NOTES

ON BOREL FIELDS OVER FINITE SETS

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1. Summary. It is shown that the number of Borel Fields over a set (S) of n elements is equal to the number of equivalence relations within S. This number is asymptotically equal to

$$(\beta + 1)^{-1/2} \exp \{n(\beta - 1 + \beta^{-1}) - 1\} \qquad \text{where} \qquad \beta \exp \beta = n.$$

2. Enumeration of Borel Fields over a finite set. Borel Fields are usually (e.g. Wald [8]) defined over a set of non-enumerably infinite elements: with quite trivial changes, the definition is applicable to finite sets, as follows:

Let $A, B, C, \cdots$ denote distinct subsets of a set S of n elements. $\mathfrak{B} = \{A, B, C, \cdots\}$ is called a Borel Field (BF) if and only if

(i) $\mathfrak{B}$ is not empty;

(ii) $A \varepsilon \mathfrak{B}$, $B \varepsilon \mathfrak{B}$ imply

$A \cap B \varepsilon \mathfrak{B}$, $A \cup B \varepsilon \mathfrak{B}$, $S - A \varepsilon \mathfrak{B}$.

It follows from the definition that a BF contains at least the empty set ($\varnothing$) and S, and is closed with respect to the formation of unions, intersections, and complements.

To enumerate the BF's, consider the subset $\mathfrak{P}$ consisting of all $P_m \varepsilon \mathfrak{B} (m = 1, 2, \cdots, r$; for some $r = 1, 2, \cdots, n)$ such that

$$P \neq \varnothing, \tag{1}$$

$$A \neq \varnothing, \quad A \neq P, \quad A \varepsilon \mathfrak{B} \quad \text{implies} \quad A \not\subset P; \tag{2}$$

in others words, no P contains an element of $\mathfrak{B}$ as a proper subset. It follows that

$$P_m \cap P_{m'} = \varnothing \qquad (\text{for } m \neq m') \tag{3}$$

and

$$\cup_m P_m = S. \tag{4}$$

If (3) were not true, the intersection, itself being an element of the BF and also a proper subset of a P, would involve a contradiction with (2); if (4) were not so, the complement of this union, being an element (other than $\varnothing$) of the BF and therefore not containing a subset of any other P, would itself be a P, namely P_{r+1}, contradicting the definition of $P = \{P_m\}$.

It is obvious that a BF defines a unique $\mathfrak{P}$; conversely a $\mathfrak{P}$ defines a unique BF as follows:

$$\mathfrak{B} = \{\varnothing; P_1, P_2, \cdots, P_r; \tbinom{r}{2} \text{ elements like } P_1 \cup P_2;$$

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$\binom{r}{3}$ elements like $P_1 \cup P_2 \cup P_3$; $\cdots$; $S\}$.

Thus every BF consists of 2^r elements, the number of BF's with 2^r elements being the same as that of $\mathfrak{P}$'s with r elements. This latter, however, is known to be $\Delta^r 0^n/r!$, where $\Delta^r 0^n$ is the leading rth difference of nth powers of the non-negative integers.

It is obvious from the foregoing that the total number of BF's over S is the same as the total number of $\mathfrak{P}$'s, namely

$$\sum_{r=1}^{n} \Delta^r 0^n/r! = G_n, \tag{5}$$

say; it is also equal to the number of equivalence relations within S. It is well known that

$$\sum_{n=0}^{n} z^n G_n/n! = \exp(e^z - 1); \tag{6}$$

in conventional symbolic notation $G_{n+1} = (1 + G)^n$. Bell [2] gives this recurrence relation as well as several realizations of G_n. We give two further simple realizations:

First, (6) shows that G_n is the nth power-moment, around zero, of the Poisson distribution with unit parameter,

$$Pr(X = x) = (ex!)^{-1}, \qquad x = 0, 1, \cdots.$$

Second, (see, for example, Fisher [4]),

$$\Delta^r 0^n/r! = \Sigma \left\{ n! \prod_{\nu=1}^{R} (\nu!)^{k_\nu} k_\nu! \right\}, \tag{7}$$

where summation takes place over all R, ν, k, such that

$$\sum_{\nu=1}^{R} \nu k_\nu = n, \tag{8}$$

$$\sum_{\nu=1}^{R} k_\nu = r. \tag{9}$$

The typical term is the number of ways n elements can be distributed corresponding to the partition of n, symbolically represented by

$$1^{k_1} 2^{k_2} \cdots \nu^{k_\nu} \cdots R^{k_R},$$

with ν, k_ν satisfying (8) and (9). Dropping the restriction due to (9), but keeping that due to (8), the sum becomes G_n.

3. Evaluation of G_n. For $n = 1$ to 20, Epstein [3] tabulates G_n, using (5).[1] He also gives an asymptotic evaluation of G_n, expressed in terms of the function $\Psi(x) = d/dx \log \Gamma(x)$ and the numbers α_n defined through the relation

$$\alpha_n \Psi(\alpha_n + 1) = n.$$

[1] For $n = 21$ to 51 an unpublished table has been prepared by Francis L. Miksa, 613 Spring Street, Aurora, Ill., U.S.A.

We shall give here a more direct asymptotic expression for G_n in terms of elementary functions; it is obtained by evaluating

$$I_n = \oint_C z^{-(n+1)} \exp(e^z)\, dz, \tag{10}$$

where C is a simple contour enclosing the origin of the z-plane. Clearly by (6) and by Cauchy's theorem,

$$G_n = \frac{n!}{2\pi i} I_n . \tag{11}$$

To obtain an asymptotic expression for I_n, we specify C in (10) by $|z| = \beta$, with $\beta = \beta(n)$ defined by

$$\beta e^\beta = n; \tag{12}$$

then C intersects the positive real axis very nearly at a point where the derivative of the integrand vanishes, and the integral can be evaluated by the method of steepest descent. By a modification of Watson's Lemma (see Jeffreys [6]) it can be shown (details are given in the Appendix) that

$$\begin{aligned} G_n &= n! \exp(n\beta^{-1} - 1)\beta^{-1}\{2\pi n(\beta + 1)\}^{-1/2} \\ &\times\{1 - (2\beta^4 + 9\beta^3 + 16\beta^2 + 6\beta + 2)(24n)^{-1}(\beta + 1)^{-3} + 0(\beta^2 n^{-2})\}; \end{aligned} \tag{13}$$

or using Stirling's formula this simplifies to

$$\begin{aligned} G_n &= (\beta + 1)^{-1/2} \exp\{n(\beta - 1 + \beta^{-1}) - 1\} \\ &\times\{1 - \beta^2(2\beta^2 + 7\beta + 10)(24n)^{-1}(\beta + 1)^{-3} + 0(\beta^2 n^{-2})\} \end{aligned} \tag{14}$$

$$= (\beta + 1)^{-1/2} \exp\{n(\beta - 1 + \beta^{-1}) - 1\}\{1 + 0(\beta n^{-1})\}. \tag{15}$$

These are the required asymptotic formulae. It should be mentioned that (15) can also be obtained from Epstein's result, with the help of Stirling's formula; but (14) would require the knowledge of Epstein's second asymptotic term which has not been determined explicitly in his paper.

The following table gives comparative values of log G_{51} as computed from the various asymptotic formulae:

log G_{51} (true value)	111.707033
from (14)	111.707084
from (15)	111.712500
from Epstein	111.706867

The true value was obtained from Miksa's value for G_{51} (l.c. footnote 1).

By a similar method as above it can be shown that for $r < n/\log n$

$$\Delta^r 0^n = r^n \exp\left\{\left(\frac{1}{2}\frac{n}{r} - r\right) e^{-n/r}\right\} \times \left\{1 + 0\left(\frac{1}{n}\right)\right\}. \tag{16}$$

This sharpens Jordan's result [7]

$$\lim_{n=\infty} r^{-n}\Delta^r 0^n = 1,$$

and establishes a connection between (5) and the known formula (see, for example Bell, [2])

$$G_n = e^{-1}\sum_{r=1}^{\infty} r^n/r!.$$

Other asymptotic formulae for $\Delta^r 0^n$ have been obtained previously by Hsu [5] and by Arfwedson [1], the former being valid when $n - r = 0(n^{1/2})$, the latter when $r = Kn$, for any constant $K < 1$.

4. Appendix. From (10) we get (with $z = \beta e^{i\varphi}$)

$$\text{(A1)}\quad \begin{aligned} I_n &= i\int_{-\pi}^{\pi} \beta^{-n} \exp\{-ni\varphi + \exp(\beta e^{i\varphi})\}\, d\varphi \\ &= i\beta^{-n}\exp(e^\beta)\left\{\int_{-\delta}^{+\delta} + \int_{\delta}^{\pi} + \int_{-\pi}^{-\delta} \exp(-ni\varphi + \exp(\beta e^{i\varphi}) - e^\beta)\right\} d\varphi, \end{aligned}$$

where $0 < \delta \leqq \pi$. We can choose

$$\text{(A2)}\qquad \delta = n^{-2/5}.$$

Then we have, for $\delta \leqq \varphi \leqq \pi$,

$$\begin{aligned} |\exp\{-ni\varphi + \exp(\beta e^{i}\varphi) - e^\beta\}| &\leqq \exp(e^{\beta\cos\delta} - e^\beta) \\ &< \exp\{-\tfrac{1}{2}\beta e^\beta(1 - \cos\delta)\} \\ &< \exp\{-cn^{1/5}\}, \end{aligned}$$

for a suitably chosen constant $c > 0$. Hence

$$\text{(A3)}\qquad \left|\int_{\delta}^{\pi}\right| < \pi\exp\{-cn^{1/5}\}$$

and similarly

$$\text{(A4)}\qquad \left|\int_{-\pi}^{-\delta}\right| < \pi\exp\{-cn^{1/5}\}$$

in (A1).

For $-\delta \leqq \varphi \leqq \delta$ the integrand in (A1) can be rewritten

$$\begin{aligned} &\exp\{-ni\varphi + \exp(\beta e^{i\varphi}) - e^\beta\} \\ &= \exp\{-ni\varphi + \exp(\beta + i\beta\varphi - \tfrac{1}{2}\beta\varphi^2 - \tfrac{1}{6}i\beta\varphi^3 + 0(\beta\varphi^4)) - e^\beta\} \\ &= \exp\{-ni\varphi + e^\beta(1 + i\beta\varphi - \tfrac{1}{2}\beta\varphi^2 - \tfrac{1}{6}i\beta\varphi^3 - \tfrac{1}{2}i\beta^2\varphi^3 \\ &\quad - \tfrac{1}{6}i\beta^3\varphi^3 + 0(\beta^4\varphi^4)) - e^\beta\} \end{aligned}$$

$$= \exp\{-\tfrac{1}{2}n\varphi^2(1+\beta)\} \times \{1 - \tfrac{1}{6}in(1+3\beta+\beta^2)\varphi^3$$

$$+ 0(n^2\beta^4\varphi^6 + n\beta^3\varphi^4)\} \tag{A5}$$

by (12), where the 0-notation refers to $n \to \infty$. Use has been made of $n\beta^3\varphi^3$ being small when $|\varphi| \leqq \delta$ and n is large; this follows from (12) and (A2).

The second term in (A5) is an odd function of φ, therefore its integral from $-\delta$ to $+\delta$ vanishes and we get

$$\int_{-\delta}^{\delta} = \int_{-\delta}^{\delta} \exp\{-\tfrac{1}{2}n\varphi^2(1+\beta)\}\, d\varphi$$

$$+ 0\left(\int_{-\infty}^{\infty} (n^2\beta^4\varphi^6 + n\beta^3\varphi^4) \exp\{-\tfrac{1}{2}n\varphi^2(1+\beta)\}\, d\varphi\right) \tag{A6}$$

$$= (\tfrac{1}{2}n(1+\beta))^{-1/2} \int_{-k}^{k} n^{-v^2}\, dv + 0(\beta^{1/2}n^{-3/2}),$$

where $k = \delta(\frac{1}{2}n(1+\beta))^{1/2}$. Now

$$\int_k^{\infty} e^{-v^2}\, dv < \int_k^{\infty} ve^{-v^2}\, dv = \tfrac{1}{2}e^{-k^2} = \tfrac{1}{2}\exp\{-\tfrac{1}{2}n(1+\beta)\delta^2\} < \tfrac{1}{2}\exp(-\tfrac{1}{2}n^{1/5}),$$

and a similar inequality holds for $\int_{-\infty}^{-k} e^{-v^2}\, dv$. Therefore replacement of the limits $\pm k$ by $\pm\infty$ in (A6) causes an error not exceeding $\exp(-\frac{1}{2}n^{1/5})$, and we get

$$\int_{-\delta}^{\delta} = (2\pi/n(1+\beta))^{1/2} + 0(\alpha^{1/2}n^{-3/2})$$

$$= (2\pi\,/\,n(1+\beta))^{1/2}\{1 + 0(\beta/n)\}. \tag{A7}$$

Summarizing (10), (11), (A1), (A3), (A4), and (A7), the leading term of (13) is obtained. The term with $0(\beta/n)$ (and if necessary, any further terms in the asymptotic expansion) can be obtained by carrying further the expansion under (A5).

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