ESTIMATES FOR GLOBAL CENTRAL LIMIT THEOREMS[1]

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1. Introduction. Let $\xi_1, \xi_2, \cdots$ be independent random variables having the same d.f. (distribution function) $F(x)$. Thus, for each $k = 1, 2, 3, \cdots$

$$\Pr\{\xi_k \leqq x\} = F(x), \qquad -\infty < x < \infty, \tag{1.1}$$

where $F(x)$ is a real monotone increasing function for which $F(-\infty) = 0$ and $F(\infty) = 1$. Let $\phi(t)$, defined by

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x), \qquad -\infty < t < \infty, \tag{1.2}$$

denote the c.f. (characteristic function) of $F(x)$. We suppose that

$$\int_{-\infty}^{\infty} x\, dF(x) = 0, \qquad \int_{-\infty}^{\infty} x^2\, dF(x) = 1, \tag{1.3}$$

so that $F(x)$ has mean 0 and standard deviation 1.

The d.f. $\tilde{F}_n(x)$ and the c.f. $\tilde{\phi}_n(x)$ of the sum $\xi_1 + \xi_2 + \cdots + \xi_n$ are such that $\tilde{\phi}_n(x) = [\phi(x)]^n$ and hence

$$[\phi(t)]^n = \int_{-\infty}^{\infty} e^{itx}\, d\tilde{F}_n(x). \tag{1.4}$$

The d.f. of the combination

$$\frac{\xi_1 + \xi_2 + \cdots + \xi_n}{n^{1/2}} \tag{1.5}$$

is then $\tilde{F}_n(n^{1/2}x)$ and we denote this by $F_n(x)$. It's c.f. is $[\phi(n^{-1/2}t)]^n$, that is,

$$[\phi(n^{-1/2}t)]^n = \int_{-\infty}^{\infty} e^{itx}\, dF_n(x). \tag{1.51}$$

The hypotheses (1.3) imply that the formulas (1.3) hold when $F(x)$ is replaced by $F_n(x)$. A special case of the central limit theorem asserts that, for each individual x in the interval $-\infty < x < \infty$,

$$\lim_{n\to\infty} F_n(x) = \Phi(x) \tag{1.6}$$

where $\Phi(x)$ is the Gaussian d.f. defined by

$$\Phi(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} e^{-u^2/2}\, du. \tag{1.61}$$

For an exposition of the above facts, see Cramér [2].

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It was recently shown by the author [1] that if $p > \frac{1}{2}$, then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} | F_n(x) - \Phi(x) |^p \, dx = 0. \tag{1.7}$$

For each $p > \frac{1}{2}$, (1.7) is a global version of the central limit theorem which complements the local version (1.6) in which values of x are considered one at a time. In fact, it was shown in [1] that it is possible to pass from one to the other of (1.6) and (1.7) by applications of theorems on convergence of sequences of d.f.'s. However, [1] did not provide a means of calculating the numbers $C_n^{(p)}$ defined by

$$C_n^{(p)} = \int_{-\infty}^{\infty} | F_n(x) - \Phi(x) |^p \, dx \tag{1.8}$$

and of determining the rapidity of the convergence to 0 of $C_n^{(p)}$ as $n \to \infty$.

Section 2 gives optimal inequalities satisfied by d.f.'s having mean 0 and standard deviation 1. Section 3 gives a formula for the constants $C_n^{(p)}$ for the interesting case in which $p = 2$. Section 4 shows that if $F(x)$ is the symmetric binomial d.f., then

$$C_n^{(2)} = \frac{1}{n}\frac{1}{6\pi^{1/2}} + O\left(\frac{1}{n^2}\right). \tag{1.9}$$

Section 5 shows that if $F(x)$ is the d.f. of a random variable uniformly distributed over the interval $-3^{1/2} < x < 3^{1/2}$, then $C_n^{(2)}$ converges to 0 much more rapidly because in this case

$$C_n^{(2)} = \frac{1}{n^2}\frac{3}{1280\pi^{1/2}} + 0\left(\frac{1}{n^3}\right). \tag{1.91}$$

Finally, inequalities are given for appraisal of the constants in (1.91) when n is fixed and not necessarily large.

2. Some optimal inequalities. In order to obtain the formula for $C_n^{(2)}$ given in Section 3 we need estimates of differences of d.f.'s satisfying (1.3). While estimates given in [1] would serve our purpose, it is of interest to know the best estimates and we proceed to derive them. We start with the following known theorem.

THEOREM 2.1. *If $F(x)$ is a d.f. for which*

$$\int_{-\infty}^{\infty} x \, dF(x) = 0; \qquad \int_{-\infty}^{\infty} x^2 \, dF(x) = 1, \tag{2.11}$$

then

$$0 \leqq F(x) \leqq \frac{1}{1 + x^2}, \qquad x \leqq 0, \tag{2.12}$$

and

$$1 - \frac{1}{1 + x^2} \leqq F(x) \leqq 1, \qquad x \geqq 0. \tag{2.13}$$

Moreover the function $(1 + x^2)^{-1}$ *is the least function such that* (2.12) *and* (2.13) *hold whenever* $F(x)$ *is a d.f. satisfying* (2.11).

This theorem gives a special case of a Tchebycheff inequality for bounds of d.f.'s having prescribed moments; for a recent treatment of the subject and for references to literature, see Royden [3]. An unmotivated proof of the theorem can be given in a few lines as follows: Let x_0 be a positive value of x for which $F(x)$ is continuous and let $y_0 = F(x_0)$. For each constant c for which $c \leqq 0$ we obtain, with the aid of (2.11),

$$\begin{aligned} 0 &\leqq \int_{-\infty}^{x_0} (x - c)^2 \, dF(x) \\ &= 1 - \int_{x_0}^{\infty} x^2 \, dF(x) + 2c \int_{x_0}^{\infty} x \, dF(x) + c^2 \int_{-\infty}^{x_0} dF(x) \\ &\leqq 1 - x_0^2(1 - y_0) + 2cx_0(1 - y_0) - c^2 y_0 . \end{aligned} \tag{2.14}$$

Clearly $y_0 > 0$, because if $y_0 = 0$, then $F(x) = 0$ when $x \leqq 0$, and (2.11) is violated. Hence we can put

$$c = -x_0(1 - y_0)/y_0 \tag{2.15}$$

in (2.14), and find that $y_0 \geqq 1 - 1/(1 + x_0)^2$. Thus (2.13) holds wherever $F(x)$ is positive and continuous and hence wherever $x \geqq 0$. To prove (2.12), we apply (2.13) to the d.f. $[1 - F(-x)]$. The last part of the theorem follows from the fact that if $F(x) = 0$ when $x < -x_0^{-1}$,

$$F(x) = 1 - \frac{1}{1 + x_0^2}, \qquad -\frac{1}{x_0} \leqq x < x_0, \tag{2.16}$$

and $F(x) = 1$ when $x \geqq x_0$, then $F(x)$ is a d.f. satisfying (2.11).

THEOREM 2.2. *If* $F(x)$ *and* $G(x)$ *are two d.f.'s for which* (2.11) *and*

$$\int_{-\infty}^{\infty} x \, dG(x) = 0, \qquad \int_{-\infty}^{\infty} x^2 \, dG(x) = 1 \tag{2.21}$$

hold, then

$$| F(x) - G(x) | \leqq \frac{1}{1 + x^2}, \qquad -\infty < x < \infty. \tag{2.22}$$

Moreover the function $(1 + x^2)^{-1}$ *is the least function such that* (2.22) *holds whenever* $F(x)$ *and* $G(x)$ *are d.f.'s satisfying* (2.11) *and* (2.21).

The conclusion (2.22) follows from (2.12), (2.13), and the analogous inequalities obtained by replacing $F(x)$ by $G(x)$. To prove the last part of the theorem, let $x_0 > 0$. Let $\epsilon > 0$. It follows from Theorem 2.1 that there is a d.f. $F(x)$ satisfying (2.11) for which

$$F(x_0) < 1 - \frac{1}{1 + x_0^2} + \frac{\epsilon}{2}. \tag{2.23}$$

If $h > 0$ and $G(x) = 0$ when $x < -(2h)^{-1/2}$, $G(x) = h$ when $-(2h)^{-1/2} \leqq x < 0$, $G(x) = 1 - h$ when $0 \leqq x < (2h)^{-1/2}$, and $G(x) = 1$ when $x \geqq (2h)^{1/2}$, then $G(x)$ is a d.f. satisfying (2.21). By making $0 < h < \epsilon/2$, we obtain

$$G(x_0) > 1 - \epsilon/2. \tag{2.24}$$

Hence

$$| F(x_0) - G(x_0) | > \frac{1}{1 + x_0^2} - \epsilon. \tag{2.25}$$

It follows that (2.22) cannot be improved when $x > 0$, and a slight modification of the argument shows that it cannot be improved when $x = 0$. That (2.22) cannot be improved when $x < 0$ follows from the fact that the inequality

$$| [1 - F(-x)] - [1 - G(-x)] | \leqq (1 + x^2)^{-1} \tag{2.26}$$

cannot be improved when $x > 0$. Thus Theorem 2.2 is proved.

In case $G(x)$ is the Gaussian d.f. $\Phi(x)$, it is possible to replace the right member of (2.22) by a smaller function of x. Theorem 2.1 and the rather crude fact that if $x < 0$, then

$$[(1 + x^2)^{-1} - \Phi(x)] \geqq [\Phi(x) - 0], \tag{2.3}$$

that is,

$$\Phi(x) \leqq \frac{1}{2(1 + x^2)} \tag{2.31}$$

imply that if $F(x)$ satisfies (2.11) then

$$| F(x) - \Phi(x) | \leqq \frac{1}{1 + x^2} - \Phi(x), \qquad x < 0. \tag{2.32}$$

For $x > 0$, the corresponding inequality is

$$| F(x) - \Phi(x) | \leqq \frac{1}{1 + x^2} - [1 - \Phi(x)], \qquad x > 0. \tag{2.33}$$

The right members of (2.32) and (2.33) cannot be replaced by smaller functions of x. If $F(x)$ satisfies (2.11), then (2.32) and (2.33) can be used to show that

$$\int_{-\infty}^{\infty} | F(x) - \Phi(x) | \, dx \leqq 2 \int_{-\infty}^{0} \left[\frac{1}{1 + x^2} - \Phi(x) \right] dx = \pi - (2/\pi)^{1/2} = 2.3437 \cdots . \tag{2.34}$$

It should be expected that the estimate in (2.34) is rather crude, but the last member cannot be reduced below $(2/\pi)^{1/2} = .79788 \cdots$ because if $F_a(x)$ is, for each $a > 1$, the d.f. satisfying (2.11) for which $F_a(x) = 0$ when $x < -a$, $F_a(x) = 1/2a^2$ when $-a \leqq x < 0$, $F_a(x) = 1 - 1/2a^2$ when $0 \leqq x < a$, and $F_a(x) = 1$

when $x \geqq a$, then

$$\lim_{a\to\infty} \int_{-\infty}^{\infty} | F_a(x) - \Phi(x) | \, dx = 2 \int_{-\infty}^{0} \Phi(x) \, dx = \left(\frac{2}{\pi}\right)^{1/2}. \tag{2.35}$$

While discussing Theorem 2.1 with the author, Aryeh Dvoretzky remarked that similar but simpler considerations should produce inequalities better than (2.12) and (2.13) when $F(x)$ is a symmetric d.f. satisfying (2.11). A d.f. $F(x)$ is called symmetric if $F(x) - \frac{1}{2} = \frac{1}{2} - F(-x)$, or $F(x) + F(-x) = 1$, for each x for which $F(x)$ is continuous. A symmetric d.f. $F(x)$ satisfies (2.11) if and only if

$$\int_{0}^{\infty} x^2 \, dF(x) = \tfrac{1}{2}. \tag{2.4}$$

We prove the following theorem.

THEOREM 2.5. *If $F(x)$ is a symmetric d.f. for which*

$$\int_{-\infty}^{\infty} x \, dF(x) = 0, \qquad \int_{-\infty}^{\infty} x^2 \, dF(x) = 1, \tag{2.51}$$

then

$$0 \leqq F(x) \leqq 1/2x^2, \qquad -\infty < x \leqq -1, \tag{2.52}$$

$$0 \leqq F(x) \leqq \tfrac{1}{2}, \qquad -1 \leqq x < 0, \tag{2.53}$$

$$\tfrac{1}{2} \leqq F(x) \leqq 1, \qquad 0 < x \leqq 1, \tag{2.54}$$

$$1 - 1/2x^2 \leqq F(x) \leqq 1, \qquad x \geqq 1. \tag{2.55}$$

Moreover the functions $\frac{1}{2}x^2$ and $\frac{1}{2}$ are optimal functions for which (2.52), (2.53), (2.54), *and* (2.55) *hold wherever $F(x)$ is a symmetric d.f. satisfying* (2.51).

Let x_0 be a positive value of x for which $F(x)$ is continuous. Then $F(x_0) \geqq F(-x_0) = 1 - F(x_0)$ and hence $F(x_0) \geqq \frac{1}{2}$. Thus (2.54) holds, and the lower bound cannot be increased because if $F(x) = 0$ when $x < -1$, $F(x) = \frac{1}{2}$ when $-1 \leqq x < 1$, and $F(x) = 1$ when $x \geqq 1$, then $F(x)$ is a symmetric d.f. satisfying (2.51). We find also that

$$\tfrac{1}{2} \geqq \int_{x_0}^{\infty} x^2 \, dF(x) \geqq \int_{x_0}^{\infty} x_0^2 \, dF(x) = x_0^2[1 - F(x_0)], \tag{2.56}$$

and hence $F(x_0) \geqq 1 - \frac{1}{2}x_0^2$. This implies (2.55). The left member of (2.55) cannot be increased because if $x_0 \geqq 1$ and $F(x) = 0$ when $x < -x_0$, $F(x) = 1/2x_0^2$ when $-x_0 \leqq x < 0$, $F(x) = 1 - 1/2x_0^2$ when $0 \leqq x < x_0$, and $F(x) = 1$ when $x \geqq x_0$, then $F(x)$ is a symmetric d.f. satisfying (2.51). The facts involving (2.52) and (2.53) follow from applying the facts involving (2.54) and (2.55) to the d.f. $[1 - F(-x)]$. Thus Theorem 2.5 is proved.

The next theorem can be obtained with the aid of Theorem 2.5 just as Theorem 2.2 was obtained with the aid of Theorem 2.1.

THEOREM 2.6. *If $F(x)$ and $G(x)$ are symmetric d.f.'s for which* (2.11) *and* (2.21) *hold, then*

$$| F(x) - G(x) | \leqq \tfrac{1}{2}, \qquad | x | \leqq 1, \tag{2.61}$$

and

$$| F(x) - G(x) | \leqq 1/2x^2, \qquad | x | \geqq 1. \tag{2.62}$$

Moreover $\frac{1}{2}$ *and* $1/2x^2$ *are the least functions of* x *such that* (2.61) *and* (2.62) *hold whenever* $F(x)$ *and* $G(x)$ *are symmetric d.f.'s satisfying* (2.11) *and* (2.21).

3. Formulas involving d.f.'s and c.f.'s. We now obtain some formulas in which we can replace $F(x)$ by $F_n(x)$ and $\phi(t)$ by $[\phi(n^{-1/2}t)]^n$. Use of (1.2) and the formula

$$e^{-t^2/2} = \int_{-\infty}^{\infty} e^{itx}\, d\Phi(x) \tag{3.1}$$

gives

$$\begin{aligned} \phi(t) - e^{-t^2/2} &= \int_{-\infty}^{\infty} e^{itx}\, d[F(x) - \Phi(x)] \\ &= - \int_{-\infty}^{\infty} [F(x) - \Phi(x)]\, d_x e^{itx} = -it \int_{-\infty}^{\infty} [F(x) - \Phi(x)] e^{itx}\, dx, \end{aligned} \tag{3.2}$$

the calculations being valid because the integrals exist, $[F(x) - \Phi(x)] \to 0$ as $| x | \to \infty$, and e^{itx} is absolutely continuous. Hence

$$\frac{i}{(2\pi)^{1/2}t} [\phi(t) - e^{-t^2/2}] = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} [F(x) - \Phi(x)] e^{itx}\, dx. \tag{3.21}$$

Since (1.3) holds and (2.21) holds when $G(x) = \Phi(x)$, it follows from Theorem 2.2 that, for each $p > 0$,

$$| F(x) - \Phi(x) |^p \leqq \frac{1}{(1 + x^2)^p}, \qquad -\infty < x < \infty. \tag{3.3}$$

The function in the right member of (3.3) being integrable over $-\infty < x < \infty$ when $p > \frac{1}{2}$, it follows that $[F(x) - \Phi(x)]$ belongs to class L_p for each $p > \frac{1}{2}$. Since $[F(x) - \Phi(x)]$ belongs to class L_2 we can use (3.21) and the inversion formulas for Fourier transforms to obtain

$$F(x) - \Phi(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{i}{(2\pi)^{1/2}t} [\phi(t) - e^{-t^2/2}] e^{-ixt}\, dt. \tag{3.4}$$

Use of either (3.21) or (3.4) and the Parseval formula for Fourier transforms gives

$$\int_{-\infty}^{\infty} | F(x) - \Phi(x) |^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\phi(t) - e^{-t^2/2}}{t} \right| dt. \tag{3.5}$$

Because $[F(x) - \Phi(x)]$ belongs to class L_1, it follows (3.21) that the left member of (3.21) is, when properly defined at $t = 0$, continuous over $-\infty < t < \infty$. Using (3.21) and (3.3) with $p = 1$ gives

$$(3.51) \qquad \left| \frac{\phi(t) - e^{-t^2/2}}{t} \right| \leqq \int_{-\infty}^{\infty} \frac{1}{1 + x^2}\, dx = \pi.$$

Since (1.2) implies that $| \phi(t) | \leqq 1$, we have also

$$(3.52) \qquad \left| \frac{\phi(t) - e^{-t^2/2}}{t} \right| \leqq \frac{2}{|t|}, \qquad t \neq 0.$$

Therefore

$$(3.53) \qquad \left| \frac{\phi(t) - e^{-t^2/2}}{t} \right|^2 \leqq h(t),$$

where $h(t) = \pi^2$ when $| t | \leqq \pi/2$ and $h(t) = 4/| t |^2$ when $| t | \geqq \pi/2$. This shows that the integrand in the right member of (3.5) is dominated by an integrable function independent of the particular d.f.'s and c.f.'s in (3.5). This fact and (3.5) show, without use of other facts relating d.f.'s $F_n(x)$ and their c.f.'s $\phi_n(x)$, that if $\phi_n(t) \to e^{-t^2/2}$ for each t, then

$$(3.6) \qquad \lim_{n\to\infty} \int_{-\infty}^{\infty} | F_n(x) - \Phi(x) |^2\, dx = 0;$$

only the Lebesgue criterion of dominated convergence for taking limits under integral signs is needed to draw the conclusion (3.6). When the conclusion (3.6) has been attained, Theorems 3.2 and 3.1 of [1] become applicable to establish (1.6) and then (1.7) for each $p > \frac{1}{2}$.

In terms of notation of the introduction, use of (3.4) and (3.5) gives

$$(3.7) \qquad F_n(x) - \Phi(x) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{[\phi(n^{-\frac{1}{2}}t)]^n - e^{-t^2/2}}{t}\, e^{-ixt}\, dt$$

and

$$(3.71) \qquad \int_{-\infty}^{\infty} | F_n(x) - \Phi(x) |^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{[\phi(n^{-1/2}t)]^n - e^{-t^2/2}}{t} \right|^2 dt.$$

It is frequently convenient to make a change of the variable of integration in the right member of (3.71) by replacing t by $n^{1/2}t$ so that (3.71) becomes

$$(3.72) \qquad \int_{-\infty}^{\infty} | F_n(x) - \Phi(x) |^2\, dx = \frac{1}{n^{1/2}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{[\phi(t)]^n - [e^{-t^2/2}]^n}{t} \right|^2 dt.$$

Since (1.2) implies that $\phi(-t)$ and $\phi(t)$ are conjugate complex numbers, the integrand in the right member of (3.72) is an even function of t and therefore

$$(3.73) \qquad \int_{-\infty}^{\infty} | F_n(x) - \Phi(x) |^2\, dx = \frac{1}{n^{1/2}} \frac{1}{\pi} \int_{0}^{\infty} \left| \frac{[\phi(t)]^n - [e^{-t^2/2}]^n}{t} \right|^2 dt.$$

Similar modifications of (3.5) and (3.71) can be made.

There are numerous formulas for inverting the Fourier-Stieltjes transformation

(3.8) $$\phi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x),$$

that is, formulas giving $F(x)$ in terms of $\phi(t)$. Some of these formulas are given in the book [2] of Cramér. In cases where it is known that $F(x)$ is a d.f. satisfying (1.3), the formula obtained from (3.4), that is

(3.81) $$F(x) = \Phi(x) + \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{\phi(t) - e^{-t^2/2}}{t} e^{-ixt}\, dt,$$

may be most convenient when one wishes to study the difference between $F(x)$ and $\Phi(x)$. In particular, it may be true that (3.71) or (3.73) is the most fruitful source of information about the left member.

4. The symmetric binomial d.f. It can be expected that, at least for the case $p = 2$, considerable information can be obtained about the constants $C_n^{(p)}$ in (1.8) for special d.f. $F(x)$ and for classes of d.f. $F(x)$ having 3 or more moments satisfying specified conditions. Before more extensive investigations are undertaken, it seems desirable to have rather precise information about the behavior of the constants $C_n^{(2)}$ for the case in which $F(x)$ is the symmetric binomial d.f. usually associated with games of heads and tails. Thus we let $F(x) = 0$ when $x < -1$, $F(x) = \frac{1}{2}$ when $-1 \leqq x < 1$, and $F(x) = 1$ when $x \geqq 1$ and seek information about the constants B_n defined by

(4.1) $$B_n = \int_{-\infty}^{\infty} | F_n(x) - \Phi(x) |^2\, dx.$$

Since $\phi(t) = \cos t$, it follows from (3.73) that

(4.11) $$B_n = \frac{1}{n^{1/2}} \frac{1}{\pi} \int_0^{\infty} | [e^{-t^2/2}]^n - [\cos t]^n |^2 \frac{dt}{t^2}.$$

Therefore

(4.12) $$B_n = B_n^{(1)} + B_n^{(2)} + B_n^{(3)} + B_n^{(4)},$$

where

(4.13) $$B_n^{(1)} = \frac{1}{n^{1/2}} \frac{1}{\pi} \int_{\pi/2}^{\infty} \frac{e^{-nt^2}}{t^2}\, dt,$$

(4.14) $$B_n^{(2)} = \frac{1}{n^{1/2}} \frac{1}{\pi} \int_{\pi/2}^{\infty} \frac{-2(\cos t)^n e^{-nt^2/2}}{t^2}\, dt,$$

(4.15) $$B_n^{(3)} = \frac{1}{n^{1/2}} \frac{1}{\pi} \int_0^{\pi/2} \left| \frac{(e^{-t^2/2})^n - (\cos t)^n}{t} \right|^2 dt,$$

(4.16) $$B_n^{(4)} = \frac{1}{n^{1/2}} \frac{1}{\pi} \int_{\pi/2}^{\infty} \frac{(\cos t)^{2n}}{t^2}\, dt.$$

As we shall show by giving more precise estimates, $B_n^{(1)}$ and $B_n^{(2)}$ approach zero with exponential rapidity, $B_n^{(3)}$ is of order at most $1/n^2$, and $B_n^{(4)}$ is of order $1/n$ so that B_n is of order $1/n$.

We find that

$$\begin{aligned} 0 < B_n^{(1)} &= \frac{1}{n^{3/2}} \frac{1}{2\pi} \int_{\pi/2}^{\infty} \frac{e^{-nt^2}}{t^3}\, 2nt\, dt \\ &< \frac{1}{n^{3/2}} \frac{4}{\pi^4} \int_{\pi/2}^{\infty} e^{-nt^2}\, 2nt\, dt = \frac{1}{n^{3/2}} \frac{4}{\pi^4} e^{-n\pi^2/4}, \end{aligned} \tag{4.21}$$

and similarly

$$| B_n^{(2)} | \leqq \frac{1}{n^{3/2}} \frac{16}{\pi^4} \int_{\pi/2}^{\infty} e^{-nt^2/2} nt\, dt = \frac{1}{n^{3/2}} \frac{16}{\pi^4} e^{-n\pi^2/8}. \tag{4.22}$$

To estimate $B_n^{(3)}$ we use the inequality

$$0 < \frac{e^{-t^2/2} - \cos t}{t^4} < \frac{1}{12}, \qquad 0 < t < \frac{\pi}{2}. \tag{4.23}$$

One way to prove (4.23) is to use the elementary power series expansions of e^x and $\cos x$ to find that the function in (4.23) has a power series expansion $\sum a_{2k}t^{2k}$ which is, when $0 < t < \pi/2$, an alternating series converging to a positive number less than a_0 which is $\frac{1}{12}$. Using (4.23) and setting momentarily $b = e^{-t^2/2}$, $a = \cos t$, we find that $a < b$ and hence

$$0 < b^n - a^n = (b - a) \sum_{k=0}^{n-1} b^{n-1-k}a^k < (b - a) \sum_{k=0}^{n-1} b^{n-1}, \tag{4.24}$$

so that

$$0 < b^n - a^n < n(b - a)b^{n-1}, \tag{4.25}$$

and hence

$$0 < \frac{b^n - a^n}{t} < \frac{nt^3}{12} e^{-(n-1)t^2/2}. \tag{4.26}$$

This and (4.15) imply that

$$B_n^{(3)} < n^{3/2} \frac{1}{144\pi} \int_0^{\pi/2} t^6 e^{-(n-1)t^2}\, dt, \tag{4.27}$$

and hence

$$B_n^{(3)} < n^{3/2} \frac{1}{144\pi} e^{\pi^2/4} \int_0^{\pi/2} t^6 e^{-nt^2}\, dt. \tag{4.28}$$

Making a change of the variable of integration in (4.28) gives

$$B_n^{(3)} < \frac{1}{n^2} \frac{1}{144\pi} e^{\pi^2/4} \int_0^{(\pi/2)n^{1/2}} t^6 e^{-t^2}\, dt. \tag{4.29}$$

Using the standard formula

$$\int_0^\infty t^{2k} e^{-t^2}\, dt = \frac{(2k)!}{2^{2k+1}k!}\, \pi^{1/2}, \tag{4.291}$$

with $k = 3$, gives

$$B_n^{(3)} < \frac{5}{268\pi^{1/2}}\, e^{\pi^2/4}\, \frac{1}{n^2}. \tag{4.292}$$

To estimate $B_n^{(4)}$, we start with (4.16) to obtain

$$B_n^{(4)} = \frac{1}{n^{1/2}} \frac{1}{\pi} \sum_{k=1}^\infty \int_{k\pi-\pi/2}^{k\pi+\pi/2} \frac{(\cos t)^{2n}}{t^2}\, dt = \frac{1}{n^{1/2}} \frac{1}{\pi} \sum_{k=1}^\infty \int_{-\pi/2}^{\pi/2} \frac{(\cos t)^{2n}}{(k\pi + t)^2}\, dt, \tag{4.3}$$

and hence

$$B_n^{(4)} = \frac{1}{n^{1/2}} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} S(t)(\cos t)^{2n}\, dt, \tag{4.31}$$

where

$$S(t) = \sum_{k=1}^\infty \frac{1}{(k\pi + t)^2}. \tag{4.32}$$

But

$$\int_{-\pi/2}^0 S(t)(\cos t)^{2n}\, dt = \int_0^{\pi/2} S(-t)(\cos t)^{2n}\, dt. \tag{4.33}$$

Therefore,

$$B_n^{(4)} = \frac{1}{n^{1/2}} \frac{1}{\pi} \int_0^{\pi/2} S_1(t)(\cos t)^{2n}\, dt, \tag{4.4}$$

where $S_1(t) = S(t) + S(-t)$ and hence

$$S_1(t) = \sum_{k=1}^\infty \left[\frac{1}{(k\pi + t)^2} + \frac{1}{(k\pi - t)^2}\right]. \tag{4.41}$$

The function $S_1(t)$ is an even function which is analytic except for poles at the the points $\pm\pi, \pm 2\pi, \cdots$. Its derivatives are easily calculated from (4.41); the first one is

$$S_1'(t) = 2 \sum_{k=1}^\infty \left[-\frac{1}{(k\pi + t)^3} + \frac{1}{(k\pi - t)^3}\right] = 4t \sum_{k=1}^\infty \frac{3k^2\pi^2 + t^2}{(k^2\pi^2 - t^2)^3}. \tag{4.42}$$

This shows that $S_1'(0) = 0$ and that $S_1'(t) > 0$ when $0 < t < \pi$, so that $S(t)$ is increasing over the interval $0 \leqq t \leqq \pi/2$ in which we are interested. Use of (4.41) and the formulas

$$\sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{6}, \qquad \sum_{k=1}^\infty \frac{1}{(2k-1)^2} = \frac{\pi^2}{8} \tag{4.43}$$

shows that

$$S_1(0) = \tfrac{1}{3}, \qquad S_1\left(\frac{\pi}{2}\right) = 1 - \frac{4}{\pi^2}. \tag{4.44}$$

While we shall not use the fact, we nevertheless pause to remark that $S_1(t)$ is an elementary function. Use of (4.41) gives

$$S_1(t) = \sum_{k=1}^{\infty} \frac{d}{dt}\left[\frac{-1}{k\pi + t} + \frac{1}{k\pi - t}\right] = \frac{d}{dt} 2t \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2 - t^2}. \tag{4.45}$$

But, as is shown in textbooks on series,

$$\sum_{k=1}^{\infty} \frac{1}{k^2\pi^2 - t^2} = \frac{1 - t\cot t}{2t^2} = \frac{\sin t - t\cos t}{2t^2 \sin t} \tag{4.46}$$

when $t \neq 0, \pm\pi, \pm 2\pi, \cdots$. Therefore, for these values of t,

$$S_1(t) = \frac{d}{dt}\left(\frac{1}{t} - \cot t\right) = \frac{1}{\sin^2 t} - \frac{1}{t^2} = \frac{t^2 - \sin^2 t}{t^2 \sin^2 t}. \tag{4.47}$$

Using (4.4) and the fact that $S_1(0) = \frac{1}{3}$, we obtain

$$B_n^{(4)} = B_n^{(5)} + B_n^{(6)}, \tag{4.5}$$

where

$$B_n^{(5)} = \frac{1}{n^{1/2}} \frac{1}{3\pi} \int_0^{\pi/2} (\cos t)^{2n}\, dt \tag{4.51}$$

and

$$B_n^{(6)} = \frac{1}{n^{1/2}} \frac{1}{\pi} \int_0^{\pi/2} [S_1(t) - S_1(0)](\cos t)^{2n}\, dt. \tag{4.52}$$

Since $S_1(t)$ is increasing over $0 < t < \pi/2$, we see that $B_n^{(6)} > 0$ and hence that $B_n^{(4)} > B_n^{(5)}$. The integral in (4.51) is elementary and well known. In fact

$$\begin{aligned} \int_0^{\pi/2} (\cos t)^{2n}\, dt &= \frac{(2n)!}{2^{2n} n!\, n!} \frac{\pi}{2} \\ &= \frac{1}{n^{1/2}} \frac{1}{2} \pi^{1/2} \left[1 - \frac{1}{8n} + 0\left(\frac{1}{n^2}\right)\right]. \end{aligned} \tag{4.53}$$

This and (4.51) give

$$B_n^{(5)} = \frac{1}{n} \frac{1}{6\pi^{1/2}} \left[1 - \frac{1}{8n} + 0\left(\frac{1}{n^2}\right)\right]. \tag{4.54}$$

To estimate $B_n^{(6)}$, we put the integral in (4.52) in the form $u(t)\, dv(t)$ where $u(t) = [S_1(t) - S_1(0)]/\sin t$ and $v(t) = -(\cos t)^{2n+1}/(2n + 1)$. Since $S_1'(0) = 0$, it follows that $u(t) \to 0$ as $t \to 0$. Hence integration by parts gives

$$B_n^{(6)} = \frac{1}{n^{1/2}} \frac{1}{\pi} \frac{1}{2n + 1} \int_0^{\pi/2} u'(t)(\cos t)^{2n+1}\, dt. \tag{4.55}$$

Properties of $S_1(t)$ imply that $u'(t)$ is bounded, say $|u'(t)| < M$, over $0 < t < \pi/2$. Since $|\cos t| \leqq 1$, it follows that

$$0 < B_n^{(6)} < \frac{1}{n^{1/2}} \frac{1}{\pi} \frac{1}{n} M \int_0^{\pi/2} (\cos t)^{2n} \, dt. \tag{4.56}$$

Use of (4.53) gives

$$0 < B_n^{(6)} < \frac{1}{n^2} \frac{1}{\pi^{1/2}} \frac{M}{2} \left[1 - \frac{1}{8n} + 0\left(\frac{1}{n^2}\right)\right]. \tag{4.57}$$

Hence $B_n^{(6)} = 0(n^{-2})$. Since also $B_n^{(1)} = 0(n^{-2})$, $B_n^{(2)} = 0(n^{-2})$, and $B_n^{(3)} = 0(n^{-2})$, it follows from (4.54) that

$$B_n = \frac{1}{n} \frac{1}{6\pi^{1/2}} + 0\left(\frac{1}{n^2}\right). \tag{4.58}$$

This is the result in (1.9) of the introduction.

We conclude this section with two remarks. It is well known that the d.f. $F_n(x)$ has jumps in the neighborhood of $x = 0$ asymptotically equal to $(2/\pi)^{1/2} n^{-1/2}$ and that the least upper bound M_n of $|F_n(x) - \Phi(x)|$ is asymptotically half of these jumps, that is, $(\frac{1}{2}\pi)^{1/2} n^{-1/2}$. The result in (4.58) shows that the set of values of x for which $|F_n(x) - \Phi(x)|$ is near its least upper bound cannot have large measure even when n is large. If we let $0 < \theta < 1$ and let $E(\theta)$ denote the set of values of x for which $|F_n(x) - \Phi(x)| > \theta(\frac{1}{2}\pi)^{1/2} n^{-1/2}$, then we have

$$\begin{aligned} \frac{1}{n} \frac{1}{6\pi^{1/2}} + 0\left(\frac{1}{n^2}\right) &= \int_{-\infty}^{\infty} | F_n(x) - \Phi(x) |^2 \, dx \\ &> \int_{E(\theta)} \frac{\theta^2}{2\pi} \frac{1}{n} \, dx = \frac{\theta^2}{2n\pi} \, | E(\theta) |, \end{aligned} \tag{4.6}$$

where $|E(\theta)|$ is the measure of $E(\theta)$, and hence

$$|E(\theta)| < \pi^{1/2}/3\theta^2 + 0(n^{-1}). \tag{4.61}$$

The average root-mean-square deviation of $F_n(x)$ from $\Phi(x)$ over the interval $-n^{1/2} \leqq x < n^{1/2}$, where $0 < F_n(x) < 1$ is

$$\begin{aligned} &\left[\frac{1}{2n^{1/2}} \int_{-n^{1/2}}^{n^{1/2}} | F_n(x) - \Phi(x) |^2 \, dx\right]^{1/2} \\ &= \frac{1}{2^{1/2} n^{1/4}} \left[\int_{-\infty}^{\infty} | F_n(x) - \Phi(x) |^2 \, dx + 0\left(\frac{1}{n^2}\right)\right]^{1/2} \\ &= \frac{1}{2^{1/2} n^{1/4}} \left[\frac{1}{6\pi^{1/2} n} + 0\left(\frac{1}{n^2}\right)\right]^{1/2} = \frac{1}{(12\pi^{1/2})^{1/2}} \frac{1}{n^{3/4}} \left[1 + 0\left(\frac{1}{n}\right)\right] \\ &= \frac{0\,216831}{n^{3/4}} \left[1 + 0\left(\frac{1}{n}\right)\right]. \end{aligned} \tag{4.7}$$

5. The uniform distribution. Let $U_1, U_2, U_3, \cdots$ be defined by

$$U_n = \int_{-\infty}^{\infty} | F_n(x) - \Phi(x) |^2 \, dx, \tag{5.1}$$

where $F(x)$ is the d.f. of a random variable uniformly distributed over $-a \leqq x \leqq a$. Thus $F(x) = 0$ when $x \leqq -a$, $F(x) = (x + a)/2a$ when $-a \leqq x \leqq a$, and $F(x) = 1$ when $x \geqq 2a$. This d.f. has mean 0, and we assume that $a = 3^{1/2}$ so that the standard deviation is 1. The c.f. is

$$\phi(t) = (\sin at)/at, \tag{5.2}$$

and it follows from (3.73) that

$$U_n = \frac{1}{n^{1/2}} \frac{1}{\pi} \int_0^{\infty} \left| \left(\frac{\sin at}{at} \right)^n - e^{-nt^2/2} \right|^2 \frac{dt}{t^2}. \tag{5.3}$$

To estimate U_n, we let

$$\delta_n = n^{-1/2} \log n \tag{5.31}$$

and set

$$U_n = U_n^{(1)} + U_n^{(2)}, \tag{5.32}$$

where

$$U_n^{(1)} = \frac{1}{n^{1/2}} \frac{1}{\pi} \int_{\delta_n}^{\infty} \left| \left(\frac{\sin at}{at} \right)^n - e^{-nt^2/2} \right|^2 \frac{dt}{t^2}, \tag{5.33}$$

$$U_n^{(2)} = \frac{1}{n^{1/2}} \frac{1}{\pi} \int_0^{\delta_n} \left| \left(\frac{\sin at}{at} \right)^n - e^{-nt^2/2} \right|^2 \frac{dt}{t^2}. \tag{5.34}$$

For each sufficiently large n we have

$$\left| \left(\frac{\sin at}{at} \right) \right| \leqq \frac{\sin a\delta_n}{a\delta_n}, \qquad e^{-nt^2/2} < e^{-n\delta_n/2}, \tag{5.4}$$

when $t \geqq \delta_n$. Hence, for these values of n,

$$\begin{aligned} U_n^{(1)} &\leqq \frac{1}{n^{1/2}} \frac{1}{\pi} \int_{\delta_n}^{\infty} \left| \left(\frac{\sin a\delta_n}{a\delta_n} \right)^n + e^{-n\delta_n^2/2} \right|^2 \frac{dt}{t^2} \\ &\leqq \frac{1}{n^{1/2}} \frac{2}{\pi\delta_n} \left[\left(\frac{\sin a\delta_n}{a\delta_n} \right)^{2n} + e^{-n\delta_n^2} \right]; \end{aligned} \tag{5.41}$$

in the last step we used the inequality $(x + y)^2 \leqq 2(x^2 + y^2)$. This and (5.31) imply that $U_n^{(1)} = 0(n^{-k})$ for each k and in particular that $U_n^{(1)} = 0(n^{-3})$. To estimate $U_n^{(2)}$, we begin by estimating the quantity

$$Q = \left| \left(\frac{\sin at}{at} \right)^n - e^{-nt^2/2} \right| \tag{5.5}$$

in the integrand in (5.34). Since $a = 3^{1/2}$, we have

$$\frac{\sin at}{at} = 1 - \frac{t^2}{2} + \frac{3t^4}{40} + 0(t^6), \tag{5.51}$$

$$\log\frac{\sin at}{at} = -\frac{t^2}{2} - \frac{t^4}{20} + 0(t^6), \tag{5.52}$$

and hence

$$\begin{aligned}\left(\frac{\sin at}{at}\right)^n &= e^{-nt^2/2}e^{-nt^4/20+0(nt^6)} \\ &= e^{-nt^2/2}\left[1 - \frac{nt^4}{20} + 0(nt^6) + 0(n^2t^8)\right].\end{aligned} \tag{5.53}$$

Therefore

$$Q = e^{-nt^2/2}\left[\frac{nt^4}{20} + 0(nt^6) + 0(n^2t^8)\right], \tag{5.54}$$

and

$$(Q/t)^2 = e^{-nt^2}\left[\frac{n^2t^6}{400} + 0(n^2t^8) + 0(n^3t^{10})\right]. \tag{5.55}$$

Hence

$$U_n^{(2)} = U_n^{(3)} + U_n^{(4)}, \tag{5.6}$$

where

$$U_n^{(3)} = \frac{1}{n^{1/2}}\frac{1}{\pi}\int_0^{\delta_n}\frac{n^2t^6}{400}e^{-nt^2}\,dt, \tag{5.61}$$

$$U_n^{(4)} = \frac{1}{n^{1/2}}\frac{1}{\pi}\int_0^{\delta_n}[0(n^2t^8) + 0(n^3t^{10})]e^{-nt^2}\,dt. \tag{5.62}$$

Using (5.31) and making a change of the variable of integration in (5.61) gives

$$U_n^{(3)} = \frac{1}{n^2}\frac{1}{400\pi}\int_0^{\log n}u^6e^{-u^2}\,du, \tag{5.63}$$

and hence

$$U_n^{(3)} = 0(n^{-3}) + \frac{1}{n^2}\frac{1}{400\pi}\int_0^{\infty}u^6e^{-u^2}\,du. \tag{5.64}$$

Using (4.291) with $k = 3$ gives

$$U_n^{(3)} = 0(n^{-3}) + \frac{1}{n^2}\frac{3}{1280\pi^{1/2}}. \tag{5.65}$$

The method used to estimate $U_n^{(3)}$ shows that $U_n^{(4)} = 0(n^{-3})$. This and the fact that $U_n^{(1)} = 0(n^{-3})$ imply that $U_n = 0(n^{-3}) + U_n^{(3)}$ and hence that

(5.66) $$U_n = 0(n^{-3}) + \frac{1}{n^2}\frac{3}{1280\pi^{1/2}}.$$

This is the result given in (1.91) in the introduction. The dominant term in (5.66) has the decimal approximations

(5.67) $$\begin{aligned}\frac{1}{n^2}\frac{3}{1280\pi^{1/2}} &= \frac{0.001322319}{n^2} = \frac{1}{756.2470n^2}\\ &= \left(\frac{0.03634}{n}\right)^2 = \frac{1}{(27.5000n)^2}.\end{aligned}$$

To complement (5.66), it is of interest to have an inequality which gives information about U_n when n has a fixed value, say 10 or 25 or 100. We obtain such an inequality by use of more precise relations between $\phi(t)$ and $e^{-t^2/2}$. Where δ is a positive number to be determined later, we start with (5.3) and put U_n in the form

(5.7) $$U_n = V_n^{(1)} + V_n^{(2)},$$

where

(5.71) $$V_n^{(1)} = \frac{1}{n^{1/2}}\frac{1}{\pi}\int_0^\delta \left|\left(\frac{\sin at}{at}\right)^n - (e^{-t^2/2})^n\right|^2 \frac{dt}{t^2},$$

(5.72) $$V_n^{(2)} = \frac{1}{n^{1/2}}\frac{1}{\pi}\int_\delta^\infty \left|\left(\frac{\sin at}{at}\right)^n - (e^{-t^2/2})^n\right|^2 \frac{dt}{t^2}.$$

In estimating $V_n^{(1)}$ we simplify formulas by setting

(5.73) $$B = e^{-t^2/2}, \qquad A = (\sin at)\,/\,at.$$

Use of the elementary power series expansions of e^x and $\sin x$ and the fact that $a = 3^{1/2}$ gives

(5.74) $$B - A = \sum_{k=2}^{\infty} (-1)^k \left[\frac{1}{2^k k!} - \frac{3^k}{(2k+1)!}\right] t^{2k},$$

and hence

(5.75) $$\frac{B-A}{t^4} = \sum_{k=0}^{\infty} (-1)^k \left[\frac{1}{2^{k+2}(k+2)!} - \frac{3^{k+2}}{(2k+5)!}\right] t^{2k}.$$

Rewriting (5.75) to display the numerical values of the coefficients of the first terms, we have

(5.76) $$\frac{B-A}{t^4} = \frac{1}{20} - \frac{13}{480}t^2 + \frac{1}{420}t^4 - \cdots.$$

We now let

(5.77) $$\delta = (24/13)^{1/2}.$$

When $0 < t < \delta$, the series in (5.75) and (5.76) is a convergent alternating series of the form $u_0 - u_1 + u_2 - \cdots$ in which $u_k > 0$ and $u_0 > u_1 > u_2 > \cdots$. Hence, when $0 < t < \delta$,

$$0 < \frac{1}{20} - \frac{13}{840}t^2 < \frac{B - A}{t^4} < \frac{1}{20}. \tag{5.78}$$

Thus $0 < A < B$ and use of

$$B^n - A^n = (B - A) \sum_{k=0}^{n-1} A^k B^{n-1-k} < n(B - A)B^{n-1} \tag{5.79}$$

gives

$$0 < \frac{B^n - A^n}{t} < n\frac{t^3}{20} e^{-(n-1)t^2/2}. \tag{5.8}$$

Use of (5.71), (5.73), and (5.79) gives

$$V_n^{(1)} < n^{3/2} \frac{1}{400\pi} \int_0^\delta t^6 e^{-(n-1)t^2}\, dt. \tag{5.81}$$

Making a change of the variable of integration in (5.81) gives, when $n > 1$,

$$V_n^{(1)} < \left(\frac{n}{n-1}\right)^{7/2} \frac{1}{n^2} \frac{1}{400\pi} \int_0^{\sqrt{n-1}\,\delta} u^6 e^{-u^2}\, du. \tag{5.82}$$

and use of (4.291) gives

$$V_n^{(1)} < \left(\frac{n}{n-1}\right)^{7/2} \frac{1}{n^2} \frac{3}{1280\pi^{1/2}}. \tag{5.83}$$

Starting with (5.72) we find that

$$V_n^{(2)} \leqq \frac{1}{n^{1/2}} \frac{2}{\pi} \int_\delta^\infty \left[\left(\frac{\sin at}{at}\right)^{2n} + e^{-nt^2}\right] \frac{dt}{t^2}. \tag{5.84}$$

Since $|\sin at| < 1$, $a = 3^{1/2}$, and $\delta > 1$ we obtain

$$V_n^{(2)} < \frac{1}{n^{1/2}} \frac{2}{\pi^\delta} \int_\delta^\infty \left[\frac{t^{-2n-1}}{3^n} + e^{-nt^2} t\right] dt = \frac{1}{n^{3/2}} \frac{1}{\pi^\delta} \left[\frac{1}{(3\delta^2)^n} + \frac{1}{(e^{\delta^2})^n}\right]. \tag{5.85}$$

But $3\delta^2 = 72/13 = 5.54 > 5.5$ and $e^{\delta^2} = e^{24/13} = e^{1.846} = 6.35 > 5.5$. Since $\pi\delta = 4.25 > 4$, it follows that

$$V_n^{(2)} < \frac{1}{n^{3/2}} \frac{1}{2(5.5)^n}. \tag{5.86}$$

From (5.7), (5.83), and (5.86) we obtain

$$U_n < \frac{1}{n^2} \left(\frac{1}{n-1}\right)^{7/2} \frac{3}{1280\pi^{1/2}} + \frac{1}{n^{3/2}} \frac{1}{2(5.5)^n}. \tag{5.9}$$

Even for values of n as small as 5, the second term in the right member of (5.9) is substantially less than the first term. When $n = 10$, the dominant term in (5.66) has the value 0.00001322. The last term in (5.9) is less than 10^{-9} and (5.9) shows that $U_{10} < 0.0000192$, the factor $[n/(n - 1)]^{7/2}$ having the value 1.4519 when $n = 10$. When $n = 100$, the dominant term in (5.66) has the value 0.0000001322 and (5.9) shows that $U_{100} < 0.0000001369$.

REFERENCES

[1] R. P. AGNEW, "Global versions of the central limit theorem," *Proceedings of the National Academy of Sciences (USA)*, Vol. 40 (1954), pp. 800–804.

[2] HARALD CRAMÉR, *Mathematical Methods of Statistics*, Princeton University Press, 1946, 575 pp.

[3] H. L. ROYDEN, "Bounds on a distribution function when its first n moments are given," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 361–376.