459	.0440910	.0222688	.0107140	.0049319	30.291
31	.1838250	.1076269	.0593038	.0309263	.0153409	.0072710	31.337
32	.2266041	.1373712	.0782631	.0421376	.0215499	.0105162	32.377
33	.2743755	.1721659	.1013986	.0563602	.0297153	.0149298	33.411
34	.3265444	.2120055	.1290503	.0740416	.0402428	.0208166	34.440
35	.3822670	.2566643	.1614300	.0955894	.0535540	.0285195	35.466
36	.4404886	.3056841	.1985863	.1213386	.0700656	.0384114	36.486
37	.5000000	.3583799	.2403785	.1515178	.0901642	.0508820	37.502
38		.4138645	.2864607	.1862174	.1141775	.0663199	38.515
39		.4710907	.3362815	.2253645	.1423449	.0850915	39.524
40		.5289093	.3890988	.2687060	.1747889	.1075159	40.531
41			.4440107	.3158027	.2114912	.1338387	41.534
42			.5000000	.3660384	.2522745	.1642069	42.536
43				.4186402	.2967949	.1986453	43.535
44				.4727136	.3445442	.2370398	44.533
45				.5272864	.3948645	.2791264	45.529
46					.4469735	.3244905	46.524
47					.5000000	.3725753	47.517
48						.4227006	48.511
49						.4740907	49.504
50						.5259093	50.496

r for which $G(m + r; p, m) = \alpha$ is not too small. For instance, in the case $k = 3$ and $r = 14$ ($\alpha = 0.0658545$), we find $r^* = 14.496$.

It may be concluded from the tables that at least for the cases $k = 3, 4, 5, 6$, $m \geqq 20$, and integers r satisfying the following inequality

$$(3.3)\quad m(k-1)/2 - 2\sqrt{m(k^2-1)/12} \leqq r \leqq m(k-1)/2 + 2\sqrt{m(k^2-1)/12},$$

the values $G(m + r; 1/k, m)$ are well approximated by

$$(3.4)\qquad N(m + r + 1/2; m(k + 1)/2, \sqrt{m(k^2 - 1)/12}.)$$

4. Applications. The distribution of the statistic S has many direct applications. There are many situations in which the parent population has the distribution (2.1) and the distribution of the sum or of the mean of a random sample is needed. As a simple example, suppose each X_j is the number scored with a throw of a perfect die, then the probability distribution function of the sum of the numbers $X_1, X_2, \cdots, X_m$ is readily seen to be $g(s; 1/6, m)$.

Besides direct applications, the statistic S, as pointed out previously, is the rank sum criterion in the goodness of fit test described in [1]. The function $G(s; 1/k, m)$ provides levels of significance and some of the more general cdf's $G(s; p, m)$ serve as power functions of the rank sum tests. For the detailed procedure of the test, the reader is referred to [1].

As another application, the distributions of S are useful in calculating the levels of significance and the power of the sequential rank sum tests. The detailed procedure is discussed in [2].

Finally, it might be worth mentioning that a special case of (2.4) can be used as a generating function in the partition and permutation problems. In fact, the function $k^m D(t; 1/k, m)$ generates the number of partitions of an integer s into an array of m integers (i.e., the number of representations of an integer s as the sum of m integers, assuming that different permutations of the same set of m integers are considered as different representations) which are greater than or equal to 1 and less than or equal to k. Thus, for the cases $1 \leqq m \leqq 20$ and $3 \leqq k \leqq 6$, such a number can be found by the use of the entries in the four tables.

5. Relation of this paper to a result of Whitfield. It is pointed out by a referee that a portion of the tables in this paper has been calculated by Whitfield [4], who gives $G(s; 1/k, m)$ to five decimal places for $3 \leqq k \leqq 8$ and $2 \leqq m \leqq 8$. In comparing the present tables with Whitfield's, the author noticed that several entries are in disagreement.

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