The second term on the right of (38) can be made arbitrarily small by making N sufficiently large. The first term can be made arbitrarily small by making n sufficiently large, since $P\{D(n - N)\} \to 0$ as $n \to \infty$. This completes the proof of (8).

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ON THE DERIVATIVES OF A CHARACTERISTIC FUNCTION AT THE ORIGIN

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1. Introduction. Let $F(x)$, $-\infty < x < \infty$, be a distribution function, and

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x)$$

its characteristic function, defined and continuous for all real t. Let k be a positive integer. If the kth moment of $F(x)$,

$$\mu_k = \int_{-\infty}^{\infty} x^k\, dF(x),$$

exists and is finite (integral absolutely convergent), $\phi(t)$ has a finite kth derivative for all real t given by

$$\phi^{(k)}(t) = i^k \int_{-\infty}^{\infty} x^k e^{itx}\, dF(x).$$

In particular,

$$\phi^{(k)}(0) = i^k \mu_k .$$

The existence and finiteness of μ_k is a sufficient condition for the existence and finiteness of $\phi^{(k)}(0)$. It can be shown (see [1]) that when k is even, this condition is also necessary; but when k is odd this is not so. Zygmund [2] has given a necessary and sufficient condition for the existence of $\phi'(0)$ and also one for the existence of a symmetric derivative of higher odd order at $t = 0$; but he imposes a certain condition (smoothness) on the characteristic function. In the following theorem the conditions are on the distribution function only.

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2. Statement of Results.

THEOREM. *Let k be an odd positive integer. Necessary and sufficient conditions for the existence of $\phi^{(k)}(0)$ are:*

$$\text{(i)} \qquad \lim_{x\to\infty} x^k\{F(-x) + 1 - F(x)\} = 0,$$

$$\text{(ii)} \qquad \lim_{T\to\infty} \int_{-T}^{T} x^k \, dF(x) \text{ exists.}$$

When these two conditions are satisfied,

$$\phi^{(k)}(0) = i^k \lim_{T\to\infty} \int_{-T}^{T} x^k \, dF(x).$$

If X is a random variable with distribution function $F(x)$, so that

$$F(x) = P\{X \leqq x\},$$

condition (i) may be stated in the form

$$\lim_{x\to\infty} x^k[P\{X \leqq -x\} + P\{X > x\}] = 0.$$

A condition which is easily proved equivalent is

$$\lim_{x\to\infty} x^k\{P\,|\,X\,|\, \geqq x\} = 0.$$

3. Two lemmas.

LEMMA 1. *If $G(x)$ is defined and non-decreasing for $x \geqq 0$, and if $k > 0$, the four statements below are equivalent, i.e., any one implies the other three.*

$$\text{(1)} \qquad \lim_{T\to\infty} T^k \int_T^{\infty} dG(x) = 0;$$

$$\text{(2)} \qquad \lim_{T\to\infty} \frac{\int_0^T x^{k+1} \, dG(x)}{T} = 0;$$

$$\text{(3)} \qquad \lim_{T\to\infty} T \int_T^{\infty} x^{k-1} \, dG(x) = 0;$$

$$\text{(4)} \qquad \lim_{T\to\infty} T \int_0^{\infty} x^{k-1} \sin^2 (x/T) \, dG(x) = 0.$$

Suppose (1) is true. Put

$$H(x) = \int_x^{\infty} dG(x) = G(\infty) - G(x).$$

Then $T^kH(T) \to 0$ when $T \to \infty$, and

$$\frac{\int_0^T x^{k+1} \, dG(x)}{T} = \frac{-\int_0^T x^{k+1} \, dH(x)}{T} = -T^kH(T) + \frac{(k+1)\int_0^T x^kH(x) \, dx}{T},$$

both terms of which $\rightarrow 0$ as $T \rightarrow \infty$ if $T^k H(T) \rightarrow 0$, and so (2) is true. Now

$$2(2T)^{-1} \int_0^{2T} x^{k+1}\, dG(x) \geqq T^{-1} \int_T^{2T} x^{k+1}\, dG(x) \geqq T^k \int_T^{2T} dG(x).$$

When (2) is true, the first term in the inequality $\rightarrow 0$ as $T \rightarrow \infty$, and therefore so does the last, i.e.,

$$W(T) = T^k\{G(2T) - G(T)\} \rightarrow 0 \text{ as } T \rightarrow \infty. \tag{5}$$

$$T^k \int_T^{\infty} dG(x) = T^k \sum_{n=1}^{\infty} \{G(2^n T) - G(2^{n-1}T)\} = \sum_{n=1}^{\infty} 2^{-(n-1)k} W(2^{n-1}T).$$

Because of (5), $W(T)$ is bounded for $T \geqq 0$, and therefore this series is uniformly convergent with respect to $T \geqq 0$. When $T \rightarrow \infty$, each term $\rightarrow 0$, and therefore (1) is true. Thus (2) implies (1).

Suppose again that (1) is true. Put

$$A(T) = \sup\, [x^k H(x);\, x \geqq T].$$

Then $A(T) \rightarrow 0$ as $T \rightarrow \infty$, and

$$\begin{aligned} T \int_T^{\infty} x^{k-1}\, dG(x) &= -T \int_T^{\infty} x^{k-1}\, dH(x) \\ &= T^k H(T) + (k-1)T \int_T^{\infty} x^{k-2} H(x)\, dx \\ &\leqq T^k H(T) + | k - 1 |\, T A(T) \int_T^{\infty} x^{-2}\, dx \\ &= T^k H(T) + | k - 1 |\, A(T), \end{aligned}$$

which $\rightarrow 0$ as $T \rightarrow \infty$. Thus (1) implies (3).

The converse of this is not actually used in this paper; but there is some interest in stating and proving it so as to round out the lemma. If $k \geqq 1$,

$$T \int_T^{\infty} x^{k-1}\, dG(x) \geqq T^k \int_T^{\infty} dG(x),$$

and so (3) implies (1) in this case. We now suppose $0 < k < 1$. Now

$$T \int_T^{2T} x^{k-1}\, dG(x) \geqq 2^{k-1} T^k \int_T^{2T} dG(x).$$

If (3) is true, the first term in the inequality $\rightarrow 0$ as $T \rightarrow \infty$, and therefore so does the second. (5) is then true, and this, as shown above, implies (1). Next,

$$T \int_0^{\infty} x^{k-1} \sin^2 (x/T)\, dG(x) = T \int_0^T + T \int_T^{\infty} = I_1 + I_2\,;$$

$$I_1 = T^{-1} \int_0^T x^{k+1} \left(\frac{\sin (x/T)}{x/T}\right)^2 dG(x);$$

$$\sin^2 1 \cdot T^{-1} \int_0^T x^{k+1}\, dG(x) \leqq I_1 \leqq T^{-1} \int_0^T x^{k+1}\, dG(x).$$

Hence $I_1 \to 0$ as $T \to \infty$ if and only if (2) is true. Thus (4) implies (2) which implies (1). Also

$$I_2 \leqq T \int_T^\infty x^{k-1}\, dG(x),$$

and so $\to 0$ as $T \to \infty$ if (3) is true. Thus (1), which implies (2) and (3), implies (4).

LEMMA 2. *When the statements* 1–4 *of Lemma* 1 *are true,*

$$T \int_0^\infty x^{k-1} \sin (x/T)\, dG(x) - \int_0^T x^k\, dG(x) \to 0 \text{ as } T \to \infty.$$

This function of T is equal to

$$T \int_T^\infty x^{k-1} \sin (x/T)\, dG(x) - \int_0^T x^k \left(1 - \frac{\sin (x/T)}{x/T}\right) dG(x),$$

which has a modulus not greater than

$$T\int_T^\infty x^{k-1}\, dG(x) + \int_0^T x^k \cdot x^2/6T^2 \cdot dG(x) \leqq T \int_T^\infty x^{k-1}\, dG(x) + \tfrac{1}{6}T^{-1} \int_0^T x^{k+1}\, dG(x).$$

This $\to 0$ as $T \to \infty$ because of (3) and (2).

4. Proof of theorem. If $\phi_0(t)$, $\phi_1(t)$ are the real and imaginary parts of $\phi(t)$,

$$\phi(t) = \phi_0(t) + i\phi_1(t),$$

$$\phi_0(t) = \int_{-\infty}^\infty \cos tx\, dF(x), \qquad \phi_1(t) = \int_{-\infty}^\infty \sin tx\, dF(x).$$

$\phi_0(t)$ is an even function of t, and $\phi_1(t)$ is an odd function of t. A derivative of $\phi_0(t)$ of odd order which exists at $t = 0$ must be zero there, and the same is true of an even derivative of $\phi_1(t)$.

Let k be an odd positive integer, and suppose that $\phi^{(k)}(0)$ exists. It follows from the last paragraph that

$$\phi^{(k)}(0) = i\phi_1^{(k)}(0),$$

and so has real part zero. $\phi^{(k-1)}(0)$ must exist and be finite. As $k - 1$ is even, this means that μ_{k-1} is finite [1]. Therefore $\phi^{(k-1)}(t)$ exists and is finite for all real t, and

$$\phi^{(k-1)}(t) = i^{k-1} \int_{-\infty}^\infty x^{k-1} e^{itx}\, dF(x),$$

$$(6) \qquad \begin{aligned} \frac{\phi^{(k-1)}(t) - \phi^{(k-1)}(0)}{t} &= i^{k-1} \int_{-\infty}^\infty x^{k-1} \frac{e^{itx} - 1}{t}\, dF(x) \\ &= -i^{k-1} \int_{-\infty}^\infty x^{k-1} \frac{\sin^2 (\frac{1}{2}tx)}{\frac{1}{2}t}\, dF(x) \\ &\qquad + i^k \int_{-\infty}^\infty x^{k-1} \frac{\sin tx}{t}\, dF(x). \end{aligned}$$

Put $G(x) = 1 - F(-x)$. This is a non-decreasing function of x. We may write

$$\begin{aligned}
\frac{\phi^{(k-1)}(t) - \phi^{(k-1)}(0)}{t} &= -i^{k-1} \int_0^\infty x^{k-1} \frac{\sin^2 (\frac{1}{2}tx)}{\frac{1}{2}t} d\{F(x) + G(x)\} \\
&\quad + i^k \left[\int_0^\infty x^{k-1} \frac{\sin tx}{t} dF(x) - \int_0^{1/t} x^k \, dF(x) \right] \\
&\quad - i^k \left[\int_0^\infty x^{k-1} \frac{\sin tx}{t} dG(x) - \int_0^{1/t} x^k \, dG(x) \right] + i^k \int_{-1/t}^{1/t} x^k \, dF(x).
\end{aligned} \tag{7}$$

Because $\phi^{(k)}(0)$ is purely imaginary, when $t \to 0$ the coefficient of i^{k-1} must $\to 0$. Hence $F(x)$ and $G(x)$ both satisfy (4) of Lemma 1 (with $T = 2/t$). Therefore they satisfy (1), i.e.,

$$T^k\{F(\infty) + G(\infty) - F(T) - G(T)\} \to 0 \text{ as } T \to \infty,$$

$$T^k\{1 - F(T) + F(-T)\} \to 0 \text{ as } T \to \infty,$$

which is equivalent to condition (i).

By Lemma 2 (with $T = 1/t$), the second and third terms on the right-hand side of (7) both $\to 0$ as $t \to 0$, and therefore

$$i^k \lim_{t \to 0} \int_{-1/t}^{1/t} x^k \, dF(x) = \phi^{(k)}(0).$$

Condition (ii) is thus necessary.

To prove that conditions (i) and (ii) are sufficient, suppose them satisfied. $F(x)$ and $G(x)$ satisfy (1) of Lemma 1 and therefore (3) also. Hence

$$\int_0^\infty x^{k-1} \, dF(x) \qquad \text{and} \qquad \int_0^\infty x^{k-1} \, dG(x)$$

are both finite, and

$$\mu_{k-1} = \int_0^\infty x^{k-1} \, d\{F(x) + G(x)\}$$

is finite. (6) is then true, and therefore (7). When $t \to 0$, the first and second terms on the right-hand side of (7) both $\to 0$, and the third term $\to$ a limit. Thus $\phi^{(k)}(0)$ exists, and

$$\phi^{(k)}(0) = i^k \lim_{T \to \infty} \int_{-T}^{T} x^k \, dF(x).$$

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