where $F_0(z) = 1$, $F_1(z) = z + 2$, and $F_n(z) = (z + 2n) F_{n-1}(z) - zF'_{n-1}(z)$, for $a > 0; x \geqq 0; n = 1, 2, 3 \cdots$. These recurrence relations could be used to compute a table of the distribution function.

4. Moments. The moments are obtainable directly from the expansion of the characteristic function

$$\frac{1}{\left(1 + \frac{t^2}{2\alpha}\right)^\lambda} = 1 - \frac{\lambda}{\alpha}\frac{t^2}{2} + \frac{\lambda(\lambda + 1)}{\alpha^2}\frac{t^4}{2!4} - \frac{\lambda(\lambda + 1)(\lambda + 2)}{\alpha^3}\frac{t^6}{3!8}.$$

We have

$$\mu_1' = 0 \qquad 0 = \mu_3' = \mu_5' = \mu_7' = \cdots$$

$$\mu_2 = \mu_2' = \frac{\lambda}{\alpha}$$

$$\mu_4 = \frac{3\lambda(\lambda + 1)}{\alpha^2}$$

$$\beta_1 = 0, \beta_2 = \frac{\mu_4}{\mu_2^2} = 3\left(1 + \frac{1}{\lambda}\right).$$

As one would expect, the variance of X increases as λ increases. It is interesting to note that β_2 is always greater than 3.

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METRICS AND NORMS ON SPACES OF RANDOM VARIABLES

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1. Introduction and summary. Let $\mathfrak{X}$ be the space of random variables defined on an abstract probability space $(\Omega, \mathfrak{A}, P)$ where we consider any two elements of $\mathfrak{X}$ which are equal a.s. (almost surely) as the same. Fréchet [2] exhibited a metric on $\mathfrak{X}$ (for example, $E[|X - Y|/(1 + |X - Y|)]$) with the property that con-

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vergence in the metric is equivalent to convergence in probability, and he showed that for some probability spaces the same cannot be done for convergence a.s. Dugué [1] showed that it is not in general possible to define a norm on $\mathfrak{X}$ such that convergence in the norm is equivalent to convergence in probability. These results are contained in and completed by the following fact which was stated without proof by the author in [5] and which follows easily from the two theorems stated and proved in this note. There exists a metric (norm) on $\mathfrak{X}$ with convergence in the metric (norm) equivalent to convergence a.s. (in probability) if, and only if, Ω is the union of countable (finite) number of disjoint atoms. After these results were obtained it was found that the equivalence of parts (ii) and (iii) of Theorem 1 had been proved by Marczewski [4], p. 121.

An atom of a probability space is a measurable set A with $P(A) > 0$, such that any measurable subset has probability 0 or $P(A)$. It is easy to show that a random variable is a.s. constant on an atom. f will always designate a real-valued function defined on $\mathfrak{X}$. Convergence in f is said to be equivalent to convergence a.s. (in probability) if, for every sequence $\{X_n\}$ of elements from $\mathfrak{X}$, $f(X_n) \to 0$ if, and only if, $X_n \to 0$ a.s. (in probability).

THEOREM 1. *The following conditions on a probability space are equivalent.*

(i) *There exists a function f, such that convergence in f is equivalent to convergence a.s.*

(ii) *For any sequence $\{X_n\}$ from $\mathfrak{X}$, if $X_n \to 0$ in probability, then $X_n \to 0$ a.s.*

(iii) *Ω is a countable union of disjoint atoms.*

THEOREM 2. *The following conditions on a probability space are equivalent.*

(a) *There exists a function f, such that convergence in f is equivalent to convergence in probability and f satisfies $|f(\alpha X)| = |\alpha| \cdot |f(X)|$ for any $X \in \mathfrak{X}$ and any real number α.*

(b) *Ω is a finite union of disjoint atoms.*

2. Proof of Theorem 1. The following well-known result (see Loève [3], p. 100, Example 7) will be used in the proof.

THEOREM A. *For any probability space, $\Omega = A + \sum_1^\infty A_i$ where all of the sets in the decomposition are disjoint and each A_i is the empty set or an atom, and for every measurable subset B of A, P takes every value between 0 and $P(B)$ for measurable subsets of B.*

(ii) implies (i) by the result of Fréchet.

To show that (i) implies (ii) assume (i) and take any sequence $X_n \to 0$ in probability. If $f(X_n) \not\to 0$ then there exists a subsequence $X_{n'}$, and an $\varepsilon > 0$ such that $|f(X_{n'})| > \epsilon$. But $X_{n'} \to 0$ in probability so that it has a subsequence $X_{n''} \to 0$ a.s. Thus $f(X_{n''}) \to 0$ contradicting $|f(X_{n'})| > \epsilon$. Therefore, $f(X_n)$ must converge to 0, hence, $X_n \to 0$ a.s.

(ii) follows easily from (iii) since a random variable is a.s. constant on an atom.

To prove that (ii) implies (iii), assume that (iii) is false. Thus in the decomposition of Theorem A, $P(A) > 0$ and for each n, $A = \sum_{i=1}^n A_{ni}$ where $P(A_{ni}) = (1/n)P(A)$ for $i = 1, 2, \cdots, n$, and the sets $A_{n1}, A_{n2}, \cdots, A_{nn}$ are disjoint.

Let X_{ni} be the characteristic function of the set A_{ni}. The sequence of random variables

$$X_{11}, X_{21}, X_{22}, X_{31}, \cdots$$

converges to 0 in probability but not a.s. so that (ii) implies (iii), completing the proof.

3. Proof of Theorem 2. To prove that (a) implies (b), assume that (a) is true and (b) is false. From Theorem A there exists a sequence A_n of events with $0 < P(A_n) \to 0$. Let X_n be the characteristic function of the set A_n. For all n, $f(X_n) \neq 0$ because if $f(X_{n_0}) = 0$, then by (a) the sequence of random variables, each of which is X_{n_0}, must converge to 0 in probability, contradicting $P(A_{n_0}) > 0$. By (a), $[f(X_n/f(X_n))] = 1$ for all n, so that the sequence of random variables $X_n/f(X_n)$ cannot converge to 0 in probability. However, it must, because $P(A_n) \to 0$. A contradiction has been reached, hence (a) implies (b).

Assuming (b) it is easy to show that $f(X) = E \mid X \mid$ is a norm on $\mathfrak{X}$ such that convergence in f is equivalent to convergence in probability. Theorem 2 is proved.

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DIVERGENT TIME HOMOGENEOUS BIRTH AND DEATH PROCESSES[1]

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1. Introduction. In a time-homogeneous birth and death process a population is considered, the size of which is given by the random variable $n(t)$ defined on the non-negative integers. If at time t the population size is n, the probability that a birth occurs in the time interval $(t, t + \Delta t)$ is $\lambda_n t + o(\Delta t)$; the probability of a death is $\mu_n t + o(\Delta t)$, and the probability of the occurrence of more than one

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