ON THE ESTIMATION OF AUTOCORRELATION IN TIME SERIES

By Z. A. Lomnicki and S. K. Zaremba

Boulton Paul Aircraft, Ltd., Wolverhampton (England)

1. Introduction. In a recent paper, F. H. C. Marriott and J. A. Pope [8] investigated, in some special cases, the bias arising in the estimation of the autocorrelation function of a discrete-parameter stochastic process when its mean is not known. M. G. Kendall [4] developed a general method for the determination of this bias in the case of an arbitrary Gaussian process.

The removal of the mean from a stochastic process may be regarded as a particular case of the elimination of a polynomial trend. The object of the present paper is to determine how the removal of this trend affects both the biases and the covariances of the estimators of the covariances and of the autocorrelation coefficients; it is not assumed that the process is necessarily Gaussian.

In the two papers mentioned above, the passage from the estimation of the covariances to that of the correlation coefficients was achieved by what may be called the method of statistical differentials. The estimator $\hat{\rho}_{k,N}$ of ρ_k was regarded as a function of certain covariance estimators and, in the derivation of relevant formulae, the difference $\hat{\rho}_{k,N} - \rho_k$ was replaced by the first differential of this function. The general validity of this kind of argument needs to be clarified, as remarked by Kendall himself in the last paragraph of his note. The same applies to the derivation of cov $(\hat{\rho}_{k,N}, \hat{\rho}_{l,N})$ by Bartlett [1] in the case in which there is no trend. In order to make rigorous this kind of argument, we prove a general theorem conceived in the same spirit as a proposition given by Cramér ([3], pp. 353–356) for functions of sample moments, and justifying the use of the method of "statistical differentials" under specified assumptions.

2. Basic definitions and assumptions.

Assumption 1. In what follows it will be assumed that $\{y_t\}$ is a discrete-parameter stochastic process composed of a determinate polynomial trend $f_m(t)$ of degree at most m, and of a linear stochastic process $\{x_t\}$:

$$y_t = f_m(t) + x_t \qquad (t = 0, \pm 1, \pm 2, \cdots), \tag{2.1}$$

where $\{x_t\}$ is of the form of

$$x_t = \sum_{s=0}^{\infty} h_s \epsilon_{t-s}, \tag{2.2}$$

the series $\sum_{t=0}^{\infty} h_t$ being absolutely convergent, and $\{\epsilon_t\}$ being a wide-sense stationary process with zero means:

$$E(\epsilon_t) = 0, \qquad E(\epsilon_t^2) = \sigma^2, \qquad E(\epsilon_t \epsilon_s) = 0 \qquad \text{for } t \neq s. \tag{2.3}$$

Received January 3, 1956.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Mathematical Statistics.* www.jstor.org

In dealing with estimation problems, some further assumptions on $\{\epsilon_t\}$ are needed, and it is customary to assume that the random variables ϵ_t are independent and have identical distributions. This assumption can be weakened by introducing the following definition:

DEFINITION 2. A stochastic process $\{\epsilon_t\}$ will be described as stationary up to the order p, with the corresponding moments behaving as if the ϵ's were independent, if all the moments up to the order p exist, if, for any integer τ and for any set of integers $t_1, t_2, \cdots, t_s$ $(s \leqq p)$,

$$E(\epsilon_{t_1+\tau}\epsilon_{t_2+\tau} \cdots \epsilon_{t_s+\tau}) = E(\epsilon_{t_1}\epsilon_{t_2} \cdots \epsilon_{t_s}),$$

and if, for any set of pairwise different integers $t_1, t_2, \cdots, t_s$ and, for any set of positive integers $\lambda_1, \lambda_2, \cdots, \lambda_s$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_s \leqq p$, we have

$$E(\epsilon_{t_1}^{\lambda_1}\epsilon_{t_2}^{\lambda_2} \cdots \epsilon_{t_s}^{\lambda_s}) = E(\epsilon_{t_1}^{\lambda_1})E(\epsilon_{t_2}^{\lambda_2}) \cdots E(\epsilon_{t_s}^{\lambda_s}).$$

ASSUMPTION 3. We assume that the process $\{\epsilon_t\}$ is stationary up to the eighth order, and that the corresponding moments behave as if the ϵ's were independent.

It follows from (2.2) that

$$R_k = \operatorname{cov}(x_t, x_{t+k}) = \sigma^2 \sum_{s=0}^{\infty} h_s h_{s+k};$$

moreover, in view of the absolute convergence of $\sum h_s$, the series $\sum_{q=-\infty}^{+\infty} R_q$ is absolutely convergent. The random variable

$$\hat{R}_{k,N} = \frac{1}{N} \sum_{t=1}^{N} x_t x_{t+k} \tag{2.4}$$

is obviously an unbiased estimator of R_k. It is also consistent; indeed, if we make

$$h_t = 0 \qquad \text{for } t < 0, \tag{2.5}$$

it follows easily from (2.2) that

$$\begin{aligned} \operatorname{cov}(x_t x_{s+k}, x_p x_{q+k}) = \kappa_4 \sum_{r=-\infty}^{+\infty} h_{t-r} h_{s+k-r} h_{p-r} h_{q+k-r} \\ + R_{t-p} R_{s-q} + R_{t-q-k} R_{s-p+k}, \end{aligned} \tag{2.6}$$

and, therefore,

$$\lim_{N\to\infty} N \operatorname{cov}(\hat{R}_{k,N}; \hat{R}_{l,N}) = \frac{\kappa_4}{\sigma^4} R_k R_l + \sum_{q=-\infty}^{+\infty} (R_q R_{q+k-l} + R_{q+k} R_{q-l}) = v_{kl}. \tag{2.7}$$

(Refer to [1]), and, in particular,

$$\lim_{N\to\infty} N \operatorname{var} \hat{R}_{k,N} = \frac{\kappa_4}{\sigma^4} R_k^2 + \sum_{q=-\infty}^{+\infty} (R_q^2 + R_{q+k} R_{q-k}) = v_{kk}, \tag{2.8}$$

where $\kappa_4 = E(\epsilon_t^4) - 3\sigma^4$.

There is no difficulty, either, in showing that (2.7) and (2.8) retain their validity for the alternative estimators of R_k:

$$C_{k,N} = \frac{1}{N-k} \sum_{t=1}^{N-k} x_t x_{t+k} = \hat{R}_{k,N-k},$$

which exhaust the information supplied by the sample of size N. Thus

$$\lim_{N\to\infty} N \operatorname{cov} (C_{k,N}; C_{l,N}) = v_{k,l} \tag{2.9}$$

and, in particular,

$$\lim_{N\to\infty} N \operatorname{var} C_{k,N} = v_{kk}. \tag{2.10}$$

The validity of (2.10) is obvious, while (2.9) follows easily from (2.7), if we note that

$$\operatorname{var} (C_{k,N} - \hat{R}_{k,N}) = O(1/N^2),$$

so that, in view of the Schwarz Inequality, of (2.8) and (2.10),

$$N[\operatorname{cov} (C_{k,N}; C_{l,N}) - \operatorname{cov} (\hat{R}_{k,N}; \hat{R}_{l,N})] = O\left(\frac{1}{N^{1/2}}\right).$$

The following proposition, concerning the asymptotic behaviour of the fourth moment of $\hat{R}_{k,N}$, will be required in Section 6:

PROPOSITION 4. Under Assumptions 1 and 3,

$$\lim_{N\to\infty} N^2 E(\hat{R}_{k,N} - R_k)^4 = 3v_{kk}^2, \tag{2.11}$$

where v_{kk} is given by (2.8).

Since the fourth moment of $\hat{R}_{k,N}$ involves the eighth moments of x_t and ϵ_t, the proof is essentially based on Assumption 3. The argument leading to (2.11) is completely elementary, but a straightforward proof is fairly laborious; a more general proposition concerning all the moments (both univariate and mixed) of the covariance estimators has been proved by the authors of the present paper, and it is hoped that this result will soon be published.

3. The bias of the covariance estimators. For any fixed k smaller than $N - m$, write $\nu = N - k$. Let $\phi_0(t), \phi_1(t), \cdots, \phi_m(t)$ be the first $m + 1$ Chebyshev polynomials orthogonal on the set $t = 1, 2, \cdots, \nu$. Of course, these polynomials depend on ν; however, in order to simplify the notations, we drop the subscript ν both here and in subsequent formulae, with the exception of those cases in which this omission could cause ambiguity. We have (see, for example, [5], pp. 159–161)

$$\phi_i(t) = \sum_{j=0}^{i} (-1)^{i-j} \frac{(i!)^2 (i+j)!(\nu - j - 1)!}{(2i)!(j!)^2(i-j)!(\nu - i - 1)!} (t-1)^{(j)} \tag{3.1}$$

and

$$\sum_{t=1}^{\nu} \phi_i(t)\phi_j(t) = \begin{cases} 0, & \text{if } i \neq j, \\ Q_i = \dfrac{(i!)^4}{(2i)!(2i+1)!}\nu(\nu^2-1)\cdots(\nu^2-i^2), & \text{if } i = j. \end{cases} \tag{3.2}$$

Expanding $f_m(t)$ into a sum of these polynomials, we find

$$f_m(t) = A_0\phi_0(t) + A_1\phi_1(t) + \cdots + A_m\phi_m(t), \tag{3.3}$$

where

$$A_i = \frac{1}{Q_i}\sum_{t=1}^{\nu} f_m(t)\phi_i(t) \qquad (i = 0, 1, \cdots, m).$$

The least-square estimators $\hat{A}_i$ of A_i are clearly

$$\hat{A}_i = \frac{1}{Q_i}\sum_{t=1}^{\nu} y_t\phi_i(t) = A_i + a_i, \tag{3.4}$$

where

$$a_i = \frac{1}{Q_i}\sum_{t=1}^{\nu} x_t\phi_i(t). \tag{3.5}$$

Similarly,

$$f_m(t+k) = B_0\phi_0(t) + B_1\phi_1(t) + \cdots + B_m\phi_m(t),$$

where

$$B_i = \frac{1}{Q_i}\sum_{t=1}^{\nu} f_m(t+k)\phi_i(t),$$

and the least-square estimator $\hat{B}_i$ of B_i are

$$\hat{B}_i = B_i + b_i, \tag{3.6}$$

where

$$b_i = \frac{1}{Q_i}\sum_{t=1}^{\nu} x_{t+k}\phi_i(t). \tag{3.7}$$

It should be noted that, while they can be expressed by means of the coefficients $A_0, A_1, \cdots, A_m$, the coefficients $B_0, B_1, \cdots, B_m$ form a different set. Evidently,

$$\sum_{i=0}^{m} A_i\phi_i(t+k) \equiv \sum_{i=0}^{m} B_i\phi_i(t) \equiv f_m(t+k),$$

but the use of the estimators $\{\hat{A}_i\}$ and $\{\hat{B}_i\}$ leads to two different estimators of the polynomial $f_m(t)$, one based on the sample values $y_1, y_2, \cdots, y_{N-k}$, and the other on $y_{k+1}, y_{k+2}, \cdots, y_N$; this is in keeping with the method applied by Kendall [4].

It is proposed to investigate, as an estimator of the covariance of the $\{x_t\}$-process, the expression

$$C_{kN}^* = \frac{1}{\nu} \sum_{t=1}^{\nu} \left[y_t - \sum_{i=0}^{m} \hat{A}_i \varphi_i(t) \right] \left[y_{t+k} - \sum_{i=0}^{m} \hat{B}_i \phi_i(t) \right],$$

which, in view of (2.1), (3.4), and (3.6), is equal to

$$C_{k,N}^* = \frac{1}{\nu} \sum_{t=1}^{\nu} \left[x_t - \sum_{i=0}^{m} a_i \phi_i(t) \right] \left[x_{t+k} - \sum_{i=0}^{m} b_i \phi_i(t) \right]. \tag{3.8}$$

LEMMA 5. *Under the assumptions of Section* 2,

$$C_{k,N}^* = C_{k,N} - \frac{1}{\nu} \sum_{i=0}^{m} a_i b_i Q_i . \tag{3.9}$$

PROOF. According to (3.8),

$$\begin{aligned} C_{k,N}^* = C_{k,N} &- \frac{1}{\nu} \sum_{t=1}^{\nu} x_{t+k} \sum_{i=0}^{m} a_i \phi_i(t) - \frac{1}{\nu} \sum_{t=1}^{\nu} x_t \sum_{i=0}^{m} b_i \phi_i(t) \\ &+ \frac{1}{\nu} \sum_{t=1}^{\nu} \sum_{i,j=0}^{m} a_i b_j \phi_i(t) \phi_j(t). \end{aligned} \tag{3.10}$$

In view of (3.2), the last sum in (3.10) is equal to

$$\frac{1}{\nu} \sum_{i=0}^{m} a_i b_i Q_i ,$$

while, owing to (3.5) and (3.7), the remaining two sums also have this value. This completes the proof of the lemma.

LEMMA 6. *With i being any fixed non-negative integer, and $\phi_i(t)$ being given by* (3.1), *if $\sum_{t=-\infty}^{+\infty} c_t$ is convergent, and if*

$$\gamma_\nu = \frac{1}{Q_i} \sum_{t,s=1}^{\nu} \phi_i(t) \phi_i(s) c_{t-s} , \tag{3.11}$$

then

$$\lim_{\nu \to \infty} \gamma_\nu = \sum_{t=-\infty}^{+\infty} c_t . \tag{3.12}$$

PROOF. Make

$$d_{\alpha,i}^{(\nu)} = \sum_{s=1}^{\nu-\alpha} \phi_i(s) \phi_i(s+\alpha) \qquad (\alpha = 0, 1, \cdots, \nu - 1); \; d_{\nu,i}^{(\nu)} = 0,$$

noting that $d_{0,i}^{(\nu)} = Q_i$. Substituting α for $t - s$ in (3.11), we find

$$\gamma_\nu = \frac{1}{d_{0,i}^{(\nu)}} \left[d_{0,i}^{(\nu)} c_0 + \sum_{\alpha=1}^{\nu-1} d_{\alpha,i}^{(\nu)} (c_\alpha + c_{-\alpha}) \right]. \tag{3.13}$$

Introducing the notation

$$S_\beta = \sum_{\alpha=-\beta}^{\beta} c_\alpha \qquad (\beta = 0, 1, \cdots, \nu - 1),$$

we can apply Abel's Transformation to the right-hand side of (3.13), obtaining

$$\gamma_\nu = \frac{1}{d_{0,i}^{(\nu)}} \sum_{\alpha=0}^{\nu-1} [d_{\alpha,i}^{(\nu)} - d_{\alpha+1,i}^{(\nu)}] S_\alpha . \tag{3.14}$$

This formula shows that the sequence $\{\gamma_\nu\}$ can be obtained from the sequence $\{S_\alpha\}$ by a linear transformation, the coefficients of which are equal to

$$\delta_{\nu_1\alpha} = \frac{1}{d_{0,i}^{(\nu)}} [d_{\alpha,i}^{(\nu)} - d_{\alpha+1,i}^{(\nu)}]. \tag{3.15}$$

According to a well-known theorem of Toeplitz (see, for example, [6]), the transformed sequence $\{\gamma_\nu\}$ is convergent to the same limit as $\{S_\nu\}$, if the following three conditions are satisfied by the coefficients of the transformation:

(i) if $\lim_{\nu\to\infty}$, $\delta_{\nu,\alpha} = 0$, for any fixed α,

(ii) if there exists a constant K such that, for every ν,

$$\sum_{\alpha=0}^{\nu-1} | \delta_{\nu,\alpha} | < K,$$

(iii) if $\lim_{\nu\to\infty} \sum_{\alpha=0}^{\nu-1} \delta_{\nu,\alpha} = 1$.

In order to prove (i) and (ii), note that

$$d_{\alpha,i}^{(\nu)} - d_{\alpha+1,i}^{(\nu)} = - \sum_{s=1}^{\nu-\alpha-1} \phi_i(s)\Delta\phi_i(s + \alpha) + \phi_i(\nu - \alpha)\phi_i(\nu). \tag{3.16}$$

According to (3.1),

$$\phi_i(s) = \sum_{j=0}^{i} f_{ij}^{(\nu)}(s - 1)^{(j)} \qquad \text{and} \qquad \Delta\phi_i(s) = \sum_{j=0}^{i} f_{ij}^{(\nu)} j(s - 1)^{(j-1)},$$

where

$$f_{ij}^{(\nu)} = (-1)^{i-j} \frac{(i!)^2(i + j)!(\nu - j - 1)!}{(2i)!(j!)^2(i - j)!(\nu - i - 1)!} \qquad (j = 0, 1, \cdots , i).$$

Thus, clearly,

$$|f_{ij}^{(\nu)}| < A\nu^{i-j}, \tag{3.17}$$

where

$$A = \max_{0\leq j\leq i} \frac{(i!)^2(i + j)!}{(2i)!(j!)^2(i - j)!}.$$

Hence

$$| \phi_i(\nu) | = \left| \sum_{j=0}^{i} f_{ij}^{(\nu)}(\nu - 1)^{(j)} \right| \leqq A \sum_{j=0}^{i} \nu^i = A(i + 1)\nu^i,$$

and, a fortiori, $|\varphi_i(\nu - \alpha)| \leqq A(i + 1)\nu^i$, so that the second term in the right-hand side of (3.16) is smaller than $A^2(i + 1)^2\nu^{2i}$. On the other hand, owing to (3.17),

$$\left| \sum_{s=1}^{\nu-\alpha-1} \phi_i(s)\Delta\phi_i(s + \alpha) \right| = \left| \sum_{j,l=0}^{i} f_{ij}^{(\nu)} f_{il}^{(\nu)} l \sum_{s=1}^{\nu-\alpha-1} (s - 1)^{(j)}(s + \alpha - 1)^{(l-1)} \right|$$

$$\leqq A^2 \sum_{j,l=0}^{i} l\nu^{2i-j-l} \sum_{s=1}^{\nu-\alpha-1} s^j(s + \alpha)^{l-1} \leqq A^2(i + 1)^3\nu^{2i}.$$

Thus there exists a constant A_0 such that

$$|d_{\alpha,i}^{(\nu)} - d_{\alpha+1,i}^{(\nu)}| < A_0\nu^{2i} \qquad (\alpha = 0, 1, \cdots, \nu - 1);$$

but according to (3.2),

$$d_{0,i}^{(\nu)} = \frac{(i!)^4}{(2i)!(2i+1)!}\nu(\nu^2 - 1)\cdots(\nu^2 - i^2),$$

and, in view of (3.15), this shows that $\lim_{\nu\to\infty}\delta_{\nu,\alpha} = 0$, and that there exists a constant K such that $\sum_{\alpha=0}^{\nu-1}|\delta_{\nu,\alpha}| < K$.

Thus Conditions (i) and (ii) of the Toeplitz Theorem are satisfied; so is Condition (iii), since $\sum_{\alpha=0}^{\nu-1}\delta_{\nu,\alpha} = 1$ identically, as can be seen by summing (3.15). Therefore,

$$\lim_{\nu\to\infty}\gamma_\nu = \lim_{\nu\to\infty}S_\nu = \sum_{\alpha=-\infty}^{+\infty}c_\alpha. \tag{3.18}$$

Hence the proof of Lemma 6 is complete.

Now it can be shown that $C_{k,N}^*$ is an asymptotically unbiased estimator of R_k and that, to order N^{-1}, the bias is equal to $-(m+1)\sum_{q=-\infty}^{+\infty}R_q/N$. More precisely we have:

PROPOSITION 7. Under the assumptions of Section 2,

$$\lim_{N\to\infty}NE(C_{k,N}^* - R_k) = -(m+1)\sum_{q=-\infty}^{+\infty}R_q. \tag{3.19}$$

PROOF. According to (3.9),

$$\nu E(C_{k,N}^* - R_k) = -\sum_{i=0}^{m}Q_iE(a_ib_i);$$

but, owing to (3.5) and (3.7)

$$Q_iE(a_ib_i) = \frac{1}{Q_i}\sum_{t,s=1}^{\nu}\phi_i(t)\phi_i(s)R_{t-s+k} \qquad (i = 0, 1, \cdots, m). \tag{3.20}$$

Applying Lemma 6, we find

$$\lim_{\nu\to\infty}Q_iE(a_ib_i) = \sum_{\alpha=-\infty}^{+\infty}R_{k+\alpha} = \sum_{q=-\infty}^{+\infty}R_q,$$

which completes the proof of (3.19).

It will be noticed that, according to (3.19), the bias of the covariance estimators based on (3.8) is negative. In the particular case when $m = 0$, (3.19) shows that this bias is asymptotically equal to the negative of the variance of the mean (see, for example, Lemma 2 in [7]), and this can be seen at once on the basis of elementary considerations. The fact that in the general case the bias is asymptotically proportional to $m + 1$ is due to the superposition of the effects of fitting the successive orthogonal polynomials, each of which contributes the same generalized mean-square error. (See Lemma 6.)

4. The covariance of the covariance estimators.

Since the elimination of a polynomial trend induces, in the covariance estimators, a bias which is merely of the order of $1/N$, one is not surprised to find that this procedure, asymptotically, does not affect the second and the fourth moments. The proof of the relevant propositions is based on the following:

LEMMA 8. *If i is any fixed nonnegative integer and $\phi_i(t)$ is given by (3.1), if $\sum_{t=-\infty}^{+\infty} c_t$ is absolutely convergent, and if*

$$\beta_\nu = \frac{1}{Q_i} \sum_{t,s=1}^{\nu} | \phi_i(s)\phi_i(t) | \, | c_{t-s} |, \tag{4.1}$$

then there exists a constant K independent of ν such that $\beta_\nu < K$.

PROOF. By making $t - s = \alpha$ and using the same argument which led to (3.13), we obtain from (4.1)

$$\beta_\nu \leqq \frac{1}{Q_i} \left[Q_i | c_0 | + \sum_{\alpha=1}^{\nu-1} (| c_\alpha | + | c_{-\alpha} |) \sum_{s=1}^{\nu-\alpha} | \phi_i(s)\phi_i(s+\alpha) | \right]; \tag{4.2}$$

but, clearly,

$$\sum_{s=1}^{\nu-\alpha} | \phi_i(s)\phi_i(s+\alpha) | = O(\nu^{2i+1}).$$

Hence, in view of the second line of (3.2), $Q_i^{-1} \sum_{s=1}^{\nu-\alpha} |\phi_i(s)\phi_i(s+\alpha)|$ is bounded by a constant independent of ν; this, owing to the convergence of $\sum_{\alpha=-\infty}^{+\infty} |c_\alpha|$, completes the proof.

PROPOSITION 9. Under the assumptions of Section 2,

$$\lim_{N\to\infty} N \operatorname{cov} (C_{k,N}^*; C_{l,N}^*) = v_{k,l}, \tag{4.3}$$

where $v_{k,l}$ is given by (2.7).

PROOF. For any fixed k and l smaller than $N - m$, write $\nu = N - k$, $\nu' = N - l$ and

$$\begin{cases} X_{ik} = \dfrac{1}{\nu} a_i b_i Q_i; & X_k = \sum_{i=0}^{m} X_{ik} \\ X_{il} = \dfrac{1}{\nu'} a_i' b_i' Q_i'; & X_l = \sum_{i=0}^{m} X_{il} \end{cases} \tag{4.4}$$

where Q_i', a_i', and b_i' are given by formulae obtained from (3.2), (3.5), and (3.7) when the sequence of polynomials $\phi_i(t)$ orthogonal on the set $t = 1, 2, \cdots, \nu$ is replaced by a similar sequence of polynomials $\psi_i(t)$ orthogonal on the set $t = 1, 2, \cdots, \nu'$.

According to (3.9),

$$C_{k,N}^* = C_{k,N} - X_k, \qquad C_{l,N}^* = C_{l,N} - X_l; \tag{4.4'}$$

hence

$$\begin{aligned} \operatorname{cov} (C_{k,N}^*; C_{l,N}^*) = \operatorname{cov} (C_{k,N}; C_{l,N}) - \operatorname{cov} (X_k; C_{l,N}) \\ - \operatorname{cov} (X_l; C_{k,N}) + \operatorname{cov} (X_k; X_l). \end{aligned} \tag{4.5}$$

But, substituting (3.5) and (3.7) in (4.4), we find

$$X_{ik} = \frac{1}{\nu} \sum_{t,s=1}^{\nu} \frac{\phi_i(t)\phi_i(s)x_t x_{s+k}}{Q_i},$$

and, in consequence,

$$\text{var } X_{ik} = \frac{1}{\nu^2} \frac{1}{Q_i^2} \sum_{t,s,p,q=1}^{\nu} \phi_i(t)\phi_i(s)\phi_i(p)\phi_i(q) \text{ cov } (x_t x_{s+k}, x_p x_{q+k}),$$

or, in view of (2.6),

$$\text{var } X_{ik} = \frac{1}{\nu^2} \frac{1}{Q_i^2} \sum_{t,s,p,q=1}^{\nu} \phi_i(t)\phi_i(s)\phi_i(p)\phi_i(q) \left[R_{t-p} R_{s-q} + R_{t-q-k} R_{s-p+k} + \kappa_4 \sum_{r=-\infty}^{+\infty} h_{t-r} h_{s+k-r} h_{p-r} h_{q+k-r} \right]. \tag{4.6}$$

This fourfold sum yields three terms, the first and the second of which are respectively equal to

$$\frac{1}{\nu^2} \left[\frac{1}{Q_i} \sum_{t,p=1}^{\nu} \phi_i(t)\phi_i(p) R_{t-p} \right]^2$$

and

$$\frac{1}{\nu^2} \left[\frac{1}{Q_i} \sum_{t,q=1}^{\nu} \phi_i(t)\phi_i(q) R_{t-q-k} \right] \left[\frac{1}{Q_i} \sum_{s,p=1}^{\nu} \phi_i(s)\phi_i(p) R_{s-p+k} \right].$$

Lemma 6 shows that each of these expressions in brackets tends to the finite limit $\sum_{q=-\infty}^{+\infty} R_q$, when $\nu \to \infty$; hence this part of var X_{ik} is $O(1/\nu^2)$.

The third sum is obviously dominated by

$$\frac{\kappa_4}{\nu^2} \left(\frac{1}{Q_i} \sum_{t,s=1}^{\nu} | \phi_i(t)\phi_i(s) | \sum_{r=-\infty}^{+\infty} | h_{t-r} h_{s+k-r} | \right)^2, \tag{4.7}$$

and the fact that this expression is also $O(1/\nu^2)$ can be proved as follows: If

$$\sum_{r=-\infty}^{+\infty} | h_{t-r} h_{s-r} | = \sum_{r=-\infty}^{+\infty} | h_{t-s-r} h_{-r} | = \mathcal{R}_{t-s}, \tag{4.8}$$

the expression in brackets in (4.7) becomes

$$\frac{1}{Q_i} \sum_{t,s=1}^{\nu} | \phi_i(t)\phi_i(s) | \mathcal{R}_{t-s-k}. \tag{4.9}$$

Obviously, the infinite sum $\sum_{\alpha=-\infty}^{+\infty} \mathcal{R}_{\alpha-k}$ is convergent, while the boundedness of (4.9) follows from Lemma 8. Thus

$$\text{var } X_{ik} = O\left(\frac{1}{\nu^2}\right). \tag{4.10}$$

From the definition of X_k and from the Triangle Inequality, it follows that var $X_k = O(1/\nu^2)$ and similarly var $X_l = O(1/\nu'^2)$; by applying the Schwarz Inequality to cov $(X_k ; C_{l,N})$, cov $(X_l ; C_{k,N})$, and cov (X_k, X_l) in (4.5), it can be seen that these covariances are respectively $O(N^{-3/2})$, $O(N^{-3/2})$, $O(N^{-2})$, which, in conjunction with (2.9), completes the proof.

COROLLARY 10. *Under the assumptions of Section* 2,

$$\lim_{N\to\infty} N \text{ var } C^*_{k,N} = v_{kk}. \tag{4.11}$$

LEMMA 11. *Under the assumptions of Section* 2,

$$E(X^4_{ik}) = O(1/N^3). \tag{4.12}$$

PROOF. Clearly

$$\begin{aligned} E(X^4_{ik}) &= \frac{1}{\nu^4}\frac{1}{Q^4_i} E\left\{\sum_{t,s=1}^{\nu} \phi_i(t)\phi_i(s)x_t x_{s+k}\right\}^4 \\ &= \frac{1}{\nu^4}\frac{1}{Q^4_i} \sum_{t_1,t_2,t_3,t_4,s_1,s_2,s_3,s_4=1}^{\nu} \phi_i(t_1)\phi_i(t_2)\phi_i(t_3)\phi_i(t_4)\phi_i(s_1)\phi_i(s_2)\phi_i(s_3)\phi_i(s_4) \\ &\qquad \cdot E(x_{t_1}x_{s_1+k}x_{t_2}x_{s_2+k}x_{t_3}x_{s_3+k}x_{t_4}x_{s_4+k}). \end{aligned} \tag{4.13}$$

Owing to (2.5), Eq. (2.2) can be written as follows:

$$x_t = \sum_{q=-\infty}^{+\infty} h_{t-q}\,\epsilon_q,$$

so that

$$\begin{aligned} &E(x_i x_j x_l x_m x_n x_p x_r x_s) \\ &= \sum_{q_1,q_2,\cdots,q_8=-\infty}^{+\infty} h_{i-q_1}h_{j-q_2}h_{l-q_3}h_{m-q_4}h_{n-q_5}h_{p-q_6}h_{r-q_7}h_{s-q_8}E(\epsilon_{q_1}\epsilon_{q_2}\cdots\epsilon_{q_8}). \end{aligned} \tag{4.14}$$

But, in view of our assumptions concerning the moments of the ϵ's, only those moments do not vanish which correspond to equal indices in sets obtained by the following types of partition of the eight indices $q_1, q_2, \cdots, q_8$: (8), (2, 6), (3, 5), (4, 4), (2, 2, 4), (2, 3, 3), (2, 2, 2, 2).

Hence the right-hand side of (4.14) becomes:

$$\left\{\begin{aligned} &\mu_8 \sum_{q=-\infty}^{+\infty} h_{i-q}h_{j-q}h_{l-q}h_{m-q}h_{n-q}h_{p-q}h_{r-q}h_{s-q} \\ &\quad + \mu_6\sigma^2 \sum\nolimits_{(28)} \sideset{}{'}\sum_{q_1,q_2=-\infty}^{+\infty} h_{i-q_1}h_{j-q_1}h_{l-q_2}h_{m-q_2}h_{n-q_2}h_{p-q_2}h_{r-q_2}h_{s-q_2} \\ &\quad + \mu_3\mu_5 \sum\nolimits_{(56)} \sideset{}{'}\sum_{q_1,q_2=-\infty}^{+\infty} h_{i-q_1}h_{j-q_1}h_{l-q_1}h_{m-q_2}h_{n-q_2}h_{p-q_2}h_{r-q_2}h_{s-q_2} \\ &\quad + \mu_4^2 \sum\nolimits_{(35)} \sideset{}{'}\sum_{q_1,q_2=-\infty}^{+\infty} h_{i-q_1}h_{j-q_1}h_{l-q_1}h_{m-q_1}h_{n-q_2}h_{p-q_2}h_{r-q_2}h_{s-q_2} \\ &\quad + \mu_4\sigma^4 \sum\nolimits_{(210)} \sideset{}{'}\sum_{q_1,q_2,q_3=-\infty}^{+\infty} h_{i-q_1}h_{j-q_1}h_{l-q_2}h_{m-q_2}h_{n-q_3}h_{p-q_3}h_{r-q_3}h_{s-q_3} \\ &\quad + \mu_3^2\sigma^2 \sum\nolimits_{(280)} \sideset{}{'}\sum_{q_1,q_2,q_3=-\infty}^{+\infty} h_{i-q_1}h_{j-q_1}h_{l-q_2}h_{m-q_2}h_{n-q_2}h_{p-q_3}h_{r-q_3}h_{s-q_3} \\ &\quad + \sigma^8 \sum\nolimits_{(105)} \sideset{}{'}\sum_{q_1,q_2,q_3,q_4=-\infty}^{\infty} h_{i-q_1}h_{j-q_1}h_{l-q_2}h_{m-q_2}h_{n-q_3}h_{p-q_3}h_{r-q_4}h_{s-q_4}, \end{aligned}\right. \tag{4.15}$$

where $\sum_{(28)}$ is extended to all the 28 partitions of the type (2, 6), $\sum_{(56)}$ to all the 56 partitions of the type (3, 5), and so on, and the prime after the sign of multiple sum denotes the exclusion of all the terms in which not all the summation indices are pairwise distinct.

Each of the multiple sums taken with respect to q_1, q_2, etc. is dominated by an expression of the type of

$$\mathcal{R}_{i-j}\mathcal{R}_{l-m}\mathcal{R}_{n-p}\mathcal{R}_{r-s},$$

where $\mathcal{R}_i$ is defined by (4.8). The only exceptions are the sums occurring in the third and the sixth lines of (4.15), which are dominated by products of the type of

$$\mathcal{R}_{i-j}\mathcal{R}_{m-n}\mathcal{R}_{p-r}\mathcal{H}^2,$$

where $\mathcal{H} = \sum_{q=0}^{\infty} | h_q |$.

When (4.15) is substituted for $E(x_i x_j x_l x_m x_n x_p x_r x_s)$, in (4.13), the contribution of the first sum of (4.15) is dominated by

$$\mu_8 \frac{1}{\nu^4}\left[\frac{1}{Q_i}\sum_{t,s=1}^{\nu} | \phi_i(t)\phi_i(s) | \mathcal{R}_{t-s-k}\right]^4.$$

But, in view of Lemma 8, the expression in brackets is bounded by a constant independent of ν, so that this contribution is $O(1/\nu^4)$. In a similar way it can be shown that all the other sums in (4.15) contribute $O(1/\nu^4)$ when substituted in (4.13), the only exceptions being the sums appearing in the third and the sixth lines. But the contributions of these sums are dominated by expressions of the type of

$$\frac{1}{\nu^4}\left[\frac{1}{Q_i}\sum_{t,s=1}^{\nu} | \phi_i(t)\phi_i(s) | \mathcal{R}_{t-s-k}\right]^3 \left[\frac{1}{Q_i}\sum_{t,s=1}^{\nu} | \phi_i(s)\phi_i(t) | \right]\mathcal{H}^2,$$

multiplied by $\mu_3\mu_5$ and $\mu_3^2\sigma^2$ respectively; here the first factor in brackets is bounded, while the second is

$$\frac{1}{Q_i}\sum_{t,s=1}^{\nu} O(\nu^{2i}) = O(\nu),$$

so that the contributions of these sums are $O(1/\nu^3)$.

Thus the total contribution of (4.15) and (4.13) is a sum of a fixed number of contributions, each of which is either $O(1/\nu^4)$ or $O(1/\nu^3)$. Hence the proof is complete.

COROLLARY 12. *Under the assumptions of Section* 2,

$$E[X_k - E(X_k)]^4 = O(1/N^3). \tag{4.16}$$

PROOF. According to Lemma 11 and to the Minkowski Inequality, $E(X_k^4) = O(1/N^3)$, and (4.16) follows at once.

PROPOSITION 13. Under the assumptions of Section 2,

$$\lim_{N\to\infty} N^2 E[C_{k,N}^* - E(C_{k,N}^*)]^4 = 3v_{kk}^2.$$

PROOF. According to (4.4′),

$$C^*_{k,N} - E(C^*_{k,N}) = (C_{k,N} - R_k) - (X_k - E(X_k)$$

and, therefore,

$$N^2E[C^*_{k,N} - E(C^*_{k,N})]^4 = N^2E(C_{kN} - R_k)^4 - 4N^2E\{(C_{k,N} - R_k)^3[X_k - E(X_k)]\}$$
$$+ 6N^2E\{(C_{k,N} - R_k)^2[X_k - E(X_k)]^2\} - 4N^2E\{(C_{k,N} - R_k)[X_k - E(X_k)]^3\}$$
$$+ N^2E[X_k - E(X_k)]^4.$$

The first term in the right-hand side tends to $3v_{kk}^2$ according to Proposition 4, and the last is $O(1/N)$ owing to Corollary 12; by repeated applications of the Schwarz Inequality we deduce from (2.10) and (4.16) that the remaining terms are respectively $O(N^{-1/4})$, $O(N^{-1/2})$, $O(N^{-3/4})$, and this completes the proof.

5. The method of "statistical differentials."

THEOREM 14. *Let $H(Y_1, Y_2, \cdots, Y_p)$ and $G(Y_1, Y_2, \cdots, Y_p)$ be any two functions vanishing at the point $(0, 0, \cdots, 0)$ and having continuous partial derivatives of the first and second orders in the neighbourhood of this point. Let $y_1^{(N)}$, $y_2^{(N)}$, $\cdots$, $y_p^{(N)}$ be any random variables with*

$$\lim_{N\to\infty} NE(y_i^{(N)}) = c_i \qquad (i = 1, 2, \cdots, p), \tag{5.1}$$

and

$$\lim_{N\to\infty} N \operatorname{cov}(y_i^{(N)}, y_j^{(N)}) = c_{ij} \qquad (i, j = 1, 2, \cdots p). \tag{5.2}$$

c_i and c_{ij} being constants. Moreover, assume

$$E[y_i^{(N)} - E(y_i^{(N)})]^4 = O(1/N^2) \qquad (i = 1, 2, \cdots, p). \tag{5.3}$$

Then, if $H(y_1^{(N)}, y_2^{(N)}, \cdots, y_p^{(N)})$ and $G(y_1^{(N)}, y_2^{(N)}, \cdots, y_p^{(N)})$ are bounded uniformly with respect to N,

$$\lim_{N\to\infty} NE[H(y_1^{(N)}, y_2^{(N)}, \cdots, y_p^{(N)})] = \sum_{i=1}^{p} \frac{\partial H}{\partial y_i} c_i + \tfrac{1}{2} \sum_{i,j=1}^{p} \frac{\partial^2 H}{\partial y_i \partial y_j} c_{ij} \tag{5.4}$$

and

$$\lim_{N\to\infty} N \operatorname{cov}[H(y_1^{(N)}, y_2^{(N)}, \cdots, y_p^{(N)}); G(y_1^{(N)} y_2^{(N)}, \cdots, y_p^{(N)})] = \sum_{i,j=1}^{p} \frac{\partial H}{\partial y_i} \frac{\partial G}{\partial y_j} c_{ij}, \tag{5.5}$$

the partial derivatives being taken at the point $(0, 0, \cdots, 0)$. (Obviously, if we make $H = G$, (5.5) yields the corresponding formula for var H).

PROOF. In the first place we note that, if δ is any positive number,

$$P[|y_i^{(N)} - E(y_i^{(N)})| \geqq \delta] = O(1/N^2), \tag{5.6}$$

which follows from (5.3) and from the generalized Chebyshev Inequality:

$$E[y_i^{(N)} - E(y_i^{(N)}]^4 \geqq \delta^4 P[|y_i^{(N)} - E(y_i^{(N)})| \geqq \delta]. \tag{5.7}$$

On the other hand, it follows from (5.1) and (5.2) that

$$(5.8)\quad \lim_{N\to\infty} NE(y_i^{(N)}y_j^{(N)}) = \lim_{N\to\infty} N[\text{cov}\,(y_i^{(N)}, y_j^{(N)}) + E(y_i^{(N)})E(y_j^{(N)})] = c_{ij}\,.$$

Since, according to the Schwarz Inequality, to (5.2), and to (5.3),

$$|E[y_i^{(N)} - E(y_i^{(N)})]^3| \leqq (E[y_i^{(N)} - E(y_i^{(N)})]^4)^{1/2}\,(\text{var}\, y_i^{(N)})^{1/2} = O(N^{-3/2}),$$

(5.3) entails

$$(5.9)\qquad E(y_i^{(N)4}) = O(1/N^2).$$

If ϵ denotes an arbitrary positive number, make

$$\eta = 2\epsilon \Big/ \left\{1 + \left[\sum_{i=1}^{p} (c_{ii})^{\frac{1}{2}}\right]^2\right\},$$

and let the positive number δ be sufficiently small to ensure that, whenever

$$|Y_i| < \delta \qquad (i = 1, 2, \cdots, p),$$

the second-order partial derivatives of H and G are continuous and differ from their respective values at the origin by less than η. Finally, let $\mathcal{E}$ be the event

$$|y_i^{(N)}| < \delta \qquad (i = 1, 2, \cdots, p),$$

and $\bar{\mathcal{E}}$ the complementary event. Clearly, both events depend on N and δ, but the abbreviated notation $\mathcal{E}$ for $\mathcal{E}_{N,\delta}$ and $\bar{\mathcal{E}}$ for $\bar{\mathcal{E}}_{N,\delta}$ should not lead to misunderstandings. Furthermore,

$$P(\bar{\mathcal{E}}) \leqq \sum_{i=1}^{p} P[|\, y_i^{(N)} \,| \geqq \delta],$$

which, according to (5.9), entails

$$(5.10)\qquad P(\bar{\mathcal{E}}) = O(1/N^2).$$

Writing $y^{(N)}$ for the vector $(y_1^{(N)}, y_2^{(N)}, \cdots, y_p^{(N)})$, and denoting by P the corresponding probability function, we have.

$$(5.11)\qquad E[H(y^{(N)}] = \int_{\mathcal{E}} H(y^{(N)})\, dP + \int_{\bar{\mathcal{E}}} H(y^{(N)})\, dP.$$

However, owing to the assumptions made, and in view of (5.10),

$$(5.12)\qquad \left| \int_{\bar{\mathcal{E}}} H(y^{(N)})\, dP \right| \leqq \max |\, H(y^{(N)}) \,| \cdot P(\bar{\mathcal{E}}) = O\left(\frac{1}{N^2}\right).$$

In the event $\mathcal{E}$, clearly,

$$(5.13)\qquad H(y^{(N)}) = \sum_{i=1}^{p} H_i(0) y_i^{(N)} + \tfrac{1}{2} \sum_{i,j=1}^{p} H_{ij}(\theta y^{(N)}) y_i^{(N)} y_j^{(N)},$$

where

$$H_i(Y) = \frac{\partial H}{\partial Y_i}, \qquad H_{ij}(Y) = \frac{\partial^2 H}{\partial Y_i \partial Y_j} \qquad (i, j = 1, 2, \cdots, p),$$

Y stands for the vector $(Y_1, Y_2, \cdots, Y_p)$ and $0 < \theta < 1$. Hence, owing to (5.11) and (5.12),

$$E[H(y^{(N)})] = \sum_{i=1}^{p} H_i(0) \int_{\mathcal{E}} y_i^{(N)} \, dP + \tfrac{1}{2} \sum_{i,j=1}^{p} \int_{\mathcal{E}} H_{ij}(\theta y^{(N)}) y_i^{(N)} y_j^{(N)} \, dP + O\left(\frac{1}{N^2}\right),$$

or

$$(5.14) \quad \begin{cases} E[H(y^{(N)})] = \sum_{i=1}^{p} H_i(0) E(y_i^{(N)}) + \tfrac{1}{2} \sum_{i,j=1}^{p} H_{ij}(0) E(y_i^{(N)} y_j^{(N)}) \\ \qquad - \sum_{i=1}^{p} H_i(0) \int_{\bar{\mathcal{E}}} y_i^{(N)} \, dP - \tfrac{1}{2} \sum_{i,j=1}^{p} H_{ij}(0) \int_{\bar{\mathcal{E}}} y_i^{(N)} y_j^{(N)} \, dP \\ \qquad + \tfrac{1}{2} \sum_{i,j=1}^{p} \int_{\mathcal{E}} [H_{ij}(\theta y^{(N)}) - H_{ij}(0)] y_i^{(N)} y_j^{(N)} \, dP + 0\left(\frac{1}{N^2}\right). \end{cases}$$

However, owing to the Schwarz Inequality and according to (5.8) and (5.10),

$$(5.15) \qquad \left| \int_{\bar{\mathcal{E}}} y_i^{(N)} \, dP \right| \leqq \{E(y_i^{(N)})^2 P(\bar{\mathcal{E}})\}^{1/2} = O(N^{-3/2}).$$

Similarly, owing to (5.9),

$$(5.16) \quad \begin{aligned} \left| \int_{\bar{\mathcal{E}}} y_i^{(N)} y_j^{(N)} \, dP \right| &\leqq \left\{ \int_{\bar{\mathcal{E}}} (y_i^{(N)} y_j^{(N)})^2 \, dP \; P(\bar{\mathcal{E}}) \right\}^{1/2} \\ &\leqq \{E(y_i^{(N)})^4 E(y_j^{(N)})^4\}^{1/4} [P(\bar{\mathcal{E}})]^{1/2} = O\left(\frac{1}{N^2}\right). \end{aligned}$$

Finally, by the Schwarz Inequality and in view of the definition of δ,

$$\begin{aligned} &\left| \int_{\mathcal{E}} [H_{ij}(\theta y^{(N)}) - H_{ij}(0)] y_i^{(N)} y_j^{(N)} \, dP \right| \\ &\qquad \leqq \left\{ \int_{\mathcal{E}} [H_{ij}(\theta y^{(N)}) - H_{ij}(0)]^2 y_i^{(N)2} \, dP \right\}^{1/2} \left[\int_{\mathcal{E}} y_j^{(N)2} \, dP \right]^{1/2} \\ &\qquad \leqq \eta [E(y_i^{(N)2}) E(y_j^{(N)2})]^{1/2}. \end{aligned}$$

Hence

$$\overline{\lim_{N \to \infty}} N \left| \int_{\mathcal{E}} [H_{ij}(\theta y^{(N)}) - H_{ij}(0)] y_i^{(N)} y_j^{(N)} \, dP \right| \leqq \eta (c_{ii} c_{jj})^{1/2},$$

and, consequently,

$$(5.17) \quad \overline{\lim_{N \to \infty}} \frac{N}{2} \left| \sum_{i,j=1}^{p} \int_{\mathcal{E}} [H_{ij}(\theta y^{(N)}) - H_{ij}(0)] y_i^{(N)} y_j^{(N)} \, dP \right| \leqq \frac{\eta}{2} \sum_{i,j=1}^{p} (c_{ii} c_{jj})^{1/2} < \epsilon.$$

Now, in view of (5.1), (5.8), (5.15), (5.16), and (5.17), we obtain from (5.14)

$$\overline{\lim_{N\to\infty}} \left| NE[H(y^{(N)})] - \sum_{i=1}^{p} H_i(0)c_i - \tfrac{1}{2} \sum_{i,j=1}^{p} H_{ij}(0)c_{ij} \right| \leqq \epsilon,$$

which proves (5.4), since the choice of ϵ was arbitrary.

An argument similar to that which led to (5.7) and (5.12) shows that

$$E[H(y^{(N)})G(y^{(N)})] = \int_{\mathcal{E}} H(y^{(N)})G(y^{(N)})\, dP + O\left(\frac{1}{N^2}\right).$$

Using the expansion of $H(y^{(N)})$ given by (5.13) and a similar expansion of $G(y^{(N)})$ based on a similar notation for the derivatives, the preceding equation can be written as follows:

$$(5.18)\quad \begin{cases} E[H(y^{(N)})G(y^{(N)})] \\ \quad = \sum_{i,j=1}^{p} H_i(0)G_j(0)E(y_i^{(N)}y_j^{(N)}) - \sum_{i,j=1}^{p} H_i(0)G_j(0)\int_{\bar{\mathcal{E}}} y_i^{(N)}y_j^{(N)}\, dP \\ \qquad + \tfrac{1}{2} \sum_{i,j,k=1}^{p} H_i(0) \int_{\mathcal{E}} G_{jk}(\theta' y^{(N)}) y_i^{(N)} y_j^{(N)} y_k^{(N)}\, dP \\ \qquad + \tfrac{1}{2} \sum_{i,j,k=1}^{p} G_i(0) \int_{\mathcal{E}} H_{jk}(\theta y^{(N)}) y_i^{(N)} y_j^{(N)} y_k^{(N)}\, dP \\ \qquad + \tfrac{1}{4} \sum_{i,j,k,l=1}^{p} \int_{\mathcal{E}} H_{ij}(\theta y^{(N)}) G_{kl}(\theta' y^N) y_i^{(N)} y_j^{(N)} y_k^{(N)} y_l^{(N)}\, dP + O\left(\frac{1}{N^2}\right), \end{cases}$$

where again $0 < \theta' < 1$.

Multiplying both sides of this equality by N and making $N \to \infty$, we obtain (5.5). Indeed, the left-land sides agree in view of (5.1), and, owing to (5.8), the first term in the right-hand side of (5.18), after multiplication by N, tends to the right-hand side of (5.5). All the other terms in the right-hand side of (5.18) can be neglected: the second is $O(N^{-2})$ according to (5.16), and simple applications of the Schwarz Inequality show that the third and the fourth terms are $O(N^{-3/2})$, while the fifth term is again $O(N^{-2})$. Thus the proof is complete.

6. The estimation of the autocorrelation coefficients.

Proposition 15. If, under the assumptions of Section 2,

$$(6.1)\qquad \rho_{k,N}^* = \frac{C_{k,N}^*}{(A_{0,N}^* B_{0,N}^*)^{1/2}},$$

where $C_{k,N}^*$ is given by (3.8) and

$$(6.2)\quad A_{0,N}^* = \frac{1}{\nu}\sum_{t=1}^{\nu}\left[y_t - \sum_{i=0}^{m} \hat{A}_i \phi_i(t)\right]^2, \qquad B_{0,N}^* = \frac{1}{\nu}\sum_{t=1}^{\nu}\left[y_{t+k} - \sum_{i=m}^{m} \hat{B}_i \phi_i(t)\right]^2,$$

then

$$(6.3)\quad \lim_{N\to\infty} NE(\rho_{k,N}^* - \rho_k) = -(m+1)(1-\rho_k) \sum_{q=-\infty}^{+\infty} \rho_q + 2 \sum_{q=-\infty}^{+\infty} (\rho_k \rho_q^2 - \rho_q \rho_{q+k})$$

PROOF. Let

$$H(Y_1, Y_2, Y_3) = \frac{Y_1 + R_k}{(Y_2 + R_0)^{1/2}(Y_3 + R_0)^{1/2}} - \frac{R_k}{R_0}. \tag{6.4}$$

According to (6.1),

$$\rho^*_{k,N} - \rho_k = H(y_1^{(N)}, y_2^{(N)}, y_3^{(N)}),$$

where

$$y_1^{(N)} = C^*_{k,N} - R_k; \qquad y_2^{(N)} = A^*_{0,N} - R_0; \qquad y_3^{(N)} = B^*_{0,N} - R_0. \tag{6.5}$$

The assumptions of Theorem 14 are clearly satisfied; (5.3) follows from Proposition 13, while the uniform boundedness of $H(y_1^{(N)}, y_2^{(N)}, y_3^{(N)})$ follows from the Schwarz Inequality. In view of Proposition 7,

$$c_1 = -(m + 1) \sum_{q=-\infty}^{+\infty} R_q.$$

Since $A^*_{0N} = C^*_{0,\nu}$ and since $B^*_{0,N}$ is an estimator of R_0 analogous to $C^*_{0,\nu}$ applied to the process $\{x_{t+k}\}$,

$$c_2 = c_3 = c_1.$$

For the same reasons, Corollary 10 applies not only to $y_1^{(N)}$, but to $y_2^{(N)}$ and $y_3^{(N)}$ as well; hence

$$c_{11} = v_{kk}, \qquad c_{22} = c_{23} = v_{00}. \tag{6.6}$$

On the other hand,

$$A^*_{0,N} = \hat{R}_{0,\nu} - \frac{1}{\nu} \sum_{i=0}^{m} a_i^2 Q_i, \qquad C^*_{k,N} = \hat{R}_{k,\nu} - \frac{1}{\nu} \sum_{i=0}^{m} a_i b_i Q_i,$$

$$B^*_{0,N} = \hat{R}_{0,\nu} - \frac{1}{\nu} \sum_{t=1}^{k} x_t^2 + \frac{1}{\nu} \sum_{t=N-k+1}^{N} x_t^2 - \frac{1}{\nu} \sum_{i=0}^{m} b_i^2 Q_i.$$

By an argument similar to that used in the proof of Proposition 9, we find

$$\operatorname{var}\left(\frac{1}{\nu} \sum_{i=0}^{m} a_i^2 Q_i\right) = O\left(\frac{1}{\nu^2}\right) \quad \text{and} \quad \operatorname{var}\left(\frac{1}{\nu} \sum_{i=0}^{m} b_i^2 Q_i\right) = O\left(\frac{1}{\nu^2}\right);$$

hence, and from (4.10), by applying the Schwarz and Triangle Inequalities, we obtain

$$N \operatorname{cov}(A^*_{0,N}; B^*_{0,N}) = N \operatorname{var} \hat{R}_{0,\nu} + O(1/\nu^{1/2}),$$

$$N \operatorname{cov}(A^*_{0,N}; C^*_{k,N}) = N \operatorname{cov}(\hat{R}_{0,\nu}; \hat{R}_{k,\nu}) + O(1/\nu^{1/2}),$$

$$N \operatorname{cov}(B^*_{0,N}; C^*_{k,N}) = N \operatorname{cov}(\hat{R}_{0,\nu}; \hat{R}_{k,\nu}) + O(1/\nu^{1/2}).$$

In view of (2.7), this implies

$$c_{12} = c_{13} = v_{0k} \quad \text{and} \quad c_{23} = v_{00}. \tag{6.7}$$

Substituting in (5.4) the values of the partial derivatives of (6.4) at the point $Y_1 = Y_2 = Y_3 = 0$, as well as the values obtained above for c_i and c_{ij} ($i, j = 1, 2, 3$), we find

$$\lim_{N\to\infty} NE(\rho^*_{k,N} - \rho_k) = -(m+1)\frac{1}{R_0^2}\sum_{q=-\infty}^{+\infty} R_q(R_0 - R_k) + \frac{2}{R_0^3}\sum_{q=-\infty}^{+\infty}(R_k R_q^2 - R_0 R_q R_{q+k}).$$

which is equivalent to (6.3).

Thus the bias of the estimators of the autocorrelation function is composed of two parts: one which is due to the bias in the covariance estimators (and is, therefore, proportional to $m + 1$), and another which is a result of the correlation between the numerator and the denominator, and is still there when there is no trend to eliminate. (See Corollary 17 below.)

REMARK 16. If $m = 0$, (6.3) yields the bias of the estimator $\rho^*_{k,N}$ of the autocorrelation function when the process is stationary with an unknown mean. This result can also be obtained directly from the formulae given by Kendall [4]. It is sufficient to note that Kendall's Eqs. (7) and (8) can be written in the following form:

$$E(A) = \rho_k - \frac{1}{\nu}\left\{\frac{1}{\nu}\sum_{j=1-\nu}^{\nu-1}(\nu - j)\rho_{k+j}\right\}; \qquad E(B) = 1 - \frac{1}{\nu}\left\{\frac{1}{\nu}\sum_{j=1-\nu}^{\nu-1}(\nu - j)\rho_j\right\},$$

where the expressions in curly brackets, as partial Cesàro sums of the infinite series $\sum_{j=-\infty}^{+\infty}\rho_{k+j}$ and $\sum_{j=-\infty}^{+\infty}\rho_j$ respectively, tend to $\sum_{j=-\infty}^{+\infty}\rho_j$.

COROLLARY 17. *If, the assumptions of Section 2 being satisfied,* $f_m(t) \equiv 0$, *and if the estimation of the autocorrelation function is based on the assumption that there is no trend, i.e., if we make*

$$\hat{\rho}_{k,n} = \frac{C_{k,N}}{\left(\frac{1}{\nu}\sum_{t=1}^{N-k} x_t^2\right)^{1/2}\left(\frac{1}{\nu}\sum_{t=k+1}^{N} x_t^2\right)^{1/2}}, \tag{6.8}$$

then

$$\lim_{N\to\infty} NE(\hat{\rho}_{k,N} - \rho_k) = 2\sum_{q=-\infty}^{+\infty}(\rho_k\rho_q^2 - \rho_q\rho_{q+k}).$$

The proof is entirely similar to that of Proposition 15, with the exception that now $c_1 = c_2 = c_3 = 0$.

PROPOSITION 18. Under the assumptions of Section 2,

$$\lim_{N\to\infty} N \operatorname{cov}(\rho^*_{k,N}; \rho^*_{l,N}) = \sum_{q=-\infty}^{+\infty}(\rho_q\rho_{q+k-l} + \rho_{q+k}\rho_{q-l} - 2\rho_l\rho_q\rho_{q+k} - 2\rho_k\rho_q\rho_{q+l} + 2\rho_k\rho_l\rho_q^2). \tag{6.9}$$

PROOF. Retaining the notation of the proof of Proposition 15, make

$$G(Y_4, Y_5, Y_6) = \frac{Y_4 + R_l}{(Y_5 + R_0)^{\frac{1}{2}}(Y_6 + R_0)^{\frac{1}{2}}} - \frac{R_l}{R_0},$$

and

$$y_4^{(N)} = C_{l,N}^* - R_l ; \qquad y_5^{(N)} = A_{0,N}^{*\prime} - R_0 ; \qquad y_l^{(N)} = B_{0,N}^{*\prime} - R_0 ,$$

where

$$A_{0,N}^{*\prime} = \frac{1}{\nu'} \sum_{t=1}^{\nu'} \left[y_t - \sum_{i=0}^{m} \hat{A}_i' \psi_i(t) \right]^2 ; \qquad B_{0,N}^{*\prime} = \frac{1}{\nu'} \sum_{t=1}^{\nu'} \left[y_{t+l} - \sum_{i=0}^{m} \hat{B}_i' \psi_i(t) \right]^2,$$

so that

$$G(y_4, y_5, y_6) = \rho_{l,N}^* - \rho_l .$$

Then Theorem 14 can be applied if H and G are regarded each as a function of the six variables $Y_1, Y_2, \cdots, Y_6$.

According to Proposition 9, $c_{14} = v_{kl}$, and, by an argument similar to that used in proving (6.7), we find

$$c_{15} = c_{16} = v_{0k}, \qquad c_{24} = c_{34} = v_{0l}, \qquad c_{25} = c_{26} = c_{35} = c_{36} = v_{00} .$$

Hence, according to (5.5),

$$\lim_{N\to\infty} N \operatorname{cov} (\rho_{k,N}^* \rho_{l,N}^*) = \frac{1}{R_0^4} \sum_{q=-\infty}^{+\infty} [R_0^2 R_q R_{q+k-l} + R_0^2 R_{q+k} R_{q-l} - 2R_0 R_l R_q R_{q+k} - 2R_0 R_k R_q R_{q+l} + 2R_k R_l R_q^2],$$

which is equivalent to (6.9).

COROLLARY 19. *Under the assumptions of Corollary* 17,

$$\lim_{N\to\infty} N \operatorname{cov} (\hat{\rho}_{k,N} \hat{\rho}_{l,N}) = \lim_{N\to\infty} N \operatorname{cov} (\rho_{k,N}^* \rho_{l,N}^*) \tag{6.10}$$

(See [1], [2].)

Finally, it should be noted that there is no difficulty in applying, with the necessary modifications, the same method to the investigation of other covariance and autocorrelation estimators, e.g., to the circular estimator of the autocorrelation function, or to $C_{k,N}^* / C_{0,N}^*$.

7. Acknowledgement. The authors wish to thank Mr. J. D. North, Chairman and Managing Director of Boulton Paul Aircraft, Ltd., for his permission to publish the present note which originated in an attempt to solve a more general problem proposed by him.

REFERENCES

[1] M. S. BARTLETT, "On the theoretical specification and sampling properties of autocorrelated time series", *J. Roy. Stat. Soc., Suppl.* Vol. 8, No. 1 (1946), pp. 27–41.

[2] M. S. BARTLETT, *An Introduction to Stochastic Processes*, Cambridge University Press, 1955.
[3] H. CRAMÉR, *Mathematical Methods of Statistics*, Princeton University Press, 1946.
[4] M. G. KENDALL, "Note on bias in the estimation of autocorrelation", *Biometrika*, Vol. 41 (1954), pp. 403–404.
[5] M. G. KENDALL, *The Advanced Theory of Statistics*, Vol. II, Ch. Griffin & Co., London, 1946.
[6] K. KNOPP, *Theorie und Anwendung der Unendlichen Reihen*, 4-te Auflage, Springer, Berlin, 1947.
[7] Z. A. LOMNICKI AND S. K. ZAREMBA, "Some applications of zero-one processes", *J. Roy. Stat. Soc. Ser. B*, Vol. 17, No. 2, pp. 243–255.
[8] F. H. C. MARRIOTT AND J. A. POPE, "Bias in the estimation of autocorrelations", *Biometrika*, Vol. 41 (1954), pp. 390–402.