SOME RESULTS USEFUL IN MULTIVARIATE ANALYSIS[1]

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1. Summary and Introduction. In this paper a few results (of a purely mathematical nature) are obtained, which are useful for studying certain distribution problems in multivariate analysis—e.g., those relating to the characteristic roots of a determinantal equation ([1], [2], [5]). In particular, the results are shown to be readily applicable to the moment problems of the sum of the roots and the distributions of the extreme roots. Most of the results given are in the form of certain recursion formulae for reducing special types of k-th order Vandermonde determinants in terms of those of orders $(k - 1)$ and $(k - 2)$. The applications of these results are given by S. N. Roy [6] and the present author [3].

2. Vandermonde's determinant. Let us first consider a type of determinant (due to Vandermonde) which plays an important role in the development of this paper. Denote by V_0 the Vandermonde's determinant of the form

$$V_0 = \begin{vmatrix} X_k^{k-1} & X_k^{k-2} & \cdots & X_k & 1 \\ X_{k-1}^{k-1} & X_{k-1}^{k-2} & \cdots & X_{k-1} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ X_2^{k-1} & X_2^{k-2} & \cdots & X_2 & 1 \\ X_1^{k-1} & X_1^{k-2} & \cdots & X_1 & 1 \end{vmatrix}, \tag{2.1}$$

where $X_1, X_2, \cdots, X_k$ are k variables. The determinant can be shown to be equal to the expression

$$V_0 = \prod_{i>j} (X_i - X_j), \tag{2.2}$$

where $\prod$ denotes the product over the k variables. The determinant V_0 has several interesting properties, of which the following will be used in this paper.

Property 1. If each of the indices of the first j columns of V_0 is increased by unity, the resulting determinant

$$V_j \text{ (say)} = (\sum X_1X_2 \cdots X_j)V_0, \tag{2.3}$$

where $\sum X_1X_2 \cdots X_j$ denotes the j-th elementary symmetric function in k variables $X_1, X_2, \cdots, X_k$.

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3. A special function and the corresponding determinant. Let us consider the integral over the domain $0 < x_1 \leqq x_2 \leqq \cdots \leqq x_k \leqq x < 1$, of a function given by

$$f(x_1, x_2, \cdots, x_k) = \prod_{i=1}^{k} \{x_i^q(1 - x_i)^r e^{tx_i}\} \prod_{i>j} (x_i - x_j), \tag{3.1}$$

where $q, r > -1$ and the t is independent of the x's. It is obvious that, in view of (2.2), $\prod_{i>j} (x_i - x_j)$ can be thrown into the form of a determinant as in (2.1). Multiply the i-th row of this determinant by $x_{k-i+1}^q(1 - x_{k-i+1})^r e^{tx_{k-i+1}}$ $(i = 1, 2, \cdots, k)$, and integrate between appropriate limits, each term of the determinant with respect to the variable it involves. Then the integral of the function $f(x_1, x_2, \cdots, x_k)$ given by (3.1) takes the form

$$\begin{vmatrix} \int_0^x x_k^{q+k-1} (1 - x_k)^r e^{tx_k} \, dx_k & \cdots & \int_0^x x_k^q(1 - x_k)^r e^{tx_k} \, dx_k \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \int_0^{x_2} x_1^{q+k-1} (1 - x_1)^r e^{tx_1} \, dx_1 & \cdots & \int_0^{x_2} x_1^q(1 - x_1)^r e^{tx_1} \, dx_1 \end{vmatrix}. \tag{3.2}$$

It has to be remembered that in expanding the determinant (3.2), the order of integration must not be changed, and hence we shall call it a "pseudo determinant."

Now let $q_1, q_2, \cdots, q_k$ be real numbers greater than -1. Let us denote by $U(x; q_k, r; \cdots; q_1, r; t)$ the pseudo-determinant

$$\begin{vmatrix} \int_0^x x_k^{q_k}(1 - x_k)^r e^{tx_k} \, dx_k & \cdots & \int_0^x x_k^{q_1}(1 - x_k)^r e^{tx_k} \, dx_k \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \int_0^{x_2} x_1^{q_k}(1 - x_1)^r e^{tx_1} \, dx_1 & \cdots & \int_0^{x_2} x_1^{q_1}(1 - x_1)^r e^{tx_1} \, dx_1 \end{vmatrix}. \tag{3.3}$$

More generally, if we replace r in the j-th column of (3.3) by

$$r_{k-j+1} (j = 1, 2, \cdots, k),$$

the resulting pseudo-determinant will be denoted by

$$U(x; q_k, r_k; q_{k-1}, r_{k-1}; \cdots; q_1, r_1; t), \tag{3.4}$$

or more explicitly by

$$U\left\{x; \begin{bmatrix} q_k, r_k & \cdots & q_1, r_1 \\ & \cdots & \\ & \cdots & \\ & \cdots & \\ q_k, r_k & \cdots & q_1, r_1 \end{bmatrix}; t\right\}. \tag{3.5}$$

Further, in the pseudo-determinant (3.3), if the indices of the i-th row alone are different from those of the rest, we denote the resulting determinant by

$$U(x;\, q_k'',\, r'';\, \cdots;\, q_1'',\, r'';\, t)^{(i)}, \tag{3.6}$$

where q_k'', r'', $\cdots$, q_1'', r'' denote the indices of the i-th row.

Since the integral of $f(x_1, x_2, \cdots, x_k)$ involves integrals of the type

$$I(x';\, q, r;\, F;\, t) = \int_0^{x'} x^q(1-x)^r F(x) e^{tx}\, dx, \tag{3.7}$$

where $F(x)$ is a function of x such that the integral in (3.7) exists, let us first consider the integral (3.7). If $F(x)$ is of the form

$$\int_0^x x_{k-1}^{q_{k-1}}(1-x_{k-1})^r e^{tx_{k-1}}\, dx_{k-1} \cdots \int_0^{x_2} x_1^{q_1}(1-x_1)^r e^{tx_1}\, dx_1, \tag{3.8}$$

the integral (3.7) may be denoted by

$$I(x';\, q, r;\, q_{k-1}, r;\, \cdots;\, q_1, r;\, t). \tag{3.9}$$

Now consider $I(x';\, q,\, r;\, F;\, t)$. Integrating (3.7) by parts we obtain the result stated in the following lemma:

LEMMA 1.

$$\begin{aligned} I(x';\, q, r;\, F;\, t) = (q + r + 1)^{-1}\{&-I_0(x';\, q, r+1;\, F;\, t) \\ &+ I(x';\, q, r+1;\, F';\, t) + qI(x';\, q-1, r;\, F;\, t) \\ &+ tI(x';\, q, r+1;\, F;\, t)\}, \end{aligned} \tag{3.10}$$

where

$$I_0(x';\, q, r+1;\, F;\, t) = x^q(1-x)^{r+1}F(x)e^{tx}\,\Big|_0^{x'}, \qquad F' = \frac{dF(x)}{dx}.$$

It may be noted that the right-hand side of (3.10) has been obtained by integrating $(1-x)^{r+q}$ and differentiating the product of $x^q/(1-x)^q$, $F(x)$ and e^{tx}, treating this product as the u term in $\int u dv$. Using Lemma 1, let us consider the integration of the function in (3.3) when $k = 2$.

THEOREM 1. *The pseudo-determinant*

$$\begin{aligned} U(x;\, q_2, r;\, q_1, r;\, t) = (q_2 + r + 1)^{-1}\{&-I_0(x;\, q_2, r+1;\, q_1, r;\, t) \\ + 2I(x;\, q_2 + q_1, 2r+1, 2t) &+ q_2U(x;\, q_2 - 1, r;\, q_1, r;\, t) \\ &+ tU(x;\, q_2, r+1;\, q_1, r;\, t)\}. \end{aligned} \tag{3.11}$$

PROOF. First, note that $U(x;\, q_2,\, r;\, q_1,\, r;\, t) = I(x;\, q_2,\, r;\, q_1,\, r;\, t) - I(x;\, q_1,\, r;\, q_2,\, r;\, t)$. Integrate the latter integrals by parts using Lemma 1 so as to reduce the index q_2 in each case by unity. The sum of all the terms thus obtained after integration gives the right-hand side of (3.11). For a more detailed proof of the theorem, the reader is referred to [3].

In (3.11) the last pseudo-determinant can further be shown to be equal to the difference of two others given by

$$U(x; q_2, r+1; q_1, r; t) = U(x; q_2, r; q_1, r; t) - U(x; q_2+1, r; q_1, r; t). \tag{3.12}$$

For integration in the general case of the function contained in the pseudo-determinant (3.3), some more results have to be used. These results are stated as lemmas in the following section. For the detailed proofs of these lemmas the reader is referred to [3].

4. Certain properties of I-functions. This section is devoted to the statement of two lemmas which will be used in the next section.

LEMMA 2. *If $(q_k', \cdots, q_1')$ denotes any permutation of $(q_k, \cdots, q_1)$, then*

$$\sum I(x; q_k', r; \cdots; q_1', r; t) = \prod_{j=1}^{k} I(x; q_j, r, t), \tag{4.1}$$

where the summation $\sum$ extends over all possible permutations.

LEMMA 3. *If $U(x; q_k'', r'', t''; \cdots; q_1'', r'', t'')^{(i)}$ denotes the pseudo-determinant in* (3.6) *with t'' for the index of the i-th row instead of t, which is the index everywhere else, then*

$$\begin{aligned}&\sum_{i=1}^{k} (-1)^{i-1} U(x; q_k'', r'', t''; \cdots; q_1'', r'', t'')^{(i)} \\ &= \sum_{j=k}^{1} (-1)^{k-j} I(x; q_j'', r'', t'') U(x; q_k, r; \cdots; q_{j+1}, r; q_{j-1}, r; \cdots; q_1, r; t).\end{aligned} \tag{4.2}$$

5. Pseudo determinant of order 3. In this section we shall prove the following theorem:

THEOREM 2. *The pseudo-determinant*

$$\begin{aligned}&U(x; q_3, r; q_2, r; q_1, r; t) \\ &= (q_3 + r + 1)^{-1}\{-I_0(x; q_3, r+1; t)U(x; q_2, r; q_1, r; t) \\ &\quad + 2I(x; q_3 + q_2, 2r+1; 2t)I(x; q_1, r, t) \\ &\quad - 2I(x; q_3 + q_1, 2r+1; 2t)I(x; q_2, r, t) \\ &\quad + q_3 U(x; q_3 - 1, r; q_2, r; q_1, r; t) \\ &\qquad + tU(x; q_3, r+1; q_2, r; q_1, r; t)\}.\end{aligned} \tag{5.1}$$

PROOF. Expand the pseudo-determinant $U(x; q_3, r; q_2, r; q_1, r; t)$ as follows:

$$\begin{aligned}&U\left\{x; \begin{pmatrix} q_3, r & & \\ & q_2, r & q_1, r \\ & q_2, r & q_1, r \end{pmatrix}; t\right\} + U\left\{x; \begin{pmatrix} & q_2, r & q_1, r \\ q_3, r & & \\ & q_2, r & q_1, r \end{pmatrix}; t\right\} \\ &\qquad + U\left\{x; \begin{pmatrix} & q_2, r & q_1, r \\ & q_2, r & q_1, r \\ q_3, r & & \end{pmatrix}; t\right\}.\end{aligned} \tag{5.2}$$

It has to be understood that in each of the pseudo-determinants in (5.2), there are no elements in the positions left blank. Each component in (5.2) stands for the product of an element of the first column and its cofactor in the third-order pseudo-determinant $U(x; q_3, r; q_2, r; q_1, r; t)$. Since the order of integration must not be changed, the product is not written explicitly. Now using Lemma 1, integrate by parts the first pseudo-determinant in (5.2) with respect to x_3, the second with respect to x_2, and the third with respect to x_1. Add the expressions obtained corresponding to each of the four terms on the right-hand side of (3.10). This yields

$$(5.3) \qquad (q_3 + r + 1)^{-1}(A^{(3)} + B^{(3)} + q_3C^{(3)} + tD^{(3)}),$$

where

$$(5.4) \qquad A^{(3)} = -I_0(x; q_3, r + 1; t)U(x; q_2, r; q_1, r; t);$$

$$(5.5) \quad B^{(3)} = 2 \sum_{i=1}^{2} (-1)^{i-1}U(x; q_3 + q_2, 2r + 1, 2t; q_3 + q_1, 2r + 1, 2t)^{(i)};$$

$$(5.6) \qquad C^{(3)} = U(x; q_3 - 1, r; q_2, r; q_1, r; t);$$

and

$$(5.7) \qquad D^{(3)} = U(x; q_3, r + 1; q_2, r; q_1, r; t).$$

Now apply Lemma 3 to the right-hand side of (5.5) with $k = 2$; we at once get the result (5.1).

6. Pseudo determinant of order k. We generalize the results of Theorem 2 in the following theorem:

THEOREM 3. *The pseudo-determinant*

$$(6.1) \qquad \begin{aligned} &U(x; q_k, r; q_{k-1}, r; \cdots; q_1, r; t) \\ &\qquad = (q_k + r + 1)^{-1}(A^{(k)} + B^{(k)} + q_kC^{(k)} + tD^{(k)}), \end{aligned}$$

where

$$(6.2) \qquad A^{(k)} = -I_0(x; q_k, r + 1; t)U(x; q_{k-1}, r; \cdots; q_1, r; t);$$

$$(6.3) \qquad \begin{aligned} B^{(k)} = 2 \sum_{j=k-1}^{1} &(-1)^{k-j-1}I(x; q_k + q_j, 2r + 1, 2t) \\ &\cdot U(x; q_{k-1}, r; \cdots; q_{j+1}, r; q_{j-1}, r; \cdots; q_1, r; t); \end{aligned}$$

$$(6.4) \qquad C^{(k)} = U(x; q_k - 1, r; q_{k-1}, r; \cdots; q_1, r; t);$$

and

$$(6.5) \qquad D^{(k)} = U(x; q_k, r + 1; q_{k-1}, r; \cdots; q_1, r; t).$$

The proof of this theorem follows step by step that of Theorem 2.

It may be noted that the pseudo-determinant in (6.5) can be expressed as a difference of two others as given below:

$$(6.6) \qquad \begin{aligned} U(x; q_k, r + 1; q_{k-1}, r; \cdots; q_1, r; t) &= U(x; q_k, r; \cdots; q_1, r; t) \\ &\quad - U(x; q_k + 1, r; q_{k-1}, r; \cdots; q_1, r; t). \end{aligned}$$

Further note that if $q_j = q_{j-1} + 1$ in Theorem 3, the pseudo-determinant (6.4) vanishes.

Now we state below another theorem which can be proved by employing techniques similar to those used to prove Theorem 3.

THEOREM 4. *If* $W(y; a_k, b; a_{k-1}, b; \cdots; a_1, b; -t)$ *denotes the pseudo-determinant*

$$(6.7)\qquad \begin{vmatrix} \int_0^{y} y_k^{a_k} e^{-ty_k}/(1+y_k)^b \, dy_k & \cdots & \int_0^{y} y_k^{a_1} e^{-ty_k}/(1+y_k)^b \, dy_k \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \int_0^{y_2} y_1^{a_k} e^{-ty_1}/(1+y_1)^b \, dy_1 & \cdots & \int_0^{y_2} y_1^{a_1} e^{-ty_1}/(1+y_1)^b \, dy_1 \end{vmatrix}$$

where $a_i > -1, b > a_i + 1, (i = 1, 2, \cdots, k)$ *and*

$$0 < y_1 \leqq y_2 \leqq \cdots \leqq y_k \leqq y < \infty;$$

then

$$(6.8)\qquad \begin{aligned} &W(y; a_k, b; a_{k-1}, b; \cdots; a_1, b; -t) \\ &\qquad = (b - a_k - 1)^{-1}(P^{(k)} + Q^{(k)} + a_k R^{(k)} - tS^{(k)}), \end{aligned}$$

where

$$(6.9)\qquad P^{(k)} = -F_0(y; a_k, b-1, -t)W(y; a_{k-1}, b; \cdots; a_1, b; -t);$$

$$(6.10)\qquad \begin{aligned} Q^{(k)} = 2 \sum_{j=k-1}^{1} &F(y; a_k + a_j, 2b - 1, -2t) \\ &\cdot W(y; a_{k-1}, b; \cdots; a_{j+1}, b; a_{j-1}, b; \cdots; a_1, b; -t), \end{aligned}$$

$$(6.11)\qquad R^{(k)} = W(y; a_k - 1, b; a_{k-1}, b; \cdots; a_1, b; -t),$$

and

$$(6.12)\qquad S^{(k)} = W(y; a_k, b-1; a_{k-1}, b; \cdots; a_1, b; -t).$$

The pseudo determinant (6.12) *can be expressed as the sum of two others, as follows*:

$$(6.13)\qquad S^{(k)} = W(y; a_k, b; \cdots; a_1, b; -t) + W(y; a_k + 1, b; \cdots, a_1, b; -t).$$

7. Applications to multivariate analysis. The results given by Theorems 3 and 4 are useful for certain distribution problems in multivariate analysis. Consider the well-known distribution of the non-zero roots ($0 < \theta_1 \leqq \theta_2 \leqq \cdots \leqq \theta_s < 1$; $s \leqq p$, the number of variates) of a determinantal equation in multivariate analysis given by R. A. Fisher [1], P. L. Hsu [2] and S. N. Roy [5]. It can be written in the form

$$(7.1)\qquad \begin{aligned} p(\theta_1, \cdots \theta_s) = C(s, m, n) \prod_{i=1}^{s} \theta_i^m (1 - \theta_i)^n \prod_{i>j} (\theta_i - \theta_j) \\ 0 < \theta_1 \leqq \cdots \leqq \theta_s < 1, \end{aligned}$$

where

$$C(s, m, n) \tag{7.2}$$
$$= \pi^{s/2} \prod_{i=1}^{s} \Gamma\{(2m + 2n + s + i + 2)/2\} \Big/ \prod_{i=1}^{s} \Gamma\{(2m + i + 1)/2\} \cdot \Gamma\{(2n + i + 1)/2\}\Gamma(i/2).$$

For the interpretation of m and n, see [4].

Now let $V^{(s)} = \sum_{i=1}^{s} \theta_i$. Consider the moment generating function of $V^{(s)}$ given by $E\{\exp tV^{(s)}\}$, where E denotes mathematical expectation. It is easy to see that, in (3.1), if we put

$$q = m, \qquad r = n, \qquad k = s, \qquad x_i = \theta_i, \qquad \text{and} \qquad x = 1, \tag{7.3}$$

multiply the resulting expression by $C(s, m, n)$ given in (7.2), and integrate with respect to the θ's over the domain $0 < \theta_1 \leqq \cdots \leqq \theta_s < 1$, we at once obtain $E\{\exp tV^{(s)}\}$. In other words, $E\{\exp tV^{(s)}\}$ is obtained from the pseudo-determinant (3.2) after substitutions (7.3) and multiplication by $C(s, m, n)$. Now apply Theorem 3 to $E\{\exp tV^{(s)}\}$. We obtain

$$\begin{aligned}(m + n + s)E(e^{tV^{(s)}}) &= 2C(s, m, n) \sum_{j=1}^{s-1} (-1)^{s-j-1} \\ &\quad \cdot \{I(1; 2m + s + j - 2, 2n + 1, 2t) \\ &\quad \times U(1; m + s - 2, n; \cdots; m + j, n; m + j - 2, n; \cdots; m, n; t)\} \\ &\quad + tC(s, m, n)U(1; m + s - 1, n + 1; m + s - 2, n; \cdots; m, n; t).\end{aligned} \tag{7.4}$$

The simplification here resulted from the fact that the A term (since $x = 1$) and (since $q_k = q_{k-1} + 1$) the C term of Theorem 3 both vanish. Now in view of the result (6.6),

$$\begin{aligned}U(1; m + s - 1, n + 1; m + s - 2, n; \cdots; m, n; t) &= (1/C(s, m, n))E\{\exp tV^{(s)}\} \\ &\quad - U(1; m + s, n; m + s - 2, n; \cdots; m, n; t).\end{aligned} \tag{7.5}$$

Further, using property 1 given in (2.3)

$$\begin{aligned}U(1; m + s - 2, n; \cdots; m + j, n; m + j - 2, n; \cdots; m, n; t) \\ = (1/C(s - 2, m, n)E\{(\textstyle\sum \theta_1 \cdots \theta_{s-j-1})e^{tV^{(s-2)}}\},\end{aligned} \tag{7.6}$$

where $\sum \theta_1 \cdots \theta_{s-j-1}$ denotes the $(s - j - 1)$-th elementary symmetric function in $(s - 2)$ variables $\theta_1 \cdots \theta_{s-2}$. Again using the same property in (7.5),

$$U(1; m + s, n; m + s - 2, n; \cdots; m, n; t) = (1/C(s, m, n))E(V^{(s)}e^{tV^{(s)}}). \tag{7.7}$$

Now make use of (7.5) to (7.7) in (7.4) and we get

$$(m+n+s-t)E(e^{tV^{(s)}}) + tE(V^{(s)}e^{tV^{(s)}}) = \{2C(s,m,n)/C(s-2,m,n)\} \sum_{j=1}^{s-1}(-1)^{s-j-1}I(1;2m+s+j-2,2n+1,2t)E\{(\sum\theta_1\cdots\theta_{s-j-1})e^{tV^{(s-2)}}\}. \tag{7.8}$$

To illustrate the use of (7.8), let us put $s = 2$. This yields

$$(m+n-t+2)E(e^{tV^{(2)}}) + tE(V^{(2)}e^{tV^{(2)}}) = 2C(2, m, n)I(1; 2m+1, 2n+1, 2t). \tag{7.9}$$

Noting that $I(1; 2m + 1, 2n + 1, 2t)$ is a confluent hypergeometric function which can be expanded as a power series in $2t$, and that $\exp tV^{(2)}$ also can be expanded as a power series in t, equating the coefficients of like powers of t on both sides of (7.9) yields

$$(m+n+i+2)\mu_i'^{(2)} - i\mu_{i-1}'^{(2)} = 2^i(2m+2)\cdots(2m+i+1)(m+n+2)/(2m+2n+4)\cdots(2m+2n+i+3) \quad (i=1,2,\cdots), \tag{7.10}$$

where $\mu_i'^{(2)}$ denotes the i-th raw moment for 2 roots. After successive substitutions of lower order moments given by the respective recurrence relations (7.10), we get

$$\mu_i'^{(2)} = \frac{(m+n+2)\Gamma(i+1)\Gamma(2m+2n+4)}{\Gamma(2m+2)\Gamma(m+n+3+i)} \sum_{j=1}^{i+1} 2^{i-j+1} \frac{\Gamma(2m+i-j+3)\Gamma(m+n+i-j+3)}{\Gamma(2m+2n+i-j+5)\Gamma(i-j+2)}. \tag{7.11}$$

Computations of a similar nature with $s = 3$ and 4 in (7.9) and further evaluation of the central moments have yielded the following results [3]:

$$\mu_1^{(s)} = s(2m+s+1)/2(m+n+s+1) \quad (s=1,2,3,4), \tag{7.12}$$

$$\mu_2^{(s)} = s(2m+s+1)(2n+s+1)(2m+2n+s+2)/4(m+n+s+1)^2(m+n+s+2)(2m+2n+2s+1) \quad (i=1,2,3,4), \tag{7.13}$$

and

$$\mu_3^{(s)} = s(n-m)(2m+s+1)(2n+s+1)(m+n+1)(2m+2n+s+2)/d, \tag{7.14}$$

where

$$d = (m+n+s+1)^3(m+n+s+2)(m+n+s+3)(2m+2n+2s)(2m+2n+2s+1), \quad (i=1,2,3,4).$$

For the corresponding $\mu_4^{(s)}$, the reader is referred to [3].

In addition to the usefulness of Theorem 3 in studying the moments of the sum of the roots as outlined above, this theorem is also useful for evaluating the cumulative distribution function of the largest root, θ_s, of the determinantal equation. For the latter purpose, in Theorem 3, multiply

$$U(x; q_k, r; q_{k-1}, r; \cdots; q_1, r; t)$$

by $C(s, m, n)$ given in (7.2), after making the following substitutions:

$$q_j = m + j - 1, \qquad r = n, \qquad k = s, \qquad x_i = \theta_i, \qquad \text{and} \qquad t = 0. \tag{7.15}$$

In this case, the C term of Theorem 3 alone vanishes. By means of Theorem 3, we reduce the cumulative distribution function involving a pseudo-determinant of order s in terms of those of orders $(s - 1)$ and $(s - 2)$. Since it has been shown [3] that the cdf of the smallest root can be obtained from that of the largest, Theorem 3 is thus useful in obtaining the cdf of either of these roots.

Again if we wish to study the moments of the criterion

$$U^{(s)} = \sum_{i=1}^{s} \theta_i/(1 - \theta_i),$$

by using Theorem 4, we will arrive at the following result:

$$\begin{aligned} (n' - m - s + t)E(e^{-tU^{(s)}}) &+ tE(U^{(s)}e^{-tU^{(s)}}) \\ &= \{2K(s, m, n')/K(s - 2, m, n')\} \sum_{j=1}^{s-1} (-1)^{s-j-1} \\ &\quad F(\infty; 2m + s + j - 2, 2n - 1, -2t) \\ &\quad E\{(\textstyle\sum \lambda_1 \cdots \lambda_{s-j-1})e^{-tU^{(s)}}\}, \end{aligned} \tag{7.16}$$

where $\lambda_i = \theta_i/(1 - \theta_i)$, which transformation in (7.1) gives $K(s, m, n')$ from $C(s, m, n)$ and $n' = m + n + s + 1$.

For a detailed study of these applications in multivariate analysis, the reader is referred to [3].

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