A MOVING SINGLE SERVER PROBLEM

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1. Introduction. An assembly line moving with uniform speed has items for service spaced along it. The single server available moves with the line while serving and against it with infinite velocity while transferring service to the next item in line. The line has a barrier in which the server may be said to be "absorbed" in the sense that service is disabled if the server moves into the barrier. The problem solved here is the following: given that a server with exponentially distributed service time starts service on the first item when it is T time units away from the barrier, what is the probability $p(k, T)$ that it completes k items of service before absorption? This is the same as determining the generating function

$$P(x, T) = \sum_{k=0}^{\infty} p(k, T)x^k. \tag{1}$$

The referee has pointed out to us an identification of this problem with that of finding the number of units of service in a busy period for the usual (stationary) single server. This may be seen as follows.

Take $\tau(t)$ as the distance from the barrier at time t, so that $\tau(0) = T$. Take the spacing between items as an independent random variable with distribution function $B(t)$. Then the graph of $\tau(t)$ as in Fig. 1 consists of lines of unit slope interrupted by jumps having the distribution $B(t)$ and occurring at t-epochs determined by the exponential distribution of service time. The graph ends when $\tau(t) = 0$ for the first time, when service is disabled.

Now consider the queueing system with a single server, Poisson arrivals, and distribution of service times $B(t)$. Take $\tau(t)$ as the waiting time of a *virtual* arrival at time t. Then the graph of $\tau(t)$ for a single busy period of the server is exactly as in Fig. 1 if the first customer served has a service time which is *given* to be T.

Note that one problem is turned into the other by interchanging service and arrival variables.

Busy periods were first considered by E. Borel [2] for the case of constant service time and with main interest in the number served, exactly as here, but with the first customer's service time the same constant as all others. Turning to the length of the busy period, D. G. Kendall [4] generalized Borel's result to arbitrary service time distribution by transforming it into a question concerning a branching process. Kendall's functional equation was carefully derived by L. Takacs [7], who also obtained a similar equation for the generating function for the number served in a busy period (with no condition on the first customer) for

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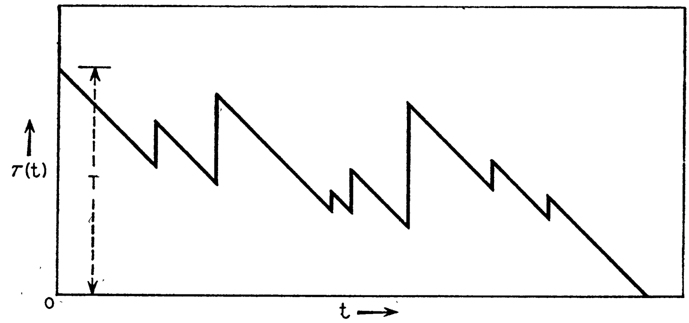

FIG. 1. Sample behavior of random variable $\tau(t)$
$\tau(t)$ = distance from barrier at time t (moving server)
= waiting time of a virtual arrival at time t (queueing system)

arbitrary distribution of service time. All of these are under the usual assumption of Poisson arrivals.

Takacs' result (l.c. Theorem 7, p. 120) in present notation is as follows:

THEOREM (Takacs). *If the generating function of number served in a busy period is*

$$F(x) = x \int_0^\infty P(x, T)\, dB(T), \tag{2}$$

and if arrivals are Poisson with average α *in unit time, then*

$$F(x) = x \int_0^\infty \exp\,[-\alpha t(1 - F(x))]\, dB(t). \tag{3}$$

This suggests that the conditional generating function $P(x, T)$ satisfies

$$P(x, T) = \exp\left\{-\alpha T \left[1 - x \int_0^\infty P(x, T)\, dB(T)\right]\right\}, \tag{4}$$

which deserves an independent derivation. It is clear that it is not a simple consequence of (3), since in the case of constant service time equal to ϵ

$$F(x) = xP(x, \epsilon)$$

and cannot possibly determine $P(x, T)$ for arbitrary T. Nevertheless Eq. (4) is correct.

Because of this, we retain our original derivation of $P(x, T)$ which is limited to the two extreme cases (of most interest to us), namely (i) constant spacing

$$\begin{aligned} B(t) &= 0, \qquad t \leqq \epsilon, \\ &= 1, \qquad t > \epsilon, \end{aligned} \tag{5}$$

and (ii) random spacing

$$B(t) = 0, \qquad t \leqq 0, \tag{6}$$

$$= 1 - e^{-\beta t}, \qquad t > 0,$$

for both of which we take the service distribution as

$$(7) \qquad A(t) = 0, \qquad t \leqq 0,$$
$$= 1 - e^{-\alpha t}, \qquad t > 0.$$

The average service time is $1/\alpha$.

We show that (4) is true in both cases and obtain explicit expressions for the probabilities $p(k, T)$ and for their moments.

2. Uniform spacing. The probability $p(0, T)$, that service on the first item begun T time units away from the barrier is not completed before absorption, is the probability that the service time is greater than T; hence

$$(8) \qquad p(0, T) = 1 - A(T) = e^{-\alpha T}.$$

For the other probabilities $p(k, T)$, $k = 1, 2, \cdots$, a recurrence may be found as follows. Suppose service on the first item is completed in the interval $t, t + dt$; then service is begun on the second item at a point $t + \epsilon$ time units away from the barrier, and it follows at once that

$$(9) \qquad \begin{aligned} p(k, T) &= \int_0^T p(k - 1, t + \epsilon)\, dA(T - t), \\ &= \int_0^T p(k - 1, t + \epsilon)\alpha e^{-\alpha(T-t)}\, dt. \end{aligned}$$

Then, using Eq. (1), the generating function $P(x, T)$ must satisfy

$$(10) \qquad P(x, T) = e^{-\alpha T} + x \int_0^T P(x, t + \epsilon)\alpha e^{-\alpha(T-t)}\, dt.$$

Suppose that this has a solution $e^{-\lambda T}$, $\lambda \equiv \lambda(x; \alpha, \epsilon)$; then (10) shows that

$$(11) \qquad e^{-\lambda T} - e^{-\alpha T} = \alpha x e^{-\lambda\epsilon}(e^{-\lambda T} - e^{-\alpha T})/(\alpha - \lambda),$$

or

$$(11a) \qquad \alpha - \lambda = \alpha x e^{-\lambda\epsilon}.$$

But this is what (4) becomes when $B(t)$ is given by (5) and $P(x, T) = e^{-\lambda T}$.

Notice that for $x = 0$, $\lambda = \alpha$, as is required by (8). Note also that all probabilities $p(k, T)$, $k = 1, 2, \cdots$ are uniquely determined by (8) and (9), and that $P(x, T)$ is an analytic function for $x < 1$. To determine it rewrite (11a) in the form

$$(\alpha\epsilon - \lambda\epsilon)e^{-(\alpha\epsilon - \lambda\epsilon)} = x\alpha\epsilon e^{-\alpha\epsilon}$$

or, what is the same thing,

$$(12) \qquad ze^{-z} = w, \qquad z = \alpha\epsilon - \lambda\epsilon, \qquad w = x\alpha\epsilon e^{-\alpha\epsilon}.$$

This is an equation familiar in Lagrange series expansions and in fact the expansion for $\exp(zT/\epsilon) = \exp(\alpha T - \lambda T)$ is given by Pólya and Szegö [5] (III Abschnitt, p. 210) in the form

$$\exp(\alpha T - \lambda T) = 1 + \sum_{k=1}^{\infty} \frac{(T/\epsilon)(T/\epsilon + k)^{k-1}}{k!} w^k, \qquad w < e^{-1}$$

or

$$\exp -\lambda T = e^{-\alpha T} + \sum_{k=1}^{\infty} \frac{T(T + k\epsilon)^{k-1}}{k!} (\alpha\epsilon^{-\alpha\epsilon})^k e^{-\alpha T} x^k. \tag{13}$$

Hence

$$p(k, T) = \frac{T(T + k\epsilon)^{k-1}}{k!} (\alpha\epsilon^{-\alpha\epsilon})^k \epsilon^{-\alpha T}, \tag{14}$$

a result which may also be obtained from (9) and mathematical induction.

For the probability $P(1, T)$ of absorption, (12) becomes

$$ze^{-z} = \alpha\epsilon e^{-\alpha\epsilon}. \tag{15}$$

The function $y(x) = xe^{-x}$ of the real variable x is zero for x zero, increases to a maximum at $x = 1$ and decreases monotonically to zero; hence the equation $a - xe^{-x} = 0$ has two real roots for $a < e^{-1}$ and in the present instance, Eq. (11), because probabilities are in question, the smaller is the proper one. For $\alpha\epsilon < 1$, this root is $\alpha\epsilon$ itself, otherwise it is denoted by z_0. Hence

$$\begin{aligned} P(1, T) &= 1, && \alpha\epsilon \leqq 1, \\ &= \exp[-(\alpha - z_0/\epsilon)T], && \alpha\epsilon > 1. \end{aligned} \tag{16}$$

It is interesting to notice that the first of these may be verified as follows. Rewrite (14) as

$$p(k, T) = e^{-\alpha T}(\alpha\epsilon^{-\alpha\epsilon})^k \left[\frac{(T + k\epsilon)^k}{k!} - \epsilon \frac{(T + k\epsilon)^{k-1}}{(k - 1)!}\right]. \tag{14a}$$

Then, by a result given by Jensen [3], namely

$$\sum_{0}^{\infty} e^{-(a+kx)} \frac{(a + kx)^k}{k!} = \frac{1}{1 - x}, \qquad |x| < 1$$

and (14a), it follows that

$$P(1, T) = \frac{1}{1 - \alpha\epsilon} - \frac{\alpha\epsilon}{1 - \alpha\epsilon} = 1, \qquad \alpha\epsilon < 1.$$

Jensen's result may also be used with (14) to show that

$$M(T) = \sum kp(k, T) = \alpha T(1 - \alpha\epsilon)^{-1}, \qquad \alpha\epsilon < 1. \tag{17}$$

For higher moments, two courses are open. First, since

$$P(1 + x, T) = \sum_{0}^{\infty} x^k M_{(k)}(T)/k! = M(x, T) \tag{18}$$

with $M_{(k)}(T)$ the kth factorial moment, it follows from (10) that

$$M(x, T) = e^{-\alpha T} + \alpha(1 + x) \int_0^T M(x, t + \epsilon)e^{-\alpha(T-t)}\, dt. \tag{19}$$

By differentiation

$$\partial M(x, T)/\partial T = \alpha[M(x, T + \epsilon) - M(x, T) + xM(x, T + \epsilon)]; \tag{20}$$

hence, equating powers of x, with a prime denoting a derivative,

$$M'_{(k)}(T) = \alpha M_{(k)}(T + \epsilon) - \alpha M_{(k)}(T) + \alpha k M_{(k-1)}(T + \epsilon), \tag{21}$$

a differential recurrence relation which may be solved step by step, and which is satisfied by $M_{(1)}(T) = M(T)$, where $M(T)$ is given by (17). The next case is

$$M'_{(2)}(T) = \alpha M_{(2)}(T + \epsilon) - \alpha M_{(2)}(T) + 2\alpha^2(T + \epsilon)(1 - \alpha\epsilon)^{-1}$$

and it turns out that

$$M_{(2)}(T) = \alpha T(\alpha\epsilon)(2 - \alpha\epsilon)(1 - \alpha\epsilon)^{-3} + M^2(T).$$

Second, from (12) by Lagrangian inversion (cf [5], l.c. 209)

$$z = w + \frac{2w^2}{2!} + \cdots + (n)^{n-1} \frac{w^n}{n!} + \cdots$$

and

$$\begin{aligned} \exp Tz/\epsilon &= \exp(\alpha T - \lambda T) \\ &= \exp(xuT + 2\epsilon T(xu)^2/2! + \cdots + (n\epsilon)^{n-1}T(xu)^n/n! + \cdots) \\ &= \sum (xu)^n Y_n(y_1, y_2, \cdots, y_n)/n! \\ &= \exp xuY, \quad \text{symbolically}, \qquad \alpha\epsilon < 1, \end{aligned} \tag{22}$$

with $u = \alpha e^{-\alpha\epsilon}$, $Y_n(y_1, y_2, \cdots, y_n)$ a multivariable polynomial introduced by Bell [1], $y_n = (n\epsilon)^{n-1}T$; and in the symbolic abbreviation the usual convention: $Y^k \equiv Y_k(y_1, y_2, \cdots, y_k)$ is followed. (The relation used in the second and third lines of (18) may be regarded as a definition of the Y polynomials).

Then

$$M(x, T) = \exp[(1 + x)uY - \alpha T], \quad \text{symbolically},$$

and again for $\alpha\epsilon < 1$

$$\begin{aligned} M_{(k)}(T) &= e^{-\alpha T}u^k D^k \exp uY, \qquad D = d/du, \\ &= e^{-\alpha T}u^k D^k \exp \alpha T \\ &= Y_k(Tu\alpha_1, Tu^2\alpha_2, \cdots, Tu^k\alpha_k), \qquad \alpha_k = D^k\alpha, \end{aligned} \tag{23}$$

the second line following from $M(0, T) = 1$, the third from the development in [6].

The derivatives α_k are readily calculated; indeed, from the initial values

$$u\alpha_1 = \alpha(1 - \alpha\epsilon)^{-1}, \qquad u^2\alpha_2 = \alpha(\alpha\epsilon)(2 - \alpha\epsilon)(1 - \alpha\epsilon)^{-3}$$

and mathematical induction, it is found that

$$u^k \alpha_k = \alpha(1 - v)^{1-2k} q_k(v), \qquad v = \alpha\epsilon \tag{24}$$

with

$$q_{k+1}(v) = [1 - k + (4k - 2)v - kv^2]q_k(v) + v(1 - v)q'_k(v),$$

the prime indicating a derivative.

It may be noticed that the variance of the number served is given by

$$\begin{aligned} \text{var} &= M_{(2)}(T) + M(T) - M^2(T) \\ &= \alpha T(1 - \alpha\epsilon)^{-3}. \end{aligned}$$

3. Random spacing. As before $p(0, T) = e^{-\alpha T}$, and the other probabilities are obtained by a recurrence derived as follows. Suppose service on the second item is begun when it is in the interval $(S, S + dS)$ in time units away from the barrier; the probability of this event is, with $a = \alpha\beta/(\alpha + \beta)$,

$$\beta\, dS \int_0^T e^{-\beta(S+t-T)} \alpha e^{-\alpha t}\, dt = a(e^{\beta T} - e^{-\alpha T})e^{-\beta S}\, dS, \qquad S > T,$$

and

$$\beta\, dS \int_{T-S}^T e^{-\beta(S+t-T)} \alpha e^{-\alpha t}\, dt = ae^{-\alpha T}(e^{\alpha S} - e^{-\beta S})\, dS, \qquad S < T.$$

Hence, just as with (9)

$$\begin{aligned} p(k, T) = {} & ae^{-\alpha T} \int_0^T (e^{\alpha S} - e^{-\beta S})p(k - 1, S)\, dS \\ & + a(e^{\beta T} - e^{-\alpha T}) \int_T^\infty e^{-\beta S} p(k - 1, S)\, dS, \qquad k > 0. \end{aligned} \tag{25}$$

It may be noticed for verifications that

$$\begin{aligned} p(1, T) &= aTe^{-\alpha T}, \\ p(2, T) &= a^2 T e^{-\alpha T}(\alpha + \beta)^{-1} + \alpha^2 T^2 e^{-\alpha T}/2!. \end{aligned}$$

The probability generating function $P(x, T)$, defined by (1), has the recurrence

$$\begin{aligned} P(x, T) = {} & e^{-\alpha T} + axe^{-\alpha T} \int_0^T (e^{\alpha S} - e^{-\beta S})P(x, S)\, dS \\ & + ax(e^{\beta T} - e^{-\alpha T}) \int_T^\infty e^{-\beta S} P(x, S)\, dS. \end{aligned} \tag{26}$$

Trying an exponential solution

$$P(x, T) = e^{-\lambda T}, \qquad \lambda \equiv \lambda(x; \alpha, \beta)$$

leads to the conditional (quadratic) equation

$$(\alpha - \lambda)(\beta + \lambda) = \alpha\beta x, \tag{27}$$

which again agrees with (4) when $B(t)$ is given by (6) and $P(x, T) = \exp -\lambda T$ as above. The solution of (27) is

$$2\lambda = \alpha - \beta + [(\alpha + \beta)^2 - 4\,\alpha\beta x]^{1/2}.$$

The positive sign must be chosen since it leads to $\lambda(0) = \alpha$ and $P(x, T) \leqq 1$ for $x \leqq 1$. Hence

$$P(x, T) = \exp -\frac{T}{2}\left[\alpha - \beta + \sqrt{(\alpha + \beta)^2 - 4\alpha\beta x}\right]. \tag{28}$$

It follows at once (taking the positive square root) that

$$\begin{aligned} P(1, T) &= 1, & \alpha &\leqq \beta, \\ &= e^{-(\alpha-\beta)T}, & \alpha &\geqq \beta. \end{aligned} \tag{29}$$

The probabilities $p(k, T)$ can be obtained easily from the generating function by noting that its second derivative may be written as

$$[(\alpha + \beta)^2 - 4\alpha\beta x]P''(x, T) = 2\alpha\beta P'(x, T) + (\alpha\beta T)^2 P(x, T). \tag{30}$$

From this follows the recurrence

$$\begin{aligned} &(k + 2)(k + 1)p(k + 2, T) \\ &\qquad = (2k + 2)(2k + 1)(a^2/\alpha\beta)p(k + 1, T) + a^2T^2p(k, T). \end{aligned} \tag{31}$$

For an explicit expression, write

$$p(k, T) = e^{-aT}\sum_{j=0}^{k-1} A_{kj}\frac{(aT)^{k-j}}{(k - j)!}b^j, \qquad b = a^2/\alpha\beta;$$

then the numbers A_{kj} are determined by the generating function recurrence

$$\begin{aligned} (1 - x)A_k(x) &= (1 - x)\sum A_{kj}x^j \\ &= A_{k-1}(x) - \frac{1}{k}\binom{2k - 2}{k - 1}x^k. \end{aligned}$$

Similarly factorial moments are determined from the following relation for the derivatives of $M(x, T) = P(1 + x, T)$:

$$[(\alpha - \beta)^2 - 4\alpha\beta x]M''(x, T) = 2\alpha\beta M'(x, T) + (\alpha\beta T)^2 M(x, T), \tag{32}$$

which leads to the recurrence

$$M_{(k+2)}(T) = (4k + 2)[\alpha\beta/(\alpha - \beta)^2]M_{(k+1)}(T) + [(\alpha\beta T)^2/(\alpha - \beta)^2]M_{(k)}(T). \tag{33}$$

Hence

$$M_{(k)}(T) = k!\sum_{j=0}^{k-1} A_{kj}\frac{(cT)^{k-j}}{(k - j)!}d^j$$

with the numbers A_{kj} as above, and $c = \alpha\beta/(\beta - \alpha)$, $d = \alpha\beta/(\beta - \alpha)^2$.

The mean and variance of the number served are

$$M(T) = \frac{\alpha\beta T}{\beta - \alpha} = \frac{\alpha T}{1 - \alpha/\beta},$$

$$\text{var}\,(T) = \frac{\alpha\beta(\beta^2 - \alpha^2)T}{(\beta - \alpha)^3} = \frac{\alpha T(1 - (\alpha/\beta)^2)}{(1 - \alpha/\beta)^3}.$$

Note the similarity to the corresponding results for uniform spacing.

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