ON A CHARACTERIZATION OF THE NORMAL DISTRIBUTION FROM PROPERTIES OF SUITABLE LINEAR STATISTICS

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1. Introduction and Summary. In recent years, problems related to the characterization of the normal distribution from the property of stochastic independence of linear functions of independent random variables have been investigated by various authors. The most general result in this direction is that obtained independently by Darmois [4] and Skitovich [14], who proved that if there exist two linear functions

$$X = \sum_{j=1}^{n} a_j x_j \quad \text{and} \quad Y = \sum_{j=1}^{n} b_j x_j \qquad \text{with} \quad a_j b_j \neq 0$$

$(j = 1, 2, \cdots n)$, such that they are stochastically independent, where x_1, $x_2, \cdots, x_n$ are n independently (but not necessarily identically) distributed proper random variables, then each x_j is normally distributed. Their methods of proof are similar in nature, both being based on the use of characteristic functions, without any assumption about the existence of moments. The same theorem has been proved independently by Basu [1], under the assumption that the random variables are identically and independently distributed and have finite moments of all orders. This result is also obtained independently by Lukacs and King [11], under the assumption that the random variables are independently (but not necessarily identically) distributed, each having finite moments up to order n, and by Linnik [10], under the assumption that the random variables have only finite variances. The special case of this theorem for $n = 2$ was proved earlier independently by Kac [6], Gnedenko [5], and Darmois [3], without any assumption about the existence of moments.

Thus we see that the problems on the consequences of stochastic independence of linear statistics have been exhaustively studied. Hence the question that naturally arises is whether similar investigations into the distribution laws of the random variables are possible under the assumption of a suitable type of stochastic dependence of the linear statistics. In this direction, the author [8] has derived a characterization of the stable law with finite expectation from the property of linearity of regression of one linear statistic on the other for the case $n = 2$. The author [7] has also obtained some characterizations of the normal distribution from the consequence of the linearity of the multiple regression of one random variable on several others, when the variables have the linear structural setup as in the case of the bi-factor theory of Spearman.

For the formulation of the problem investigated in the present paper, we require a precise definition of the terms *conditional distribution*, *linearity of re-*

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gression, and *homoscedasticity*. Let $F(x, y)$ and $F_0(x)$ denote respectively the distribution function of the two-dimensional random variable (x, y) and the marginal distribution function of x. Then we define the conditional distribution function of y for fixed x by $F_x(y)$ so that it satisfies the relation

$$\int_{-\infty}^{x} F_\xi(y)\, dF_0(\xi) = F(x, y). \tag{1.1}$$

In the present investigation, the following assumptions on the distributions of the random variables will be made:

Assumption 1. The conditional distribution of y for fixed x as defined in (1.1) is assumed to exist, wherever needed.

Assumption 2. Each of the random variables concerned has a finite second moment. This assumption allows us to take derivatives of the characteristic functions of the corresponding random variables up to and including the second order.

Assumption 3. Without any loss of generality in the proof, it is assumed that the expectation of each of the random variables concerned is equal to zero.

The role of these assumptions is to ensure the existence of the expectation and the variance of the conditional distribution of y for fixed x, which may be defined respectively as

$$E_x(y) = \int_{-\infty}^{\infty} y\, dF_x(y), \tag{1.2}$$

$$\begin{aligned} V_x(y) &= \int_{-\infty}^{\infty} \{y - E_x(y)\}^2\, dF_x(y) \\ &= S_x(y) - \{E_x(y)\}^2, \end{aligned} \tag{1.3}$$

where

$$S_x(y) = \int_{-\infty}^{\infty} y^2\, dF_x(y).$$

In this case, the regression of y on x is said to be linear, if the relation

$$E_x(y) = \beta x \tag{1.4}$$

is satisfied for all x, except for a set of probability measure zero, as the expectations of both the random variables x and y are already assumed to be zero. The constant β in equation (1.4) is called the coefficient of regression of y on x. Similarly the conditional distribution of y for fixed x is said to be homoscedastic, if the conditional variance $V_x(y)$ as defined in (1.3) is a constant σ_0^2 not involving x. Thus if the regression of y on x is linear and given by βx and the conditional variance of y for fixed x is a constant σ_0^2, we have the relation

$$S_x(y) = \sigma_0^2 + \beta^2 x^2 \tag{1.5}$$

to be satisfied for all x, in addition to the condition (1.4). For simplicity in notation, throughout the present paper we shall use the term the conditional distribution of y for fixed x is L.R.H. (β, σ_0^2), meaning thereby that the regression of y on x is linear and given by βx and that the conditional variance of y for fixed x is σ_0^2, being free of x, i.e., equivalent to the statement that both the relations (1.4) and (1.5) are simultaneously satisfied.

In the following sections we shall derive some characterizations of the normal distribution from the property of linearity of regression and homoscedasticity of suitable linear statistics. The main theorem (Theorem 3.2) is given in Section 3. The proof of this theorem uses as a starting point a very simple set of necessary and sufficient conditions for the linearity of regression and homoscedasticity (Lemma 2.1) and finally a theorem of Linnik (Lemma 2.3) on an analytical extension of Cramér's theorem [2] on the normal law. Several important corollaries are deduced in the subsequent section.

2. Certain useful lemmas. We shall now give some important lemmas which are instrumental in the proof of the theorems.

LEMMA 2.1. (Necessary and sufficient conditions for the linearity of regression and homoscedasticity). *Let x and y be two proper random variables each having a finite variance. Then the necessary and sufficient conditions for the conditional distribution of y for fixed x to be* L.R.H. (β, σ_0^2) *are that the equations*

$$(2.1) \qquad \left.\frac{\partial \varphi(u, v)}{\partial v}\right]_{v=0} = \beta \frac{d\varphi(u, 0)}{du}$$

and

$$(2.2) \qquad \left.\frac{\partial^2 \varphi(u, v)}{\partial v^2}\right]_{v=0} = -\sigma_0^2 \varphi(u, 0) + \beta^2 \frac{d^2\varphi(u, 0)}{du^2}$$

are to be satisfied for all real u, where $\varphi(u, v)$ and $\varphi(u, 0)$ represent respectively the characteristic functions of the distribution of (x, y) and the marginal distribution of x.

This lemma helps us in introducing the differential equation connecting the characteristic functions of the variables concerned and has been proved independently by Rao [12] and Rothschild and Mourier [13].

LEMMA 2.2. *Let x and y be two proper random variables each having a finite variance. Then if the conditional distribution of y for fixed x is* L.R.H. (β, σ_0^2), *the conditional distribution of $by (b \neq 0)$ for fixed $ax (a \neq 0)$ is* L.R.H. $(\beta', \sigma_0'^2)$ *where $\beta' = b\beta/a$ and $\sigma_0'^2 = b^2\sigma_0^2$.*

LEMMA 2.3. (Analytical extension of Cramér's theorem on the normal law). *Let $X_1, X_2, \cdots, X_n$ be n independent proper random variables and let further $\varphi_j(t)$ denote the characteristic function of the distribution of $X_j (j = 1, 2, \cdots, n)$ If now the functions $\varphi_j(t)$ satisfy the equation*

$$(2.3) \qquad \prod_{j=1}^{n} \{\varphi_j(t)\}^{\alpha_j} = e^{Q(t)},$$

for all real t in a certain neighbourhood $|t| < \delta$, $(\delta > 0)$ of the origin, where α_j's are some positive numbers and $Q(t)$ a quadratic polynomial in t, then each X_j is normally distributed.

PROOF. We give below a short proof of this very interesting lemma which is due to Linnik [9].[1]

Without any loss of generality in the proof, we can take the quadratic polynomial $Q(t)$ to be of the form $Q(t) = iat - \frac{1}{2}t^2$ and work with the characteristic functions $\theta_j(t)$ of the symmetric random variables $x_j = X_j - X_j'(j = 1, 2, \cdots, n)$, where X_j' is distributed independently of X_j and has the same distribution as X_j. Then it can be easily shown that the characteristic functions $\theta_j(t)$ satisfy the equation

$$\prod_{j=1}^{n} \{\theta_j(t)\}^{\alpha_j} = e^{-t^2} \tag{2.4}$$

for all real t in a suitably chosen neighbourhood of the origin. Again noting that each of the characteristic functions $\theta_j(t)$ is real, we have

$$\begin{aligned} \theta_j(t) &= \int_{-\infty}^{\infty} \cos tx \, dF_j(x) \\ &= 1 - 2\int_{-\infty}^{\infty} \sin^2 \frac{tx}{2} \, dF_j(x) \\ &\leqq \exp\left\{-2\int_{-\infty}^{\infty} \sin^2 \frac{tx}{2} \, dF_j(x)\right\}, \qquad j = 1, 2, \cdots, n, \end{aligned} \tag{2.5}$$

whence, using the equation (2.4), we get

$$\sum_{j=1}^{n} \alpha_j \int_{-\infty}^{\infty} \sin^2 \frac{tx}{2} \, dF_j(x) \leqq \frac{t^2}{2}, \tag{2.6}$$

thus yielding for every j, the inequality

$$\int_{-\infty}^{\infty} \frac{\sin^2 (tx/2)}{t^2} \, dF_j(x) \leqq \frac{1}{2\alpha_j} \tag{2.7}$$

holding for all real t in a certain neighbourhood of the origin. Then by using Fatou's theorem, it follows from (2.7) that the second moment of each x_j exists.

Next we shall show by induction that each x_j has finite moments of all orders. Let us suppose that each x_j has finite moments up to an even order $2k$. Then differentiating both sides of the equation (2.4) with respect to t, $2k$ times we get

$$S_1(t) + S_2(t) + S_3(t) = \mathfrak{G}_{2k}(t)e^{-t^2}, \tag{2.8}$$

where $S_1(t)$ contains all the derivatives of order $2k$ of the functions $\theta_j(t)$, $S_2(t)$ contains all the derivatives of only odd order not greater than $2k - 1$, $S_3(t)$ contains only derivatives of even order not greater than $2k - 2$, and $\mathfrak{G}_{2k}(t)$ is a polynomial of degree $2k$ in t. We also note that $S_2(0) = 0$.

We should further note that for each term on the left hand side of (2.8) which, except for a constant coefficient, is a product of the derivatives of the func-

[1] The proof of this lemma is given in "A. A. Zinger and Yu. V. Linnik—On an analytical generalization of a theorem of Cramér and its application, *Vestnik Leningrad Univ.*, Vol. 10 (1955), pp. 51–56." In this paper the authors have also given an alternative proof of Darmois-Skitovich theorem using this lemma.

tions $\theta_j(t)$, if p_{jr} is the order of the derivative of $\theta_j(t)$ and q_{jr} the corresponding power with which it appears, then

$$\sum_{j,r} p_{jr} q_{jr} = 2k, \tag{2.9}$$

where $\theta_j(t)$ itself is to be considered as a derivative of the order zero.

Now from (2.8) we get easily

$$\begin{aligned} &\frac{S_1(t) - S_1(0)}{t^2} + \frac{S_2(t)}{t^2} + \frac{S_3(t) - S_3(0)}{t^2} \\ &\qquad = \frac{\mathfrak{G}_{2k}(t) - \mathfrak{G}_{2k}(0)}{t^2} e^{-t^2} - \mathfrak{G}_{2k}(0) \frac{1 - e^{-t^2}}{t^2}. \end{aligned} \tag{2.10}$$

Then it is easy to verify that as $t \to 0$, the expression on the right-hand side as well as each of the second and third summands on the left-hand side of Eq. (2.10) tends to a finite limit. Consequently $[S_1(t) - S_1(0)]/t^2$ must also tend to a finite limit as t tends to zero.

Moreover, noting that

$$S_1(t) = \sum_{j=1}^{n} \alpha_j \theta_j^{(2k)}(t) \frac{\prod_{r=1}^{n} \{\theta_r(t)\}^{\alpha_r}}{\theta_j(t)}, \tag{2.11}$$

it can be shown from the above result that

$$-\frac{1}{2} \sum_{j=1}^{n} \alpha_j \frac{\theta_j^{(2k)}(t) - \theta_j^{(2k)}(0)}{t^2} = (-1)^k \sum_{j=1}^{n} \alpha_j \int_{-\infty}^{\infty} x^{2k} \frac{\sin^2 \frac{tx}{2}}{t^2} dF_j(x) \tag{2.12}$$

has a finite limit as $t \to 0$. Again proceeding in the same way as in (2.6) and using Fatou's theorem, we prove that each x_j has a finite moment of order $2k + 2$, thus completing the induction.

We shall next show that each $\theta_j(t)$ is an entire function of t. Without any loss of generality in the proof, we can, by making a suitable change of scale of the variables if necessary, take each of the indices $\alpha_j \geqq 1$. Now raising both sides of Eq. (2.4), to the power $2k$, we differentiate $2k$ times with respect to t and then put $t = 0$, thus obtaining

$$S_1^*(0) + S_3^*(0) = (-1)^k \frac{(2k)!\,k^k 2^k}{k!}. \tag{2.13}$$

Denoting by μ_{2kj} the moment of the order $2k$ of the random variable x_j, we have

$$S_1^*(0) = (-1)^k \sum_{j=1}^{n} 2k\alpha_j \mu_{2kj}, \tag{2.14}$$

while $S_3^*(0)$ consists of terms, each containing the moments of even order (the order being at most $2k - 2$) with a positive coefficient having the same sign

$(-1)^k$ by virtue of (2.9). Then noting that $\alpha_j \geqq 1$ and $k^k < e^k k!$, we get easily, using (2.13) and (2.14) together, the inequality

$$\mu_{2kj} < \sum_{j=1}^{n} 2k\alpha_j \mu_{2kj} < \frac{(2k)!\,k^k 2^k}{k!} < (2k)!(2e)^k, \tag{2.15}$$

whence it follows that each $\theta_j(t)$ has a power series development about $t = 0$ with a radius of convergence not less than $1/\sqrt{2e}$, the series representation being of the form

$$\theta_j(t) = \sum_{k=0}^{\infty} (-1)^k \frac{\mu_{2kj}}{(2k)!} t^{2k}, \qquad j = 1, 2, \cdots, n. \tag{2.16}$$

We now consider the behaviour of each of $\theta_j(t)$ for purely imaginary values of t. Substituting $t = iv$ (v real) in (2.16) and then making the variable transformation $w = v^2 (w \geqq 0)$, we can easily verify that the functions $\xi_j(w)$ which reduce to $\theta_j(t)$, if we put $w = -t^2$, satisfy the equation

$$\prod_{j=1}^{n} \{\xi_j(w)\}^{\alpha_j} = e^w \tag{2.17}$$

and have the power series development

$$\xi_j(w) = \textstyle\sum_{k=0}^{\infty} [(\mu_{2kj})/(2k)!]w^k$$

about $w = 0$, the radius of convergence being not less than $1/2e$ and the coefficients being all positive. Now let $w_0 > 0$ be a point within the radius of convergence of each of the $\xi_j(w)$. Then taking $\eta_j(W) = \xi_j(w_0 + W)/\xi_j(w_0)$, it is easy to verify that the functions $\eta_j(W)$ satisfy the equation

$$\prod_{j=1}^{n} \{\eta_j(W)\}^{\alpha_j} = e^W, \tag{2.18}$$

which is completely analogous to the equation (2.17) and has the power-series development

$$\eta_j(W) = \sum_{k=0}^{\infty} \frac{\xi_j^{(k)}(w_0)}{\xi_j(w_0)} \frac{W^k}{k!}$$

in a certain neighbourhood of $W = 0$, the coefficients being all positive.

Then proceeding exactly in the same way as above, we can show that the radius of convergence of each of the series development for $\eta_j(W)$ is not less than $1/2e$. Hence each $\xi_j(w)$, having no singularities for $0 \leqq w < w_0 + 1/2e$, has a power-series development with radius of convergence not less than $w_0 + 1/2e$ ($\xi_j(w)$ is a series with positive coefficients). This fact evidently leads to the conclusion that each of $\xi_j(w)$ and hence each of $\theta_j(t)$ is an entire function. The remainder of the proof is exactly similar to that of the theorem of Cramér [2].

3. The theorems. We are now in a position to prove the following theorems:

THEOREM 3.1. *Let* (x_j, y_j) $j = 1, 2, \cdots, n$ *be* n *independently (but not neces-*

sarily identically) distributed two-dimensional proper random variables each having a finite variance such that the conditional distribution of y_j for fixed x_j is L.R.H. (β_j, σ_{j0}^2) *for* $j = 1, 2, \cdots, n$. *Let* $X = \sum_{j=1}^n a_j x_j$ *and* $Y = \sum_{j=1}^n b_j y_j$ *be two linear functions with* $a_j b_j \neq 0$ $(j = 1, 2, \cdots, n)$, *then the conditional distribution of Y for fixed X is always* L.R.H. (β, σ_0^2), *whenever the relation*

$$\frac{b_1\beta_1}{a_1} = \frac{b_2\beta_2}{a_2} = \cdots = \frac{b_n\beta_n}{a_n} = \beta$$

is satisfied.

PROOF. For convenience in procedure, let us substitute $\xi_j = a_j x_j$ and $\eta_j = b_j y_j$ for $j = 1, 2, \cdots, n$. Then we can write $X = \sum_{j=1}^n \xi_j$ and $Y = \sum_{j=1}^n \eta_j$, and further using Lemma 2.2, we get that the conditional distribution of η_j for fixed ξ_j is L.R.H. $(\beta_j', \sigma_{j0}'^2)$ where $\beta_j' = b_j\beta_j/a_j$ and $\sigma_{j0}'^2 = b_j^2\sigma_{j0}^2$; $j = 1, 2, \cdots, n$.

Let $\varphi_j(u, v)$ and $\varphi_j(u, 0)$ denote respectively the characteristic functions of the distribution of (ξ_j, η_j) and the marginal distribution of $\xi_j (j = 1, 2, \cdots, n)$ and similarly $\Phi(u, v)$ and $\Phi(u, 0)$, those of the distribution of (X, Y) and the marginal distribution of X respectively.

Then we can write

$$\Phi(u, v) = E\{\exp(iuX + ivY)\} = E\{\exp(iu \sum_j \xi_j + iv \sum_j \eta_j)\} = \prod_{j=1}^n \varphi_j(u, v). \tag{3.1}$$

Again it is given that the conditional distribution of η_j for fixed ξ_j is L.R.H. $(\beta_j', \sigma_{j0}'^2)$, for all $j = 1, 2, \cdots, n$. Hence applying Lemma 2.1 and using the conditions (2.1) and (2.2), we get

$$\left.\frac{\partial\varphi_j(u, v)}{\partial v}\right]_{v=0} = \beta_j' \frac{d\varphi_j(u, 0)}{du}, \tag{3.2}$$

$$\left.\frac{\partial^2\varphi_j(u, v)}{\partial v^2}\right]_{v=0} = -\sigma_{j0}'^2\varphi_j(u, 0) + \beta_j'^2 \frac{d^2\varphi_j(u, 0)}{du^2}, \qquad j = 1, 2, \cdots, n. \tag{3.3}$$

Now differentiating both sides of (3.1) with respect to v, r times $(r = 1, 2)$ and then putting $v = 0$ and using the relations (3.2) and (3.3), we get

$$\left.\frac{\partial\Phi(u, v)}{\partial v}\right]_{v=0} = \sum_{j=1}^n \beta_j' \frac{d\varphi_j(u, 0)}{du} \prod_{k\neq j} \varphi_k(u, 0), \tag{3.4}$$

$$\left.\frac{\partial^2\Phi(u, v)}{\partial v^2}\right]_{v=0} = -\prod_{j=1}^n \varphi_j(u, 0) \sum_{j=1}^n \sigma_{j0}'^2 + \sum_{j=1}^n \beta_j'^2 \frac{d^2\varphi_j(u, 0)}{du^2} \prod_{k\neq j} \varphi_k(u, 0) + \sum_{j\neq k} \beta_j'\beta_k' \frac{d\varphi_j(u, 0)}{du} \frac{d\varphi_k(u, 0)}{du} \prod_{l\neq j,k} \varphi_l(u, 0). \tag{3.5}$$

Again putting $v = 0$ on both sides of (3.1) and then differentiating both sides with respect to u, r times ($r = 1, 2$), we get

$$\frac{d\Phi(u, 0)}{du} = \sum_{j=1}^{n} \frac{d\varphi_j(u, 0)}{du} \prod_{k \neq j} \varphi_k(u, 0), \tag{3.6}$$

$$\begin{aligned} \frac{d^2\Phi(u, 0)}{du^2} &= \sum_{j=1}^{n} \frac{d^2\varphi_j(u, 0)}{du^2} \prod_{k \neq j} \varphi_k(u, 0), \\ &+ \sum_{j \neq k} \frac{d\varphi_j(u, 0)}{du} \frac{d\varphi_k(u, 0)}{du} \prod_{l \neq j,k} \varphi_l(u, 0). \end{aligned} \tag{3.7}$$

Now under the conditions of the theorem, we have $\beta_1' = \beta_2' = \cdots = \beta_n' = \beta$. Then substituting this in (3.4) and (3.5) and finally comparing with (3.6) and (3.7), we get

$$\left.\frac{\partial\Phi(u, v)}{\partial v}\right]_{v=0} = \beta \frac{d\Phi(u, 0)}{du}, \tag{3.8}$$

$$\left.\frac{\partial^2\Phi(u, v)}{\partial v^2}\right]_{v=0} = -\Phi(u, 0) \sum_{j=1}^{n} \sigma_{j0}'^2 + \beta^2 \frac{d^2\Phi(u, 0)}{du^2}. \tag{3.9}$$

Then the proof of the theorem follows at once using Lemma 2.1 to (3.8) and (3.9).

From the above theorem, it follows easily that if there exist two linear functions $X = \sum_{j=1}^{n} a_j x_j$ and $Y = \sum_{j=1}^{n} b_j x_j$, with $a_j b_j \neq 0$ ($j = 1, 2, \cdots, n$) where $x_1, x_2, \cdots, x_n$ are n independently (but not necessarily identically) distributed proper random variables each having a finite variance, then the conditional distribution of Y for fixed X is always L.R.H. (β, σ_0^2) whenever the relation $b_1/a_1 = b_2/a_2 = \cdots = b_n/a_n = \beta$ is satisfied.

In the following we shall establish the normality of the random variables x_j's from the property of the linearity of regression and homoscedasticity of the conditional distribution of Y for fixed X as introduced in Theorem 3.1, under some conditions. For this purpose let σ_j^2 denote the variance of the random variable x_j ($j = 1, 2, \cdots, n$). Then the coefficient of regression of Y on X will be given by $\beta = \sum a_j b_j \beta_j \sigma_j^2 / \sum a_j^2 \sigma_j^2$, the summation extending over all the indices j for which $(b_j \beta_j)/a_j \neq \beta$. We now state the following theorem.

THEOREM 3.2. *With the same notations and assumptions as those used in theorem* 3.1, *the necessary and sufficient condition for the conditional distribution of* Y *for fixed* X *to be* L.R.H. (β, σ_0^2) *is that*

(i) *each* x_j *for which* $(b_j\beta_j)/a_j \neq \beta$ *is normally distributed, while each* y_j *and the other* x_j*'s have arbitrary distributions.*

$$\beta = \sum{}' a_j b_j \beta_j \sigma_j^2 / \sum{}' a_j^2 \sigma_j^2 \tag{ii}$$

and

$$\sigma_0^2 = \sum_{j=1}^{n} b_j^2 \sigma_{j0}^2 + \sum{}' \left(\frac{b_j \beta_j}{a_j} - \beta\right)^2 a_j^2 \sigma_j^2,$$

where $\sum'$ *stands for the summation over all the indices* j *for which* $(b_j\beta_j)/a_j \neq \beta$.

PROOF.

Necessity: First of all, substituting $\xi_j = a_j x_j$ and $\eta_j = b_j y_j$ as in Theorem 3.1, for $j = 1, 2, \cdots, n$, we get $X = \sum_{j=1}^{n} \xi_j$ and $Y = \sum_{j=1}^{n} \eta_j$.

Since it is given that the conditional distribution of Y for fixed X is L.R.H. (β, σ_0^2), we get on using the conditions (2.1) and (2.2) of Lemma 2.1,

$$\left.\frac{\partial \Phi(u, v)}{\partial v}\right]_{v=0} = \beta \frac{d\Phi(u, 0)}{du}, \tag{3.10}$$

$$\left.\frac{\partial^2 \Phi(u, v)}{\partial v^2}\right]_{v=0} = -\sigma_0^2 \Phi(u, 0) + \beta^2 \frac{d^2\Phi(u, 0)}{du^2}. \tag{3.11}$$

Next using the relations (3.1), (3.4), (3.5), (3.6), and (3.7) together in the equations (3.10) and (3.11), we have

$$\sum_{j=1}^{n} \beta_j' \frac{d\varphi_j(u, 0)}{du} \prod_{k \neq j} \varphi_k(u, 0) = \beta \left[\sum_{j=1}^{n} \frac{d\varphi_j(u, 0)}{du} \prod_{k \neq j} \varphi_k(u, 0) \right], \tag{3.12}$$

$$\begin{aligned} -\prod_{j=1}^{n} \varphi_j(u, 0) \sum_{j=1}^{n} \sigma_{j0}'^2 &+ \sum_{j=1}^{n} \beta_j'^2 \frac{d^2\varphi_j(u, 0)}{du^2} \prod_{k \neq j} \varphi_k(u, 0) \\ &+ \sum_{j \neq k} \beta_j' \beta_k' \frac{d\varphi_j(u, 0)}{du} \frac{d\varphi_k(u, 0)}{du} \prod_{l \neq j,k} \varphi_l(u, 0) = -\sigma_0^2 \prod_{j=1}^{n} \varphi_j(u, 0) \\ &+ \beta^2 \left[\sum_{j=1}^{n} \frac{d^2\varphi_j(u, 0)}{du^2} \prod_{k \neq j} \varphi_k(u, 0) + \sum_{j \neq k} \frac{d\varphi_j(u, 0)}{du} \frac{d\varphi_k(u, 0)}{du} \prod_{l \neq j,k} \varphi_l(u, 0) \right]. \end{aligned} \tag{3.13}$$

Now noting that each of the characteristic functions is continuous for all real u and equal to unity at the origin, we restrict the values of u to a suitably chosen neighborhood $|u| < \delta$ $(\delta > 0)$ of the origin, such that each of the factors occurring in the product $\prod_{j=1}^{n} \varphi_j(u, 0)$ is different from zero, and then divide both sides of (3.12) and (3.13) by $\prod_{j=1}^{n} \varphi_j(u, 0)$, thus obtaining

$$\sum_{j=1}^{n} \beta_j' \frac{d\varphi_j(u, 0)}{du} \Big/ \varphi_j(u, 0) = \beta \sum_{j=1}^{n} \frac{d\varphi_j(u, 0)}{du} \Big/ \varphi_j(u, 0), \tag{3.14}$$

$$\begin{aligned} -\sum_{j=1}^{n} \sigma_{j0}'^2 &+ \sum_{j=1}^{n} \beta_j'^2 \frac{d^2\varphi_j(u, 0)}{du^2} \Big/ \varphi_j(u,0) \\ &+ \sum_{j \neq k} \beta_j' \beta_k' \left\{ \frac{d\varphi_j(u, 0)}{du} \Big/ \varphi_j(u, 0) \right\} \left\{ \frac{d\varphi_k(u, 0)}{du} \Big/ \varphi_k(u, 0) \right\} \\ &= -\sigma_0^2 + \beta^2 \left[\sum_{j=1}^{n} \frac{d^2\varphi_j(u, 0)}{du^2} \Big/ \varphi_j(u, 0) \right. \\ &\left. + \sum_{j \neq k} \left\{ \frac{d\varphi_j(u, 0)}{du} \Big/ \varphi_j(u,0) \right\} \left\{ \frac{d\varphi_k(u, 0)}{du} \Big/ \varphi_k(u,0) \right\} \right] \end{aligned} \tag{3.15}$$

Then using the transformation $\psi_j(u) = \ln \varphi_j(u, 0)$ for $j = 1, 2, \cdots, n$ and expressing (3.14) and (3.15) in terms of the derivatives of $\psi_j(u)$, we have

$$\sum_{j=1}^{n} \beta_j' \frac{d\psi_j}{du} = \beta \sum_{j=1}^{n} \frac{d\psi_j}{du}, \tag{3.16}$$

$$\begin{aligned} -\sum_{j=1}^{n} \sigma_{j0}'^2 + \sum_{j=1}^{n} \beta_j'^2 \frac{d^2\psi_j}{du^2} + \left(\sum_{j=1}^{n} \beta_j' \frac{d\psi_j}{du}\right)^2 \\ = -\sigma_0^2 + \beta^2 \left[\sum_{j=1}^{n} \frac{d^2\psi_j}{du^2} + \left(\sum_{j=1}^{n} \frac{d\psi_j}{du}\right)^2\right]. \end{aligned} \tag{3.17}$$

Using (3.16), (3.17) further simplifies to

$$-\sum_{j=1}^{n} \sigma_{j0}'^2 + \sum_{j=1}^{n} \beta_j'^2 \frac{d^2\psi_j}{du^2} = -\sigma_0^2 + \beta^2 \sum_{j=1}^{n} \frac{d^2\psi_j}{du^2}. \tag{3.18}$$

Again differentiating both sides of (3.16) once more with respect to u, we get

$$\sum_{j=1}^{n} \beta_j' \frac{d^2\psi_j}{du^2} = \beta \sum_{j=1}^{n} \frac{d^2\psi_j}{du^2}. \tag{3.19}$$

Then using (3.18) and (3.19) together, we have

$$\begin{aligned} \sum_{j=1}^{n} (\beta_j' - \beta)^2 \frac{d^2\psi_j}{du^2} &= \sum_{j=1}^{n} \beta_j'^2 \frac{d^2\psi_j}{du^2} - 2\beta \sum_{j=1}^{n} \beta_j' \frac{d^2\psi_j}{du^2} + \beta^2 \sum_{j=1}^{n} \frac{d^2\psi_j}{du^2} \\ &= \sum_{j=1}^{n} \beta_j'^2 \frac{d^2\psi_j}{du^2} - \beta^2 \sum_{j=1}^{n} \frac{d^2\psi_j}{du^2} \\ &= -\left(\sigma_0^2 - \sum_{j=1}^{n} \sigma_{j0}'^2\right) = C \qquad \text{(say)}. \end{aligned} \tag{3.20}$$

Finally integrating Eq. (3.20) with respect to u, we get

$$\prod_{j=1}^{n} \{\varphi_j(u, 0)\}^{(\beta_j'-\beta)^2} = e^{Q(u)}, \tag{3.21}$$

which holds for all u in the interval $|u| < \delta$ ($\delta > 0$) where $Q(u)$ is a quadratic polynomial in u.

Then using the theorem of Linnik (Lemma 2.3), it follows at once from (3.21) that each ξ_j, for which $\beta_j' \neq \beta$, is normally distributed. Thus each x_j, for which $(b_j\beta_j)/a_j \neq \beta$, is normally distributed.

Sufficiency: Without any loss of generality, we assume that for the first r pairs ($r \leqq n$) of the random variables (x_j, y_j), the relation $(b_j\beta_j)/a_j \neq \beta$ is satisfied so that for the remaining $n - r$ pairs, $b_j\beta_j/a_j = \beta$. Then from the conditions given in Theorem 3.2, the distribution of each x_j is normal for $j = 1, 2, \cdots, r$ and arbitrary for $j = r + 1, \cdots, n$. That is, putting $\xi_j = a_jx_j$ and $\eta_j = b_jy_j$, we get the distribution of ξ_j as normal for $j = 1, 2, \cdots, r$ and arbitrary for $j = r + 1, r + 2, \cdots, n$. We are also given that $\beta_j' \neq \beta$ ($j = 1, 2, \cdots, r$), while $\beta_{r+1}' = \beta_{r+2}' = \cdots = \beta_n' = \beta$.

Let us write $X = X_0 + \xi_{r+1} + \cdots + \xi_n$ and $Y = Y_0 + \eta_{r+1} + \cdots + \eta_n$, where $X_0 = \sum_{j=1}^{r} \xi_j$ and $Y_0 = \sum_{j=1}^{r} \eta_j$.

Let $\Phi_0(u, v)$ and $\Phi_0(u, 0)$ denote the characteristic functions of the distribution of (X_0, Y_0) and the marginal distribution of X_0 respectively and further $\sigma_j'^2$ the variance of ξ_j for $j = 1, 2, \cdots, n$.

Then it can be shown, proceeding exactly in the same way as in (3.4) through (3.7) above, that

$$\left.\frac{\partial\Phi_0(u, v)}{\partial v}\right]_{v=0} \Big/ \Phi_0(u, 0) = -u \sum_{j=1}^{r} \beta_j' \sigma_j'^2; \tag{3.22}$$

$$\left.\frac{\partial^2\Phi_0(u, v)}{\partial v^2}\right]_{v=0} \Big/ \Phi_0(u, 0) = -\sum_{j=1}^{r} \sigma_{j0}'^2 - \sum_{j=1}^{r} \beta_j'^2 \sigma_j'^2 + u^2 \left(\sum_{j=1}^{r} \beta_j' \sigma_j'^2\right)^2; \tag{3.23}$$

$$\frac{d\Phi_0(u, 0)}{du} \Big/ \Phi_0(u, 0) = -u \sum_{j=1}^{r} \sigma_j'^2; \tag{3.24}$$

$$\frac{d^2\Phi_0(u, 0)}{du^2} \Big/ \Phi_0(u, 0) = -\sum_{j=1}^{r} \sigma_j'^2 + u^2 \left(\sum_{j=1}^{r} \sigma_j'^2\right)^2, \tag{3.25}$$

where

$$\Phi_0(u, 0) = \exp\left\{-\tfrac{1}{2}u^2 \sum_{j=1}^{r} \sigma_j'^2\right\}.$$

Then using (3.22) and (3.24) together, we get

$$\left.\frac{\partial\Phi_0(u, v)}{\partial v}\right]_{v=0} = \beta \frac{d\Phi_0(u, 0)}{du}, \tag{3.26}$$

where β is given by

$$\beta = \sum_{j=1}^{r} \beta_j' \sigma_j'^2 \Big/ \sum_{j=1}^{r} \sigma_j'^2 .$$

Again eliminating u^2 from both the equations (3.23) and (3.25) and using the value of β as obtained in (3.26), we have

$$\begin{aligned}\left.\frac{\partial^2\Phi_0(u, v)}{\partial v^2}\right]_{v=0} \Big/ \Phi_0(u, 0) &= -\sum_{j=1}^{r} \sigma_{j0}'^2 - \sum_{j=1}^{r} \beta_j'^2 \sigma_j'^2 \\ &\quad + \beta^2 \left[\frac{d^2\Phi_0(u, 0)}{du^2} \Big/ \Phi_0(u, 0) + \sum_{j=1}^{r} \sigma_j'^2\right],\end{aligned} \tag{3.27}$$

which after a little simplification reduces to

$$\left.\frac{\partial^2\Phi_0(u, v)}{\partial v^2}\right]_{v=0} = -\sigma_0'^2 \Phi_0(u, 0) + \beta^2 \frac{d^2\Phi_0(u, 0)}{du^2}, \tag{3.28}$$

where

$$\sigma_0'^2 = \sum_{j=1}^{r} \sigma_{j0}'^2 + \sum_{j=1}^{r} (\beta_j' - \beta)^2 \sigma_j'^2.$$

Then using Lemma 2.1 to (3.26) and (3.28) it follows easily that the conditional distribution of Y_0 for fixed X_0 is L.R.H. $(\beta, \sigma_0'^2)$, where β and $\sigma_0'^2$ are already defined in (3.26) and (3.28) respectively.

Again since $\beta'_{r+1} = \beta'_{r+2} = \cdots = \beta'_n = \beta$, it follows by applying Theorem 3.1 that the conditional distribution of Y for fixed X is L.R.H. (β, σ_0^2) where β is the same as that defined in (3.26) and

$$\sigma_0^2 = \sum_{j=1}^{n} \sigma_{j0}'^2 + \sum_{j=1}^{r} (\beta_j' - \beta)^2 \sigma_j'^2 .$$

Hence the proof of the theorem.

4. Some Corollaries. We shall now deduce some important corollaries in this section.

COROLLARY 4.1. *Let there exist two linear functions*

$$X = \textstyle\sum_{j=1}^{n} a_j x_j \quad \text{and} \quad Y = \sum_{j=1}^{n} b_j x_j \quad \text{with} \quad a_j b_j \neq 0 \qquad (j = 1, 2, \cdots, n)$$

where $x_1, x_2, \cdots, x_n$ are n independently (but not necessarily identically) distributed proper random variables each having a finite variance σ_j^2 and zero expectation. Then the necessary and sufficient condition for the conditional distribution of Y for fixed X to be L.R.H. (β, σ_0^2) *is that*

(i) *each x_j for which $b_j/a_j \neq \beta$ is normally distributed, while the remaining x_j's have arbitrary distributions*

(ii) $$\beta = \textstyle\sum' a_j b_j \sigma_j^2 / \sum' a_j^2 \sigma_j^2$$

and

$$\sigma_0^2 = \sum{}' \left(\frac{b_j}{a_j} - \beta\right)^2 a_j^2 \sigma_j^2 ,$$

the summation extending over all the indices j such that $b_j/a_j \neq \beta$.

COROLLARY 4.2. *Let (x_j, y_j) $j = 1, 2, \cdots, n$ be n independently (but not necessarily identically) distributed two-dimensional proper random variables each having a finite variance and zero expectation, such that the conditional distribution of y_j for fixed x_j is* L.R.H. (β_j, σ_{j0}^2) *for $j = 1, 2, \cdots, n$. If there exist two linear functions $X = \sum_{j=1}^{n} a_j x_j$ and $Y = \sum_{j=1}^{n} b_j y_j$ with $a_j b_j \neq 0$ $(j = 1, 2, \cdots, n)$ such that they are stochastically independent, then each x_j for which $\beta_j \neq 0$ is normally distributed, while the remaining x_j's and each y_j have arbitrary distributions.*

COROLLARY 4.3. *Let (x_j, y_j) $j = 1, 2$, be two independently (but not necessarily identically) distributed two-dimensional proper random variables each having a finite variance such that the conditional distribution of y_j for fixed x_j is* L.R.H. (β_j, σ_{j0}^2), *$j = 1, 2$. If there exist two linear functions $X = a_1x_1 + a_2x_2$ and $Y = b_1y_1 + b_2y_2$ with $a_j b_j \neq 0$ $(j = 1, 2)$, then the necessary and sufficient condition for the conditional distribution of Y for fixed X to be* L.R.H. (β, σ_0^2) *is that each x_j is normal whenever $b_1\beta_1/a_1 \neq b_2\beta_2/a_2$.*

PROOF.

Necessity: First of all we substitute $\xi_j = a_j x_j$ and $\eta_j = b_j y_j$ $(j = 1, 2)$ and then proceed in exactly the same way as in Theorem 3.2. Then the equation (3.12) reduces to

$$(\beta_1' - \beta) \frac{d\varphi_1(u, 0)}{du} \varphi_2(u, 0) + (\beta_2' - \beta)\varphi_1(u, 0) \frac{d\varphi_2(u, 0)}{du} = 0. \tag{4.1}$$

Now we shall show that under the conditions $\beta_1' \neq \beta_2'$, neither $\beta_1' - \beta$ nor $\beta_2' - \beta$ can be equal to zero. For if $\beta_1' - \beta = 0$, while $\beta_2' - \beta \neq 0$, the equation (4.1) becomes

$$\varphi_1(u, 0) \frac{d\varphi_2(u, 0)}{du} = 0. \tag{4.2}$$

Thus in a suitably chosen neighbourhood $|u| < \delta$ $(\delta > 0)$, of the origin where $\varphi_1(u, 0) \neq 0$, we have

$$\frac{d\varphi_2(u, 0)}{du} = 0, \tag{4.3}$$

thus leading to the conclusion that the distribution of ξ_2 is improper, the whole mass being concentrated at the origin. Similarly if $\beta_2' - \beta = 0$, while $\beta_1' - \beta \neq 0$, it can be shown in an exactly similar manner that the distribution of ξ_1 is improper. Hence the only alternative left is when both $\beta_1' - \beta = 0$ and $\beta_2' - \beta = 0$ simultaneously, but in this case we have $\beta_1' = \beta_2'$, which is contrary to the conditions of the theorem. The rest of the proof is as in Theorem 3.2.

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