SOME USES OF QUASI-RANGES[1]

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1. Summary. Confidence intervals for, and tests of hypotheses about, the interquantile distance are obtained, using one or two properly chosen quasi-ranges. Consistency (of the estimates and tests) is proved. Applications are also given to making inferences about the standard deviations of distributions whose cdf's are of the form $F((x - \mu)/\lambda)$.

2. Introduction. Let $F(x)$ be the cdf (cumulative distribution function) of a given distribution. For a fixed p, $0 < p < 1$, any ξ_p satisfying

$$F(\xi_p - 0) \leqq p \leqq F(\xi_p) \tag{1}$$

is called a quantile of order p (or p-quantile) of the given distribution. If there exist two—and hence infinitely many—such ξ_p's, then one of them is chosen as the p-quantile. Let ξ_q be the q-quantile, where $p < q < 1$. The difference $\xi_q - \xi_p$ is called an interquantile distance. For two reasons we are interested in methods of inference about $\xi_q - \xi_p$. First, the quantity itself is sometimes used as a measure of dispersion of the distribution. (An example is $\xi_{.75} - \xi_{.25}$, known as the interquartile range.) Secondly, for many familiar distributions, $\xi_q - \xi_p$ differs from the standard deviation of the distribution only by a constant factor; consequently any inference about the former can be readily transformed into one about the latter. (See Section 4C.)

Let a random sample of size n be drawn and $x_1, x_2, \cdots, x_n$ be the corresponding order statistics (in ascending order of magnitude). For any integers r and s where $1 \leqq r \leqq s \leqq n$, the difference $x_s - x_r$ is called a quasi-range. (Conventionally, $x_n - x_1$ is called a range and $x_s - x_r$, a quasi-range, only if $s = n - r + 1$, where $1 < r < s$. See, for example, [1].) Symmetric quasi-ranges $(x_{n-r+1} - x_r)$ and their linear combinations are useful in statistical inference. In fact, many uses of them are well known. (See, for example, [1], [5], and the references cited there.) In this paper we shall see some distribution-free methods of using quasi-ranges (not necessarily symmetric) in making inferences about $\xi_q - \xi_p$. Confidence intervals for $\xi_q - \xi_p$ are obtained of the form $(x_v - x_u, x_s - x_r)$. To test the hypothesis, say $\xi_q - \xi_p = d$, we may then use as a critical region: $x_s - x_r < d$ or $x_v - x_u > d$. If the integers r, s, u, and v satisfy respectively $B_n(s - 1, q) - B_n(r - 1, p) \geqq 1 - \alpha$ and $- B_n(v - 1, q) + B_n(u - 1, p) \geqq 1 - \alpha$, where $B_n(r, p)$ is the binomial cdf defined by (2), then the corresponding confidence interval is with confidence coefficient at least $1 - 2\alpha$, and the test is of significance level at most 2α. If there exists more than

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one set of integers satisfying the inequalities, two optimal methods of selection are suggested. For large samples, however, these methods are shown to be equivalent, assuming that the parent cdf $F(x)$ is continuous at $x = \xi_p$ or ξ_q. Furthermore, it is shown that if $F(x)$ meets some additional continuity requirements, then the statistics $x_s - x_r$ and $x_v - x_u$, obtained by the optimal methods of selection, are consistent estimates of $\xi_q - \xi_p$; and the corresponding test is consistent with respect to the alternative $\xi_q - \xi_p \neq d$.

Some applications are given in Section 4.

3. Consistency. Let $F(x)$ be the cdf of a given distribution. Suppose that for given p and q, where $0 < p < q < 1$, ξ_p and ξ_q are uniquely defined. Let $x_1, x_2, \cdots, x_n$ be the order statistics of a random sample of size n.

LEMMA 1. *If r, s, u, and v are integers such that $1 \leqq r \leqq s \leqq n$ and $1 \leqq u \leqq v \leqq n$, and*

$$B_n(r, p) = \sum_{i=0}^{r} \binom{n}{i} p^i (1 - p)^{n-i}, \tag{2}$$

then

$$P(x_s - x_r \geqq \xi_q - \xi_p) \geqq B_n(s - 1, q) - B_n(r - 1, p) = L, \tag{3}$$

$$P(x_v - x_u \leqq \xi_q - \xi_p) \geqq -B_n(v - 1, q) + B_n(u - 1, p) = L', \tag{4}$$

where $P(A)$ is the probability of the event A.

If $F(x)$ is continuous at $x = \xi_p$ or ξ_q, then

$$P(x_s - x_r \geqq \xi_q - \xi_p) \leqq L + 1 = U, \tag{3'}$$

$$P(x_v - x_u \leqq \xi_q - \xi_p) \leqq L' + 1 = U'. \tag{4'}$$

PROOF. Let $P(A, B)$ denote the probability of simultaneous occurrence of the events A and B. Clearly $P(x_s - x_r \geqq \xi_q - \xi_p) \geqq P(x_s \geqq \xi_q, x_r \leqq \xi_p) \geqq P(x_s \geqq \xi_q) + P(x_r \leqq \xi_p) - 1$. Now $P(x_s \geqq \xi_q) = B_n(s - 1, F(\xi_q - 0)) \geqq B_n(s - 1, q)$, since for fixed n and r, $B_n(r, p)$ is a decreasing function of p([3], p. 127). Similarly $P(x_r \leqq \xi_p) \geqq 1 - B_n(r - 1, p)$. Therefore we have (3). To prove (3′), apply the same method to $P(x_s - x_r < \xi_q - \xi_p)$. Likewise we obtain the other inequalities.

It can be shown easily (by (11)) that if n is sufficiently large, then for any α where $0 < \alpha < 1$, there exists at least one set of integers r, s, u, and v for which

$$L \text{ and } L' \geqq 1 - \alpha. \tag{5}$$

The corresponding $x_s - x_r$ and $x_v - x_u$ are then respectively confidence upper and lower bounds for $\xi_q - \xi_p$ with confidence coefficients at least $1 - \alpha$. If there are two or more sets of integers satisfying (5), the following methods of selection may be used. (i) Select the pair of r and s which minimizes $s - r$, and that of u and v which maximizes $v - u$. This method has some intuitive appeal. But quite often there exists more than one pair of r and s and one pair of u and v which satisfy the requirements. In such cases, selection perhaps should be made in accordance with practical consideration. (See a similar case in [7], p. 15.) Further, in the case of $q = 1 - p$, the quasi-ranges selected by this method are, in general, not necessarily symmetric or nearly so. But if $\alpha < \cdot 5$ and n is large, then at least one $x_s - x_r$ and one $x_v - x_u$ are symmetric or nearly so. (See Section 4B.) (ii) This method, to be described later, determines the integers $r, \cdots$, and v uniquely for given p, q, n, and α. It is more or less a natural generalization of the method of using symmetric quasi-ranges for the case $q = 1 - p$. These two methods are not identical in general, but become equivalent (in the sense of (16)) when sample size increases indefinitely.

LEMMA 2. *For integers t and w, where $0 \leqq t, w \leqq n - 1$, and*

$$c = q - p, \tag{6}$$

choose

$$r = [(n - t)p/(1 - c)] + 1, \qquad s = r + t, \tag{7}$$

$$u = [(n - w)p/(1 - c)] + 1, \qquad v = u + w, \tag{8}$$

where $[a]$ denotes the integral part of a. Then L and L', defined in (3) and (4), are respectively non-decreasing and non-increasing functions of t and w. Further, let c_1 and c_2 be such that $0 < c_1 < c < c_2 < 1$. If $t = [n\, c_2]$ and $w = [n\, c_1]$, then

$$\lim_{n \to \infty} L = \lim_{n \to \infty} L' = 1. \tag{9}$$

On the other hand, if $t = [n\, c_1]$ and $w = [n\, c_2]$, then

$$\lim_{n \to \infty} U = \lim_{n \to \infty} U' = 0, \tag{10}$$

provided that $F(x)$ is continuous at $x = \xi_p$ or ξ_q.

PROOF. From (7) and (8), it can be seen that $1 \leqq r, s, u, v \leqq n$. For example, $s \leqq n$ because $s \leqq (n - t)\, p/(1 - c) + t + 1 < n + 1$. Hence L and L' are well-defined functions of t and w. Now r is a non-increasing function of t. But if t is increased by 1, r is decreased at most by 1. Hence s is a non-decreasing function of t, consequently so is L. In a similar way we show that L' is a non-increasing function of w.

It is well known ([2], p. 200, and [4], p. 193) that as $n \to \infty$,

$$B_n(r, p) - \Phi(x) \to 0, \tag{11}$$

uniformly in r, $0 \leqq r \leqq n$, where $x = (r - np)/\sqrt{np(1 - p)}$ and

$$\Phi(x) = \int_{-\infty}^{x} (2\pi)^{-1/2} e^{-t^2/2}\, dt. \tag{12}$$

As $n \to \infty$, it can be shown that if $t = [n\, c_2]$, and r and s are defined by (7),

$$\begin{aligned} (r - 1 - np)/n^{1/2} &\to -\infty, \\ (s - 1 - nq)/n^{1/2} &\to \infty, \end{aligned} \tag{13}$$

for $r - 1 - np = n(c - c_2)p/(1 - c) + 0\ (1)$ and $s - 1 - nq = nb + 0\ (1)$, where $b = (1 - c_2)p/(1 - c) + c_2 - q = (c_2 - c)\ (1 - q)/(1 - c) > 0$. Combining (3), (11), and (13), we obtain $\text{Lim}_{n\to\infty} L = 1$. Likewise we prove the rest of (9) and (10).

LEMMA 3. *Corresponding to given n and α $(0 < \alpha < 1)$, let r_1 and s_1 (u_1 and v_1) be a pair of integers which minimizes $s - r$ (maximizes $v - u$) among all pairs of integers r and s (u and v) such that $1 \leqq r(u) \leqq s\ (v) \leqq n$ and $L(L') \geqq 1 - \alpha$. Let $t_2(w_2)$ be the least (greatest) integer among $0, 1, \cdots$, and $n - 1$ such that if r_2 and $s_2(u_2$ and $v_2)$ are defined accordingly by (7) ((8)), $L(L') \geqq 1 - \alpha$. (From Lemma 2, such r_i, s_i, u_i and v_i, $i = 1, 2$, exist for any α if n is sufficiently large.) For fixed p_i and q_i, $i = 1, 2$, where $p_1 < p < p_2$ and $q_1 < q < q_2$, define*

$$(14) \qquad k_i = [n\ p_i] + 1, \qquad m_i = [n\ q_i] + 1.$$

Assume that $F(x)$ is continuous at $x = \xi_p$ or ξ_q. Then for sufficiently large n and $i = 1, 2$,

$$(15) \qquad \begin{aligned} k_1 &\leqq r_i\,, u_i \leqq k_2\,, \\ m_1 &\leqq s_i\,, v_i \leqq m_2\,. \end{aligned}$$

As a consequence of (15), *we have*

$$(16) \qquad r_1 \sim r_2\,, \qquad s_1 \sim s_2\,, \qquad u_1 \sim u_2\,, \qquad \text{and} \qquad v_1 \sim v_2\,.$$

($r_1 \sim r_2$ means $\text{Lim}_{n\to\infty} r_1/r_2 = 1$.)

PROOF. By (9) and (10), $[n\ c_1] \leqq t_2 \leqq [n\ c_2]$ for any fixed c_1 and c_2 for which $c_1 < c < c_2$, provided that n is sufficiently large. Choose c_1 and c_2 sufficiently close to c, then $r_2 \geqq n(1 - c_2)\ p/(1 - c) \geqq np_1 + 1 \geqq k_1$ and

$$s_2 \leqq n[(1 - c_1)\ p/(1 - c) + c_2] + 0\ (1) \leqq nq_2 \leqq m_2\,.$$

Similarly we have $r_2 \leqq k_2$ and $s_2 \geqq m_1$. In the same way we show that u_2 and v_2 satisfy (15). Further, from (11), $s_1 \geqq m_1$ and $r_1 \leqq k_2$ for large n. Suppose that $s_1 > m_2$ for some n however large. Let $p_1' < p < p_2'$ and $q < q_2' < q_2$; and p_1' and p_2', and q_2' be respectively so close to p and q that $q_2' - p_1' < q_2 - p_2'$. Let $k_i' = [n\ p_i'] + 1$, $i = 1, 2$, and $m_2' = [n\ q_2'] + 1$. Then for large n, $r_1 \leqq k_2'$, $r_2 \geqq k_1'$, and $s_2 \leqq m_2'$. Therefore $s_1 - r_1 > m_2 - k_2' > m_2' - k_1' \geqq s_2 - r_2$. This, however, contradicts the definitions of r_1 and s_1. Hence $s_1 \leqq m_2$. Similarly we show that $r_1 \geqq k_1$ and the rest of (15).

The following lemma is a known fact ([2], p. 369). We state it without a proof.

LEMMA 4. *Let a continuous distribution be given with* cdf *$F(x)$ and* pdf *(probability density function) $f(x)$. Suppose that for $0 < p < q < 1$, ξ_p and ξ_q are uniquely defined; $f(\xi_p)$, $f(\xi_q) \neq 0$; and $f'(x)$, the derivative, is continuous in some neighborhoods of $x = \xi_p$ and ξ_q. If $k = [n\ p] + 1$ and $m = [n\ q] + 1$ (we assume that $n\ p$ and $n\ q$ are not integers), and x_k and x_m are the corresponding order statistics of a sample of size n, then as $n \to \infty$, $x_m - x_k$ has an asymptotically normal distribution with mean $\xi_q - \xi_p$ and variance $O(1/n)$.*

As a consequence of the previous lemmas, we have

THEOREM. *Let a continuous distribution be given whose* cdf *and* pdf *satisfy the continuity conditions stated in Lemma* 4. *For given n and α, let r_i, s_i, u_i, and v_i, $i = 1, 2$, be the integers defined in Lemma* 3. *If x_{r_i}, etc., are respectively the r_ith, etc., order statistics of a sample of size n, then $x_{s_i} - x_{r_i}$ and $x_{v_i} - x_{u_i}$, $i = 1, 2$, are consistent estimates of $\xi_q - \xi_p$.*

PROOF. Following Lemmas 3 and 4, for given δ, $\epsilon > 0$, if p_1, q_2 and δ' are properly chosen, $k_1 = [n\,p_1] + 1$, and $m_2 = [n\,q_2] + 1$, then, for sufficiently large n, $P(x_{s_2} - x_{r_2} > \xi_q - \xi_p + \delta) \leqq P\,(x_{m_2} - x_{k_1} > \xi_q - \xi_p + \delta) \leqq P(x_{m_2} - x_{k_1} > \xi_{q_2} - \xi_{p_1} + \delta') \leqq \epsilon$. In a similar way, we easily complete the proof.

4. Applications.

A. *Confidence intervals and tests of hypotheses.* In Section 3 we proved, for given α and sufficiently large n, the existence of the integers r_i, s_i, u_i, and v_i, $i = 1, 2$, defined in Lemma 3. To actually find these integers, we may use, for example, [6] and [8]. Then $x_{s_i} - x_{r_i}$ and $x_{v_i} - x_{u_i}$ are respectively confidence upper and lower bound for $\xi_q - \xi_p$ with confidence coefficients at least $1 - \alpha$, and $(x_{v_i} - x_{u_i}, x_{s_i} - x_{r_i})$, a confidence interval with confidence coefficient at least $1 - 2\alpha$.

Let $H_0{:}\xi_q - \xi_p = d$. Then the tests, using as critical regions: $x_{s_i} - x_{r_i} < d$; $x_{v_i} - x_{u_i} > d$; and $x_{s_i} - x_{r_i} < d$ or $x_{v_i} - x_{u_i} > d$, $i = 1, 2$, are respectively: (i) of significance levels at most α, α, and 2α, and (ii) consistent with respect to the alternatives $\xi_q - \xi_p < d$; $\xi_q - \xi_p > d$; and $\xi_q - \xi_p \neq d$, provided that the continuity conditions of Lemma 4 are satisfied. A test, for testing a given hypothesis, is said to be consistent with respect to a certain alternative, if, whenever the alternative is true, the power of the test tends to unity as sample size tends to infinity.

As an example, let us find confidence upper bounds for $\xi_{.6} - \xi_{.3}$ with confidence coefficients at least .95, using a random sample of size 50. It is easy to see that $L \geqq 1 - \alpha$ of (5) is equivalent to

$$B_n(i, p) + B_n(j, 1 - q) \leqq \alpha, \tag{17}$$

where $i = r - 1$ and $j = n - s$; and $s - r$ is minimized if $i + j$ is maximized. Now $n = 50$, $p = .3$, $q = .6$, $1 - q = .4$, and $\alpha = .05$. The largest integers i and j for which $B_{50}\,(i, .3)$ and $B_{50}\,(j, .4) \leqq .05$ are 9 and 13. For $n = 50$, and $i = 9, 8$ respectively, the largest integers j's for which (17) holds are 11 and 13. Therefore $i + j$ is maximized if $i = 8$ and $j = 13$. Hence $r_1 = 9$ and $s_1 = 37$. Further, if $t = 28$, then by (7), $r = 10$, $s = 38$, $i = 9$, and $j = 12$ for which (17) does not hold. But if $t = 29$, then $r = 10$, $s = 39$, $i = 9$, $j = 11$, and (17) holds. Hence $r_2 = 10$ and $s_2 = 39$.

B. *A special case.* To make inferences about $\xi_q - \xi_p$ when $q = 1 - p$, it seems most natural to use symmetric quasi-ranges $(x_{n-r+1} - x_r)$. If $q = 1 - p$, $s = n - r + 1$, and $v = n - u + 1$, then (5) becomes

$$B_n(r - 1, p) \leqq \alpha/2, B_n(u - 1, p) \geqq 1 - \alpha/2. \tag{18}$$

TABLE 1

Values of r' and u'

$p = .25$ and $q = .75$

n	α											
	.25		.10		.05		.025		.01		.005	
10	1(r')	5(u')	—	—	—	—	—	—	—	—	—	—
20	3	8	2	9	2	10	1	—	1	—	—	—
30	5	11	4	13	3	13	3	14	2	15	2	—
40	7	14	6	16	5	17	4	17	4	18	3	19
50	9	17	8	19	7	20	6	21	5	22	5	23
60	11	20	10	22	9	23	8	24	7	25	6	26
70	13	23	12	25	11	26	10	27	9	28	8	29
80	16	25	14	27	13	29	12	30	11	31	10	32
90	18	28	16	30	15	32	14	33	12	34	12	36
100	20	31	18	33	17	35	16	36	14	38	14	39

Let r' and u' be respectively the largest and smallest integers r and u for which (18) holds. Let $s' = n - r' + 1$ and $v' = n - u' + 1$. Then $x_{s'} - x_{r'}$, $x_{v'} - x_{u'}$, and $(x_{v'} - x_{u'}, x_{s'} - x_{r'})$ are confidence upper and lower bounds, and interval for $\xi_{1-p} - \xi_p$ with confidence coefficients $1 - \alpha$, $1 - \alpha$, and $1 - 2\alpha$. To find r' and u', we may use [6] and [8]. Table 1 below is obtained in this way. There $p = .25$ and $q = .75$. If, for example, $n = 30$ and $\alpha = .05$, then $r' = 3$ and $n - r' + 1 = 28$. Therefore $P(x_{28} - x_3 \geqq \xi_{.75} - \xi_{.25}) \geqq .95$. Likewise, $P(x_{18} - x_{13} \leqq \xi_{.75} - \xi_{.25} \leqq x_{28} - x_3) \geqq .90$.

A question then follows. Are the quasi-ranges, obtained by applying the general methods (in Section 4A) to this particular case ($q = 1 - p$), symmetric and identical to the corresponding ones obtained by the methods just described? More precisely, if $q = 1 - p$, are the integers r_i, s_i, u_i, and v_i, $i = 1, 2$, (defined in Lemma 3) equal respectively to r', s', u', and v' (defined by (18) with the same n, α, and p)? We have the following answers. (i) $r' = r_2$ or $r_2 - 1$, $s' = s_2$, $u' = u_2$, and $v' = v_2$ or $v_2 - 1$. In other words, $x_{s_2} - x_{r_2}$ and $x_{v_2} - x_{u_2}$ are either identical to $x_{s'} - x_{r'}$ and $x_{v'} - x_{u'}$, or only slightly different from them. (ii) Generally, no similar relations exist between r' and r_1, etc. In the first place, the integers r_1, s_1, u_1 and v_1 are not always uniquely determined. Sometimes none of the corresponding quasi-ranges is symmetric or nearly so. (For example, if $n = 100$, $p = .1$, $q = .9$, and $\alpha = .99$, then, following (17), $r_1 = 4, 5, 6, 15, 16, 17$; and $s_1 = 84, 85, 86, 95, 96, 97$.) If, however, $\alpha < .5$ and n is sufficiently large, then it can be shown that one set of $r_1, \cdots$, and v_1 coincides with $r_2, \cdots$, and v_2. Therefore for $\alpha < .5$ and large n, at least one $x_{s_1} - x_{r_1}$ and one $x_{v_1} - x_{u_1}$ are either identical to $x_{s'} - x_{r'}$ and $x_{v'} - x_{u'}$ or only slightly different from them.

We shall now prove (i) and (ii). If $q = 1 - p$, then from (7), $r = [(n - t)/2] + 1$. If, corresponding to given n and α, $(n - t_2)/2$, where t_2 is defined in Lemma 3, is not an integer, then $s_2 = n - r_2 + 1$. Otherwise $s_2 = n - (r_2 - 1) + 1$. In the first case, we see that (18) holds for $r = r_2$ but not for $r = r_2 + 1$. (This is because (5), $L \geqq 1 - \alpha$, holds for $t = t_2$, $r = r_2$, and $s = s_2$; while $L < 1 - \alpha$ holds for $t = t_2 - 1$, $r = r_2 + 1$, and $s = s_2$.) Hence $r' = r_2 - 1$. In the second case, (18) holds for $r = r_2 - 1$, but not for $r = r_2$. Hence $r' = r_2 - 1$. Furthermore, if $\alpha < .5$ and n is sufficiently large, then no i and j satisfying (17) can be greater than np. Using the fact that $\binom{n}{i} p^i(1 - p)^{n-i}$ is an increasing function of i for all $0 \leqq i < (n + 1)p$, it is easy to see that one of those pairs of i and j for which (17) holds and $i + j$ is maximized must be such that $i = j$ or $i = j + 1$. The integers r_1 and s_1, corresponding to this pair of i and j, are equal respectively to r_2 and s_2. ($r_1 = i + 1$, $s_1 = n - r_1 + 1$ or $n - r_1 + 2$. Let $t = s_1 - r_1$ in (7), then $r = r_1$ and $s = s_1$. Hence $t_2 = s_1 - r_1$, $r_2 = r_1$, and $s_2 = s_1$.) Finally, in a similar way, we show the statements concerning u_i, v_i, u', and v'.

C. *The standard deviation*. Let $f(x)$ be a pdf of the form $(1/b) f_0((x - a)/b)$, where $-\infty < x < \infty$, and a and $b > 0$ are the parameters. If m, m_0, σ^2, and σ_0^2 are respectively the means and variances of $f(x)$ and $f_0(x)$, then $m = a + bm_0$ and $\sigma^2 = b^2\sigma_0^2$. If ξ_p and ξ_p^0 are the p-quantiles of $f(x)$ and $f_0(x)$, then $\xi_p = a + b\xi_p^0$. It follows that $\xi_q - \xi_p = b(\xi_q^0 - \xi_p^0)$ and

$$\sigma = (\xi_q - \xi_p)/c_0, \tag{19}$$

where $c_0 = (\xi_q^0 - \xi_p^0)/\sigma_0$ depends on p, q, and $f_0(x)$, but not on a and b. Therefore for given p, q, and $f_0(x)$, any inference about $\xi_q - \xi_p$ can be readily transformed into one about σ. Thus if $x_s - x_r$ is a confidence bound for $\xi_q - \xi_p$, then $(x_s - x_r)/c_0$ is a confidence bound for σ.

For each of the following types of distribution, Table 2 gives their standard deviations, the c_0's defined by (19), and the values of the c_0's corresponding to $p = .25$ and $q = .75$. The pdfs $f(x)$ are

(20) Normal: $(1/\sigma\sqrt{2\pi}) \exp[-(x - \mu)^2/2\sigma^2]$, $\quad -\infty < x < \infty$;

(21) Laplace: $(1/2\lambda) \exp(-|x - \mu|/\lambda)$, $\quad -\infty < x < \infty$;

(22) Triangular: $(1/\lambda)[1 - |x - \mu|/\lambda]$, $\quad |x - \mu| \leqq \lambda$;

(23) Rectangular: $1/2h$, $\quad |x - a| \leqq h$;

(24) Exponential: $(1/\lambda) \exp[-(x - \mu)/\lambda]$, $\quad x \geqq \mu$.

TABLE 2

s.d.	c_0	$c_0(.25, .75)$
σ	$\xi_q^0 - \xi_p^0$	1.35
$\lambda\sqrt{2}$	$(1/\sqrt{2})[\pm\log p_1 - \pm\log q_1]$	0.98
$\lambda/\sqrt{6}$	$\sqrt{6}[\pm(1 - \sqrt{q_1}) - \pm(1 - \sqrt{p_1})]$	1.44
$h/\sqrt{3}$	$2\sqrt{3}(q - p)$	1.73
λ	$\log(1 - p)/(1 - q)$	1.10

For the normal distribution of (20), c_0 is not explicitly given in terms of p and q, but ξ_p^0 (and similarly ξ_q^0) satisfies $\Phi(\xi_p^0) = p$, where Φ is given by (12), and can be found by a normal probability table. Further, in the formulas for the c_0's corresponding to Laplace and triangular distributions; p_1 (and similarly q_1) is defined to be $1 - |1 - 2p|$; and the $+$ or $-$ sign associated with p_1 should be used according as $p \geqq \frac{1}{2}$ or $\leqq \frac{1}{2}$. Finally, as an example, let a sample of size 50 be drawn from the exponential distribution of (24). Let $p = .25$, $q = .75$, and $\alpha = .025$. From Table 1, $r' = 6$ and $u' = 21$. From Table 2, the standard deviation is λ and $c_0 = 1.10$. Therefore $((x_{30} - x_{21})/1.10, (x_{45} - x_6)/1.10)$ is a confidence interval for λ with confidence coefficient at least .95.

REFERENCES

[1] J. H. Cadwell, "The distribution of quasi-ranges in samples from a normal population," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 603–613.
[2] H. Cramér, *Mathematical Methods of Statistics*, Princeton University Press, 1946.
[3] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 1, John Wiley & Sons, 1951.
[4] P. Lévy, *Calcul des Probabilités*, Gauthiers-Villars, Paris, 1955.
[5] F. Mosteller, "On some useful 'inefficient' statistics." *Ann. Math. Stat.*, Vol. 17 (1946), pp. 377–408.
[6] H. G. Romig, *50–100 Binomial Tables*, John Wiley & Sons, 1953.
[7] S. S. Wilks, "Order Statistics," *Bull. Amer. Math. Soc.*, Vol. 54 (1948), pp. 6–50.
[8] *Tables of the Binomial Probability Distribution*, National Bureau of Standards, Washington, D. C., 1950.