A METHOD OF CONSTRUCTING PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

By J. W. Archbold and N. L. Johnson

University College, London

1. Summary. Partially balanced incomplete block designs were introduced by Bose and Nair [1], who described a number of methods of constructing such designs. Among these methods there is one based on incidence properties of finite geometries. This uses the finite geometries associated with the Galois field $GF(p^n)$ with addition and multiplication (mod p). By weakening the geometrical structure (or, equivalently, by weakening the rules of addition and multiplication), it is possible to obtain new designs.

A basic feature of a finite projective geometry is that the coordinates are elements of a finite field. What we do here is to allow the coordinates to belong instead to a linear associative algebra $\mathcal{A}$, of finite order n and with modulus, over a finite field F. The procedure is summarized below and explained with more detail in regard to two designs. (For accounts of a similar geometrical theory, using an infinite field, see [7], [8], [9].)

2. Introduction. It is well known [6] that the elements of $\mathcal{A}$ can be regularly represented by $n \times n$ matrices with elements in F; such matrices are here said to belong to $\mathcal{A}$. Corresponding to the fact that $\mathcal{A}$ has order n, there is a set of $n \times n$ matrices, $U_1, \cdots, U_n$, over F such that the elements of $\mathcal{A}$ are represented by those and only those matrices of the form $\lambda_1 U_1 + \cdots + \lambda_n U_n$, with $\lambda_1, \cdots, \lambda_n$, in F, and the existence of a modulus means that the $n \times n$ unit matrix U belongs to the set.

A *coordinate matrix* X is a matrix of n rows and $n(h + 1)$ columns partitioned into $h + 1$ submatrices:

$$X = (X_0\, X_1 \cdots X_h),$$

where $X_0, X_1, \cdots, X_h$ belong to $\mathcal{A}$. X defines a *class of equivalent coordinate matrices*, which consists of all matrices AX with A in $\mathcal{A}$ and of rank n. A class has rank r when any (and therefore every) member has rank r.

A *projective space of dimension h and rank r over* $\mathcal{A}$ is a set, $\mathcal{S}_h^r(\mathcal{A})$, of elements (its *points*) in one-to-one correspondence with the classes of equivalent coordinate matrices of rank r over $\mathcal{A}$.

A set of k points, with coordinate matrices $X^1, \cdots, X^k$ (the superscripts being used to distinguish between coordinate matrices), is said to be *linearly dependent over* $\mathcal{A}$ when there exist matrices $A_1, \cdots, A_k$ belonging to $\mathcal{A}$ and not all 0 such that

$$A_1X^1 + \cdots + A_kX^k = 0.$$

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When $\mathfrak{A}$ is not itself a field, it is possible for two points to be distinct yet linearly dependent (with A_1, A_2 both being not null). It may be noted that this kind of possibility does not occur in ordinary geometry.

A *prime of rank s in* $S_h^r(\mathfrak{A})$ consists of all points subject to a relation $X_0L_0 + \cdots + X_hL_h = 0$, where $L_0, \cdots, L_h$ belong to $\mathfrak{A}$ and the matrix

$$\begin{pmatrix} L_0 \\ \cdot \\ \cdot \\ \cdot \\ L_h \end{pmatrix}$$

has rank s. A prime may or may not be an S_{h-1}^t. Two primes in, say, an $S_2^r(\mathfrak{A})$ can meet in more than one point.

To obtain an incidence diagram showing which points of $S_h^r(\mathfrak{A})$ lie on which primes of given rank s, we need a finite algebra $\mathfrak{A}$ and must therefore take F to be a finite field. We examine below the simplest cases which arise when F is a $GF(2)$.

3. Algebras of dual numbers. The simplest kinds of finite algebra with modulus are algebras of *dual numbers* over $GF(2)$. Here, the finite groundfield has just the two elements 0 and 1, and the algebra is of order 2, having as a base two elements u and e such that

$$u^2 = u, \qquad ue = eu = e, \qquad e^2 = \alpha e + \beta u,$$

with α and β in $GF(2)$; u is the modulus, and there are just four elements in the algebra, namely, 0, u, e, and $f = u + e$.

There are four cases to consider, according to the values given to α and β:

(i) If $\alpha = \beta = 0$, $e^2 = 0$. We have then the *parabolic* dual numbers. In the regular matrix representation,

$$U_1 = U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The non-zero elements multiply according to the table

$$\begin{gathered} u^2 = u, \qquad e^2 = 0, \qquad f^2 = u, \\ ef = fe = e, \\ ue = eu = e, \\ uf = fu = f. \end{gathered}$$

(ii) If $\alpha = 0, \beta = 1$, then $e^2 = u$ and $(e + u)^2 = 0$. The elements u and f form an alternative base to the algebra, which is thus seen to be isomorphic with (i) and therefore has nothing new for us.

(iii) If $\alpha = 1, \beta = 0$, then $e^2 = e$ and $f^2 = (e + u)^2 = f$, while $ef = fe = 0$. This gives a new algebra for which

$$U_1 = U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

(iv) If $\alpha = \beta = 1$, then $e^2 = e + u$. This algebra is a field, the inverses of u, e, f being u, f, e; it has no interest here.

4. The parabolic case. The matrices U, E, F representing u, e, f are here

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

There is just one coordinate matrix $(X_0\ X_1 \cdots X_h)$ of rank 0, namely, $(0\ 0 \cdots 0)$. There are $2^{h+1} - 1$ possible matrices of rank 1; no two are equivalent and in each of them every X_i is either E or 0. There remain $4^{h+1} - (2^{h+1} - 1) - 1$ possible matrices of rank 2; these fall into pairs of equivalent matrices $(Y_0\ Y_1 \cdots Y_h)$ and $F(Y_0\ Y_1 \cdots Y_h)$. Hence, $\mathcal{S}_h^1(\mathfrak{A})$ and $\mathcal{S}_h^2(\mathfrak{A})$ contain, respectively, $2^{h+1} - 1$ and $\frac{1}{2}(4^{h+1} - 2^{h+1})$ points. For $h = 2$, these numbers are 7 and 28.

Confining our attention now to the case $h = 2$, the 28 points $P_1, \cdots, P_{28}$ in $\mathcal{S}_2^2(\mathfrak{A})$ may be assigned coordinate matrices as follows:

1. $(U00)$	2. $(UE0)$	3. $(U0E)$	4. (UEE)
5. $(0U0)$	6. $(0UE)$	7. $(EF0)$	8. (EFE)
9. $(00U)$	10. (EEU)	11. $(0EU)$	12. $(E0U)$
13. $(UU0)$	14. (UFE)	15. (UUE)	16. $(UF0)$
17. $(0UU)$	18. (EFF)	19. (EFU)	20. $(0UF)$
21. (UUU)	22. (FUF)	23. (UFF)	24. (FFU)
25. $(U0U)$	26. $(F0U)$	27. (FEF)	28. (UEF)

Details regarding this tableau will be found in [5]. It is enough here to point out, as regards its structure, that any two matrices in the same row are linearly dependent and that any entry, say $(X_0\ X_1\ X_2)$, is related to the entry $(Y_0\ Y_1\ Y_2)$ beneath it by the transformation $Y_0 = X_2$, $Y_1 = X_0 + X_2$, $Y_2 = X_1$.

The same coordinate matrices, written as columns, and numbering, specify the 28 primes $\pi_1, \cdots, \pi_{28}$ of rank 2.

Diagram 1 shows which points lie on which primes. The numbers down the left-hand side of the diagram can be taken as referring to the primes and the numbers along the top as referring to the points. If π_i contains P_j, the fact is registered by placing a cross where the row corresponding to π_i meets the column corresponding to P_j. It will be recognized that this design is a group-divisible PBIB (as defined by Bose & Connor [2]; see also [3], [4]), with parameters

$$v = b = 28, \qquad m = 7, \qquad n = 4, \qquad r = k = 6, \qquad \lambda_1 = 2, \qquad \lambda_2 = 1.$$

This corresponds to the fact that the primes divide into 7 sets of four and the primes in any one set have the property that any two of them meet in two distinct and yet linearly dependent points. Such a set has been called a *quadrilateral of rank 1*. Any two primes belonging to different quadrilaterals of rank 1 meet in just one point.

The 28 points are divided, in a dual manner, into 7 *quadrangles of rank 1*. Any two points in one such quadrangle are joined by two primes which are

DIAGRAM 1

Incidence diagram for the primes of rank 2 in $\mathcal{S}_2^2(\mathfrak{A})$

		A				B				C				D				E				F				G			
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
	9	x	x			x		x						x			x												
c	11	x	x				x		x						x	x													
	12			x	x	x		x							x	x													
	10			x	x		x		x					x			x												
	1					x	x			x		x						x			x								
a	3					x	x				x		x						x	x									
	2							x	x	x		x							x	x									
	4							x	x		x		x					x			x								
	13									x	x			x		x						x			x				
d	16									x	x				x		x						x	x					
	15											x	x	x		x							x	x					
	14											x	x		x		x					x			x				
	21													x	x			x		x						x			x
	24													x	x				x		x						x	x	
f	23															x	x	x		x							x	x	
	22															x	x		x		x					x			x
	17	x			x													x	x			x		x					
e	18		x	x														x	x				x		x				
	19		x	x																x	x	x		x					
	20	x			x															x	x		x		x				
	25					x			x													x	x			x		x	
g	28						x	x														x	x				x		x
	27						x	x																x	x	x		x	
	26					x			x															x	x		x		x
	5	x		x						x			x													x	x		
b	8		x		x						x	x														x	x		
	6	x		x							x	x																x	x
	7		x		x					x			x															x	x

distinct but linearly dependent. Any two points in different quadrangles of rank 1 are joined by just one prime.

The two divisions of points and primes determine a subdivision of the diagram into 49 4×4 squares. Those which contain crosses contain them in the form of a PBIB, and there are three types of such subsidiary designs. These marked squares are themselves arranged as the elements in the well-known pattern associated with the ordinary finite projective geometry over $GF(2)$. The patterns inside the 4×4 squares reflect the structure of the base of the algebra, while the pattern of the 4×4 squares reflects the structure of the groundfield.

DIAGRAM 2

Incidence diagram for the primes of rank 1 in $\mathcal{S}_2^2(\mathcal{A})$

	1 2 3 4	5 6 7 8	9 10 11 12	13 14 15 16	17 18 19 20	21 22 23 24	25 26 27 28
1	x x x x	x x x x		x x x x			
2		x x x x	x x x x		x x x x		
3			x x x x	x x x x		x x x x	
4				x x x x	x x x x		x x x x
5	x x x x				x x x x	x x x x	
6		x x x x				x x x x	x x x x
7	x x x x		x x x x				x x x x

There are seven primes of rank 1, $\sigma_1, \cdots, \sigma_7$ in $\mathcal{S}_2^2(\mathcal{A})$, with coordinate matrices $\begin{pmatrix} L_0 \\ L_1 \\ L_2 \end{pmatrix}$ as follows:

$$\sigma_1\colon \begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix}, \quad \sigma_2\colon \begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix}, \quad \sigma_3\colon \begin{pmatrix} E \\ E \\ 0 \end{pmatrix}, \quad \sigma_4\colon \begin{pmatrix} E \\ E \\ E \end{pmatrix}, \quad \sigma_5\colon \begin{pmatrix} 0 \\ E \\ E \end{pmatrix}, \quad \sigma_6\colon \begin{pmatrix} E \\ 0 \\ E \end{pmatrix}, \quad \sigma_7\colon \begin{pmatrix} 0 \\ E \\ 0 \end{pmatrix}.$$

Each prime of rank 1 consists of the 4 sides of a quadrilateral of rank 1 and thus contains 12 points of the space. The incidence diagram is given in Diagram 2.

This design is, of course, a simple form of group-divisible PBIB of the type described by Bose and Connor [2].

In $\mathcal{S}_2^1(\mathcal{A})$ there are 7 points, $Q_1, \cdots, Q_7$, with the coordinate matrices

$$Q_1 : (00E), \qquad Q_2 : (E00), \qquad Q_3 : (EE0),$$
$$Q_4 : (EEE), \qquad Q_5 : (0EE), \qquad Q_6 : (E0E), \qquad Q_7 : (0E0).$$

There are 28 primes of rank 2 and the incidence diagram for these is obtained by interchanging the rows and columns in the diagram for primes of rank 1 in $\mathcal{S}_2^2(\mathcal{A})$.

Each prime of rank 1 in $\mathcal{S}_2^1(\mathcal{A})$ contains all of the points $Q_1, \cdots, Q_7$. The incidence diagram for these primes is therefore just a 7×7 array of crosses.

5. The non-parabolic case. The matrices U, E, F representing u, e, f are now

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad F = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

There is just one coordinate matrix $(X_0\, X_1 \cdots X_h)$ of rank 0. There is one family of $2^{h+1} - 1$ possible matrices of rank 1, no two being equivalent, in each of which every X_i is either E or 0; another such family is obtained by replacing E everywhere by F; and there are no more primes of rank 1. There remain $4^{h+1} - 2(2^{h+1} - 1) - 1 = (2^{h+1} - 1)^2$ possible matrices of rank 2 and no two are equivalent. Hence, $\mathcal{S}_h^1(\mathcal{A})$ contains two families each of $2^{h+1} - 1$ points and $\mathcal{S}_h^2(\mathcal{A})$ contains $(2^{h+1} - 1)^2$ points. For $h = 2$, these numbers are 7 and 49.

Again confining attention to the case $h = 2$, the 49 points, $P_1, \cdots, P_{49}$, in $\mathcal{S}_2^2(\mathcal{A})$ may be assigned coordinate matrices as follows:

1. $(U00)$	2. $(UF0)$	3. $(U0F)$	4. $(EF0)$	5. (EFF)	6. (UFF)	7. $(E0F)$
8. $(UU0)$	9. (UUF)	10. $(EU0)$	11. (EEF)	12. (UEF)	13. (EUF)	14. $(UE0)$
15. (UUU)	16. (EUU)	17. (EEU)	18. (UEE)	19. (EUE)	20. (UEU)	21. (UUE)
22. $(0UU)$	23. (FEU)	24. (FEE)	25. (FUE)	26. $(0EU)$	27. $(0UE)$	28. (FUU)
29. $(U0U)$	30. (EFE)	31. (UFE)	32. (UFU)	33. $(U0E)$	34. $(E0U)$	35. (EFU)
36. $(0U0)$	37. $(0EF)$	38. (FUF)	39. $(0UF)$	40. $(FU0)$	41. $(FE0)$	42. (FEF)
43. (000)	44. $(F0E)$	45. $(0FU)$	46. $(F0U)$	47. (FFU)	48. (FFE)	49. $(0FE)$

There are also 49 primes of rank 2, $\pi_1, \cdots, \pi_{49}$, to which we may assign coordinate matrices according to the above scheme by writing the matrices as columns instead of rows.

The array of points has been organized in the following manner. The matrix of products $(X_0E\ X_1E\ X_2E)$ is the same for all matrices $(X_0\ X_1\ X_2)$ in any given row of the array. (We could alternatively have used F in this connection in place of E and so have obtained an alternative display of the points.) Then, also, any entry $(Y_0\ Y_1\ Y_2)$ in the array is derived from the entry $(X_0\ X_1\ X_2)$ immediately above it (the first row counts as following the seventh cyclically) by means of the homographic substitution of period 7:

$$Y_0 = X_0 + X_2, \qquad Y_1 = X_0, \qquad Y_2 = X_1.$$

Diagram 3 shows the incidences of the points and primes of rank 2. To save space, only the first seven rows of the design are shown here. Each further set of rows can be obtained by moving each mark 7 columns cyclically to the right and 7 rows downwards (see the 28 × 28 pattern shown earlier), and repeating this process. In the completed design, the rows (which represent the primes) are numbered as follows:

26	21	35	6	42	44	10
28	20	33	4	37	48	13
23	18	30	7	40	45	9
24	19	34	2	39	47	12
22	15	29	1	36	43	8
25	16	32	3	41	49	11
27	17	31	5	38	46	14

Each point lies on 9 primes and each prime contains 9 points. Any given prime is met just once by each of 36 primes and 3 times by each of 12 primes. These 12

DIAGRAM 3

Part of the incidence diagram for points and primes of rank 2 in $\mathcal{S}_2^2(\mathfrak{a})$

	1	2	3	4	5	6	7	8 —14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	22—35	36—42	43—49
26	x	x		x								x	x		x			x	x		x				
28		x	x		x					x				x	x	x	x		x						
23			x	x		x			x				x	x			x				x	x			
24				x	x		x			x	x		x			x				x	x				
22	x				x	x			x	x		x				x		x				x			
25		x				x	x		x		x				x				x	x		x			
27	x		x				x				x	x		x			x	x		x					

primes fall into 6 pairs, those in any pair meeting the given prime in the same 3 points, while each point on the given prime belongs to 2 of the 6 pairs of primes. A corresponding dual arrangement is obtained by starting with any point.

These designs are PBIB with two associate classes but are not group divisible. For these designs,

$$b = v = 49, \quad r = k = 9, \quad \lambda_1 = 1, \quad \lambda_2 = 3,$$
$$p^1_{11} = 25, \quad p^1_{12} = p^1_{21} = 10, \quad p^1_{22} = 2,$$
$$p^2_{11} = 30, \quad p^2_{12} = p^2_{21} = 6, \quad p^2_{22} = 5.$$

The dual design is of identical form.

DIAGRAM 4

Incidence diagram for the primes of rank 1 in $\mathcal{S}_2^2(\mathfrak{C})$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	x	x	x	x	x	x	x								x	x	x	x	x	x	x	x	x	x	x	x	x	x
2								x	x	x	x	x	x	x								x	x	x	x	x	x	x
3															x	x	x	x	x	x	x							
4																						x	x	x	x	x	x	x
5	x	x	x	x	x	x	x																					
6	x	x	x	x	x	x	x	x	x	x	x	x	x	x														
7								x	x	x	x	x	x	x	x	x	x	x	x	x	x							
8	x	x		x				x		x				x				x	x		x			x	x		x	
9		x	x		x			x				x	x			x				x	x	x	x		x			
10			x	x		x			x	x		x			x				x	x			x				x	x
11				x	x		x			x	x		x			x	x		x			x				x	x	
12	x				x	x			x				x	x	x	x		x				x		x				x
13		x				x	x	x	x		x				x		x				x				x	x		x
14	x		x				x				x	x		x			x	x		x			x	x		x		

	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49
1																					
2	x	x	x	x	x	x	x														
3	x	x	x	x	x	x	x	x	x	x	x	x	x	x							
4								x	x	x	x	x	x	x	x	x	x	x	x	x	x
5	x	x	x	x	x	x	x								x	x	x	x	x	x	x
6								x	x	x	x	x	x	x							
7															x	x	x	x	x	x	x
8		x	x		x			x				x	x			x				x	x
9	x		x				x				x	x		x			x	x		x	
10	x	x		x				x		x				x				x	x		x
11		x				x	x	x	x		x				x		x				x
12				x	x		x			x	x		x			x	x		x		
13			x	x		x			x	x		x			x				x	x	
14	x				x	x			x				x	x	x	x		x			

There are 14 primes of rank 1, $\sigma_1, \cdots, \sigma_{14}$, with coordinate matrices as follows:

$$1:\begin{pmatrix}0\\E\\E\end{pmatrix}\quad 2:\begin{pmatrix}E\\E\\E\end{pmatrix}\quad 3:\begin{pmatrix}E\\0\\E\end{pmatrix}\quad 4:\begin{pmatrix}E\\0\\0\end{pmatrix}\quad 5:\begin{pmatrix}0\\E\\0\end{pmatrix}\quad 6:\begin{pmatrix}0\\0\\E\end{pmatrix}\quad 7:\begin{pmatrix}E\\E\\0\end{pmatrix}$$

$$8:\begin{pmatrix}0\\0\\F\end{pmatrix}\quad 9:\begin{pmatrix}F\\F\\F\end{pmatrix}\quad 10:\begin{pmatrix}F\\0\\F\end{pmatrix}\quad 11:\begin{pmatrix}F\\0\\0\end{pmatrix}\quad 12:\begin{pmatrix}0\\F\\F\end{pmatrix}\quad 13:\begin{pmatrix}F\\F\\0\end{pmatrix}\quad 14:\begin{pmatrix}0\\F\\0\end{pmatrix}$$

The incidence diagram is shown in Diagram 4. Each prime of rank 1 contains 21 points to which 7 primes of rank 2 contribute equally; this may be expected from such identities as

$$\begin{pmatrix}E\\0\\0\end{pmatrix} = \begin{pmatrix}U\\0\\0\end{pmatrix}E = \begin{pmatrix}U\\F\\0\end{pmatrix}E = \begin{pmatrix}U\\0\\F\end{pmatrix}E = \begin{pmatrix}U\\F\\F\end{pmatrix}E = \begin{pmatrix}E\\0\\F\end{pmatrix}E = \begin{pmatrix}E\\F\\0\end{pmatrix}E = \begin{pmatrix}E\\F\\F\end{pmatrix}E.$$

These 7 primes together contribute all the points of the associated prime of rank 1 each 3 times.

Each prime of rank 1 is met 7 times by each of the 6 other primes with which it is grouped in the diagram and 9 times with each of the 7 primes of the other group.

This design is a PBIB with $b = 14, v = 49, r = 6, k = 21, \lambda_1 = 4, \lambda_2 = 2$, and

$$p_{11}^1 = 5, \quad p_{12}^1 = p_{21}^1 = 6, \quad p_{22}^1 = 30,$$

$$p_{11}^2 = 2, \quad p_{12}^2 = p_{21}^2 = 10, \quad p_{22}^2 = 25.$$

DIAGRAM 5

Incidence diagram for primes of rank 1 in $S_2^1(\mathfrak{A})$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	x	x		x				x	x	x	x	x	x	x
3		x	x		x			x	x	x	x	x	x	x
5			x	x		x		x	x	x	x	x	x	x
6				x	x		x	x	x	x	x	x	x	x
4	x				x	x		x	x	x	x	x	x	x
7		x				x	x	x	x	x	x	x	x	x
2	x		x				x	x	x	x	x	x	x	x
14	x	x	x	x	x	x	x	x		x	x			
12	x	x	x	x	x	x	x		x		x	x		
9	x	x	x	x	x	x	x			x		x	x	
8	x	x	x	x	x	x	x				x		x	x
11	x	x	x	x	x	x	x	x				x		x
13	x	x	x	x	x	x	x	x	x				x	
10	x	x	x	x	x	x	x		x	x				x

Its dual (regarding columns as "blocks" and rows as "varieties") is a group-divisible PBIB with $b = 49$, $v = 14$, $r = 21$, $k = 6$, $\lambda_1 = 7$, $\lambda_2 = 9$, and

$$p_{11}^1 = 5, \quad p_{12}^1 = p_{21}^1 = 0, \quad p_{22}^1 = 7,$$
$$p_{11}^2 = 0, \quad p_{12}^2 = p_{21}^2 = 6, \quad p_{22}^2 = 0.$$

The space $\mathcal{S}_2^1(\mathcal{Q})$ contains 14 points, $Q_1, \cdots, Q_{24}$, with coordinate matrices

1. $(0EE)$	2. (EEE)	3. $(E0E)$	4. $(E00)$	5. $(0E0)$	6. $(00E)$	7. $(EE0)$
8. $(00F)$	9. (FFF)	10. $(F0F)$	11. $(F00)$	12. $(0FF)$	13. $(FF0)$	14. $(0F0)$

The previous diagram, with rows and columns interchanged, shows the incidences between the points of this space and its 49 primes of rank 2.

Diagram 5 shows the incidences for the primes of rank 1 (which we can take to be $\sigma_1, \cdots, \sigma_{14}$ as above). Each prime of rank 1 contains 10 points. It is met 8 times by each of the other 6 primes in its own set and 6 times by each of the 7 primes in the other set.

It is, of course, of no special interest as a PBIB design.

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