DISTRIBUTIONS OF ROOTS OF QUADRATIC EQUATIONS WITH RANDOM COEFFICIENTS[1]

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General Problem. The problem under consideration is, given the joint p.d.f. of the coefficients of an algebraic equation which can be expressed in polynomial form, to determine the joint p.d.f. of the roots and their marginal p.d.f.'s. Complete results are obtainable for the quadratic.

1. Introduction. Consider an algebraic equation which can be written in polynomial form as

$$\eta^n - \xi_1\eta^{n-1} + \xi_2\eta^{n-2} - \cdots + (-1)^n\xi_n = 0, \tag{1.1}$$

where the coefficients, $\xi_i(i = 1, \cdots, n)$, are real or complex random variables with a given joint p.d.f. The roots of (1.1), $\eta_i(i = 1, \cdots, n)$, are random variables which have a p.d.f. that depends upon the p.d.f. of the coefficients. To obtain the joint p.d.f. of the η_i it is apparent that we must consider the two cases, when the coefficients are real and complex, separately. Furthermore, when the coefficients are real the roots may be either real or complex and hence require separate treatment. The case where the ξ_i are complex random variables was considered in a note by M. A. Girshick [1]. When the ξ_i are real, the η_i may be real or complex. For real η_i the functional form of their p.d.f. is obtained by a change of variables in the p.d.f. of the ξ_i by the use of the relationships

$$\xi_1 = \sum_{i=1}^{n} \eta_i, \qquad \xi_2 = \sum_{i<j} \eta_i \cdot \eta_j, \cdots, \xi_n = \prod_{i=1}^{n} \eta_i, \tag{1.2}$$

with Jacobian, J, given by $\Pi_{i<j}(\eta_i - \eta_j)$. For complex η_i the treatment is similar, but a new set of relationships must be found to replace (1.2). In this case, we must be able to express the ξ_i as functions of the real and imaginary parts of the η_i separately.

2. Limitations. We can now see that there are two major problems involved in determining the p.d.f. of the roots of (1.1) explicitly. The functional form of the p.d.f. can be obtained without difficulty. However, we must be able to determine what regions of the coefficient space will give rise to real roots and what regions will give complex roots. Secondly, after having identified these regions we must be able to define their transforms into the root space. At present, complete results are obtainable only for the quadratic.

3. Quadratic. For $n = 2$ we have

$$\eta^2 - \xi_1\eta + \xi_2 = 0, \tag{3.1}$$

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where ξ_1 and ξ_2 are real random variables and hence may be any real-valued, Borel-measurable functions of real random variables.

The roots, η_1 and η_2, of (3.1) are random variables associated with ξ_1 and ξ_2 by the relationships

$$\eta_1 = \frac{\xi_1}{2} + \sqrt{\frac{\xi_1^2}{4} - \xi_2}, \qquad \eta_2 = \frac{\xi_1}{2} - \sqrt{\frac{\xi_1^2}{4} - \xi_2} \tag{3.2}$$

or

$$\xi_1 = \eta_1 + \eta_2, \qquad \xi_2 = \eta_1 \cdot \eta_2 \tag{3.3}$$

η_1 and η_2 are either both real or are complex conjugates. From (3.2) we see at once that all points belonging to the "interior" of the parabola $\xi_2 = \xi_1^2 / 4$ will give complex roots, while the remainder of the (ξ_1, ξ_2)-plane, which consists of points on and "outside" of the parabola, will give real roots.

We now consider the joint p.d.f., $f(x, y)$, of ξ_1 and ξ_2, where $f(x, y)$ is of the continuous type. By truncating along the parabola $\xi_2 = \xi_1^2 / 4$, we obtain conditional p.d.f.'s relative to the hypotheses $\xi_2 > \xi_1^2 / 4$ and $\xi_2 \leqq \xi_1^2 / 4$. If we let $P(R) = P(\xi_2 \leqq \xi_1^2 / 4)$ and $P(C) = P(\xi_2 > \xi_1^2 / 4)$, then $P(R)$ and $P(C)$ are the probabilities of real and complex roots, respectively, and are given by

$$P(R) = \iint_{y \leqq x^2/4} f(x, y)\, dy\, dx \qquad \text{and} \qquad P(C) = \iint_{y > x^2/4} f(x, y)\, dy\, dx.$$

The conditional or truncated p.d.f.'s [2] are

$$f(x, y \mid C) = f(x, y)/P(C), \, y > \frac{x^2}{4}; \qquad f(x, y \mid R) = f(x, y)/P(R), \, y \leqq \frac{x^2}{4}.$$

For $\xi_2 \leqq \xi_1^2 / 4$, the roots of (3.1) are real and have a joint p.d.f. which is uniquely determined by the p.d.f. of the coefficients ξ_1 and ξ_2. We will let $g(v_1, v_2 \mid R)$ denote the p.d.f. of the real roots. The functions (3.2) and (3.3) satisfy the sufficient conditions given by Cramér [2] for a change of variables in a continuous type density function. Therefore, we have

$$g(v_1, v_2 \mid R) = f(v_1 + v_2, v_1 v_2)\, |J| \,/\, P(R)$$

for all $v_1 \geqq v_2$, where $|J| = (v_1 - v_2)$.

Let $g_1(v_1 \mid R)$ and $g_2(v_2 \mid R)$ be the marginal density functions of the real roots η_1 and η_2, respectively. These are given by

$$g_1(v_1 \mid R) = \int_{-\infty}^{v_1} g(v_1, v_2 \mid R)\, dv_2, \qquad \text{and} \qquad g_2(v_2 \mid R) = \int_{v_2}^{\infty} g(v_1, v_2 \mid R)\, dv_1.$$

For $\xi_2 > \xi_1^2 / 4$, the roots of (3.1) are complex conjugates. Let $\eta_1 = \alpha + \beta i$, then $\eta_2 = \alpha - \beta i$. α and β are defined by the functions

$$\alpha = \xi_1/2, \qquad \beta = \sqrt{\xi_2 - \frac{\xi_1^2}{4}}, \tag{3.4}$$

or

$$\xi_1 = 2\alpha, \qquad \xi_2 = \alpha^2 + \beta^2.$$

α and β have a joint p.d.f. which is uniquely determined by the p.d.f. of ξ_1 and ξ_2, $f(x, y \mid C)$. Let $h_1(X, Z \mid C)$ denote the p.d.f. of α and β. The functions (3.4) satisfy the conditions stated by Cramér [2], so that we may find $h_1(X, Z \mid C)$ by a change of variables in $f(x, y \mid C)$. Therefore, we have

$$h_1(X, Z \mid C) = f(2X, X^2 + Z^2)\, |J| \,/\, P(C)$$

for all X and all $Z > 0$, where $|J| = 4Z$.

Similarly, if we let $h_2(X, Z \mid C)$ be the joint p.d.f. of α and $-\beta$, we will have

$$h_2(X, Z \mid C) = f(2X, X^2 + Z^2)\, |J'| \,/\, P(C)$$

for all X and all $Z < 0$, where $|J'| = -4Z$.

4. Examples. Numerous cases were considered [3] to the extent of expressing the marginal p.d.f.'s as integrals. For these cases ξ_1 and ξ_2 were categorized according to type of interval over which their p.d.f. was greater than zero. There are twelve different interval types as follows: $(-\infty, \infty)$, $(0, \infty)$, $(-\infty, 0)$, (A, ∞), $(-\infty, -A)$, $(-A, \infty)$, $(-\infty, A)$, (A, B), $(-A, -B)$, $(-A, B)$, $(0, A)$, $(-A, 0)$, where $A > 0$, $B > 0$. The various combinations were considered using the normal, gamma, and rectangular density functions, respectively, and assuming independence for ξ_1 and ξ_2 for convenience in obtaining their joint p.d.f.'s. Some dependent cases were also considered.

4.1. Example. *Bivariate Normal.* Let $f(x, y)$ be the general bivariate normal p.d.f., $n(x, y; \mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$. Then

$$\begin{aligned} g(v_1, v_2 \mid R) = \frac{(v_1 - v_2)}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}P(R)} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{v_1 + v_2 - \mu_1}{\sigma_1}\right)^2\right.\right. \\ \left.\left. - 2\rho\left(\frac{v_1 + v_2 - \mu_1}{\sigma_1}\right)\left(\frac{v_1v_2 - \mu_2}{\sigma_2}\right) + \left(\frac{v_1v_2 - \mu_2}{\sigma_2}\right)^2\right]\right\} \\ -\infty < v_2 \leqq v_1, \qquad -\infty < v_1 < \infty, \end{aligned} \tag{4.1.1}$$

where

$$P(R) = \int_{-\infty}^{\infty}\int_{-\infty}^{x^2/4} n(x, y; \mu_1, \mu_2, \sigma_1, \sigma_2, \rho)\, dy\, dx.$$

If we let $u = (x - \mu_1) / \sigma_1$ and $w = (y - \mu_2) / \sigma_2$ we have

$$P(R) = \int_{-\infty}^{\infty}\int_{-\infty}^{\theta'(u)} n(u, w; \rho)\, dw\, du,$$

where

$$\theta'(u) = \frac{1}{4\sigma_2}\left[(\sigma_1 u + \mu_1)^2 - 4\mu_2\right].$$

On completing the square on w in the exponent, and substituting

$$t = \frac{w - u\rho}{\sqrt{1 - \rho^2}}, \qquad dt = \frac{dw}{\sqrt{1 - \rho^2}},$$

we obtain

$$P(R) = \int_{-\infty}^{\infty} \int_{-\infty}^{\theta(u)} (2\pi)^{-1} \exp\{-\tfrac{1}{2}(t^2 + u^2)\}\, dt\, du,$$

where $\theta(u) = [\theta'(u) - \rho u] / \sqrt{1 - \rho^2}$. Finally, we may write

$$P(R) = \int_{-\infty}^{\infty} \varphi(u)\, \Phi[\theta(u)]\, du, \tag{4.1.2}$$

where

$$\Phi(z) = \int_{-\infty}^{z} \varphi(t)\, dt$$

is the cumulative normal probability function and $\varphi(t)$ is the standardized normal density function. We must employ numerical methods of integration to find the value of $P(R)$ for a given $n(x, y)$, i.e., for a given set of values for μ_1, μ_2, σ_1, σ_2, and ρ.

For the complex roots we have

$$h_1(X, Z \mid C) = \frac{4Z}{P(C)}\, n(2X, X^2 + Z^2), \qquad Z > 0,$$

and

$$h_2(X, Z \mid C) = h_1(X, -Z \mid C), \qquad Z < 0,$$

where $P(C) = 1 - P(R)$.

The marginal distributions of the real roots have for density functions

$$g_1(v_1 \mid R) = \int_{-\infty}^{v_1} g(v_1, v_2 \mid R)\, dv_2, \qquad -\infty < v_1 < \infty, \tag{4.1.3}$$

$$g_2(v_2 \mid R) = \int_{v_2}^{\infty} g(v_1, v_2 \mid R)\, dv_1, \qquad -\infty < v_2 < \infty. \tag{4.1.4}$$

On substituting (4.1.1) in (4.1.3), expanding the terms in the exponent, and collecting terms w.r.t. powers of v_2, we obtain

$$\begin{aligned} g_1(v_1 \mid R) = \int_{\infty}^{v_1} & \frac{(v_1 - v_2)}{2\pi\sigma_1\sigma_2\sqrt{1 - \rho^2}P(R)} \\ & \cdot \exp\left\{-\frac{1}{2(1 - \rho^2)}\, [m_1^2(v_1)\cdot v_2^2 - 2m_1(v_1)\cdot m_2(v_1)\cdot v_2 + m_3(v_1)]\right\} dv_2, \end{aligned} \tag{4.1.5}$$

where

$$m_1^2(v_1) = (v_1 / \sigma_2 - \rho / \sigma_1)^2 + (1 - \rho^2) / \sigma_1^2,$$

TABLE 1

$\rho = 0, \pm.2, \pm.4, \pm.6, \pm.8, \pm.9$

μ_1	μ_2	σ_1	σ_2
0	0	1	1
3	10	1	2
10	10	1	1
3	3	1	1
10	3	2	1
−10	3	2	1

TABLE 2

$\mu_1 = \mu_2 = 0, \sigma_1 = \sigma_2 = 1$

ρ	$P(R)$
.9	.5237 449
.8	.5453 219
.6	.5698 161
.4	.5872 947
.2	.5873 160
0	.5890 214
−.2	.5873 160
−.4	.5872 947
−.6	.5698 161
−.8	.5453 219
−.9	.5237 449

$$m_1(v_1)\cdot m_2(v_1) = \rho v_1^2 / \sigma_1\sigma_2 - (1 / \sigma_1^2 + \rho\mu_1 / \sigma_1\sigma_2 - \mu_2 / \sigma_2^2)v_1 + (\mu_1 / \sigma_1^2 - \rho\mu_2 / \sigma_1\sigma_2),$$

and

$$m_3(v_1) = (v_1 / \sigma_1 - \mu_1 / \sigma_1 + \rho\mu_2 / \sigma_2)^2 + (1 - \rho^2)\mu_2^2 / \sigma_2^2 .$$

If we carry out the same procedure on (4.1.4) w.r.t. v_1, we arrive at

$$g_2(v_2 \mid R) = \int_{v_2}^{\infty} \frac{(v_1 - v_2)}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}P(R)} \cdot\exp\left\{-\frac{1}{2(1-\rho^2)}[m_1^2(v_2)\cdot v_1^2 - 2m_1(v_2)\cdot m_2(v_2)\cdot v_1 + m_3(v_2)]\right\} dv_1. \tag{4.1.6}$$

Equations (4.1.2), (4.1.5), and (4.1.6) were evaluated for the various sets of the parameters shown in Table 1 using the ElectroData digital computer of the Statistical Laboratory, Purdue University [3]. Table 2 gives the values of $P(R)$ for the case $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$; a few representative graphs of the marginal p.d.f.'s, $g_1(v_1 \mid R)$ are shown in Fig. 1 for the same case. The curves for $g_2(v_2 \mid R)$ are mirror images of those for g_1, the symmetry being due to the fact that $g_1(v_1 \mid R; \mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = g_2(-v_2 \mid R; -\mu_1, \mu_2, \sigma_1, \sigma_2, -\rho)$ and $v_1 = -v_2$, since $n(x, y; \mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = n(-x, y; -\mu_1, \mu_2, \sigma_1, \sigma_2, -\rho)$. The tails of the g_2 curves are shown as dashed lines in Fig. 1.

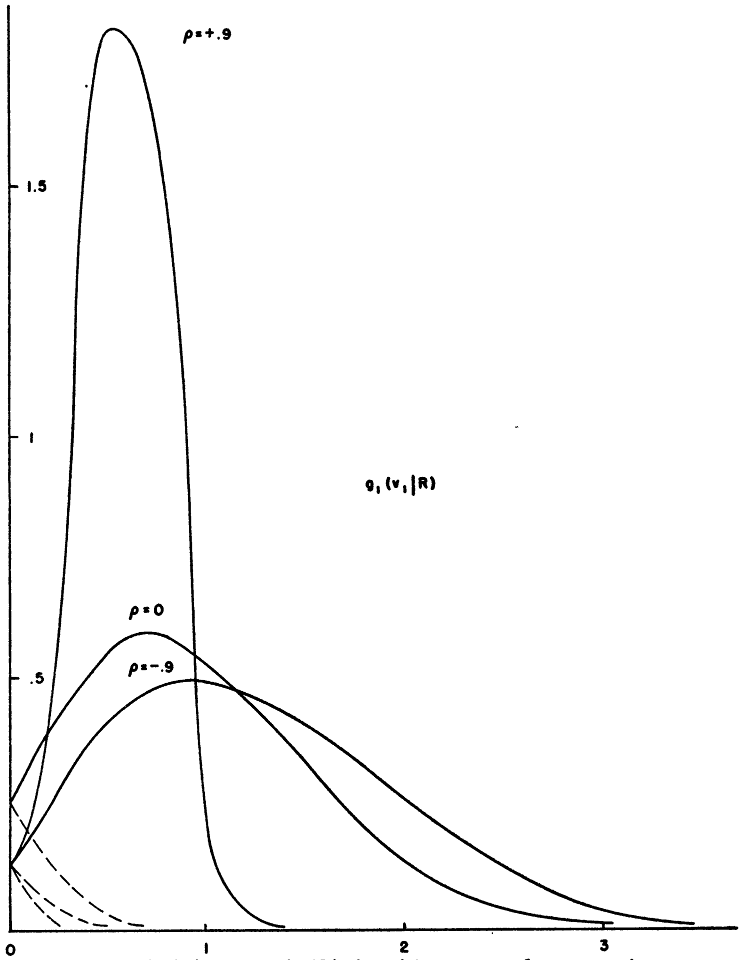

Fig. 1 — $g_1(v_1 \mid R)$ and $g_2(v_2 \mid R)$ for 4.1, $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$

FIG. 1

4.2 Example. *A Gamma Type*. Let $f(x, y) = \exp\{-x - y\}$, $x \geqq 0$, $y \geqq 0$. Then

$$P(R) = \int_0^\infty \int_0^{x^2/4} \exp\{-x - y\}\, dy\, dx = 1 - 2e\sqrt{\pi}[1 - \Phi(\sqrt{2})] \doteq .24,$$

and

$$g(v_1, v_2 \mid R) = \frac{(v_1 - v_2)}{.24} \exp\{-(v_1 + v_2 + v_1 v_2)\}, \qquad 0 \leqq v_2 \leqq v_1, 0 \leqq v_1 \leqq \infty;$$

$$h_1(X, Z \mid C) = \frac{4Z}{.76} \exp\{-(2X + X^2 + Z^2)\}, \qquad X \geqq 0, Z > 0;$$

$$h_2(X, Z \mid C) = \frac{-4Z}{.76} \exp\{-(2X + X^2 + Z^2)\}, \qquad X \geqq 0, Z < 0.$$

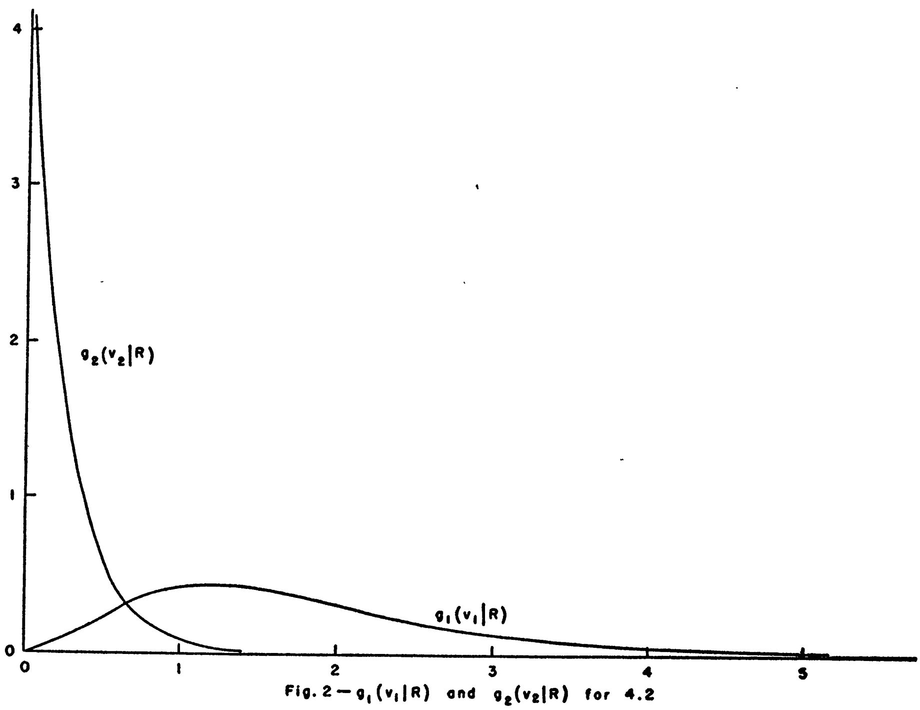

FIG. 2

Finally,

$$g_1(v_1 \mid R) = \int_0^{v_1} g(v_1, v_2 \mid R)\, dv_2 = \frac{1}{.24}\left[\frac{v_1^2 + v_1 - 1}{(1 + v_1)^2} \exp\{-v_1\} + (1 + v_1)^{-2} \exp\{-(v_1^2 + 2v_1)\}\right], \qquad 0 \leqq v_1 < \infty,$$

and

$$g_2(v_2 \mid R) = \int_{v_2}^{\infty} g(v_1, v_2 \mid R)\, dv_1 = \frac{1}{.24}(1 + v_2)^{-2} \exp\{-(v_2^2 + 2v_2)\}, \qquad 0 \leqq v_2 < \infty.$$

The frequency curves, $g_1(v_1 \mid R)$ and $g_2(v_2 \mid R)$, are plotted in Fig. 2.

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