For large values of N the variance, skewness and kurtosis of r are

$$\sigma^2 = \frac{1-\rho^2}{N+2} + O(N^{-2}),$$
$$(21) \qquad \sqrt{\beta_1} = \mu_3/\sigma^3 = o(N^{-1}),$$
$$\beta_2 = \mu_4/\sigma^4 = 3 + o(N^{-1}).$$

These last results are to be expected since it is well known that r has an asymptotic normal distribution.

4. Final remarks. The above results should be adequate, as Leipnik has suggested, for serial correlation problems when $N \geqq 20$. In particular the expressions for the moments of r will be of assistance in evaluating the moments of functions of r; for example, the variance stabilizing transformation $z = \sin^{-1} r$, which will be treated in a future paper.

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ON A DECISION PROCEDURE BASED ON THE TUKEY STATISTIC

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1. Summary. In this paper a decision procedure based on the Tukey Studentized range ([5], [6], [8]) has been shown to be an optimum procedure for a particular type of slippage of means of univariate normal populations based on a common but unknown variance. The method given here is similar to that used by Paulson [2] and Truax [7].

2. Introduction. Let $x_{ij}(i = 1, 2, \cdots, k; j = 1, 2, \cdots, n)$ be the elements of k independent samples of size n from normal populations with means μ_i and variance $\sigma^2(i = 1, 2, \cdots, k)$. Let

$$\bar{x}_i = \textstyle\sum_{j=1}^{n} (x_{ij}/n), \qquad s^2 = \sum_{i=1}^{k} \sum_{j=1}^{n} (x_{ij} - \bar{x}_i)^2/k(n-1),$$

$\bar{x}_{\max} = \max(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k)$ and $\bar{x}_{\min} = \min(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k)$. Let D_{00} denote the decision that the k means are all equal, and let

$$D_{ij}(i \neq j; i, j = 1, 2, \cdots, k)$$

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denote the decision that D_{00} is incorrect and $\mu_i = \mu_{\min}$ and $\mu_j = \mu_{\max}$. We will say that the pair (μ_i, μ_j) has slipped by an amount $\Delta(\Delta > 0)$ if $\mu_1 = \mu_2 = \cdots = \mu_{i-1} = \mu_{i+1} = \cdots = \mu_{j-1} = \mu_{j+1} = \cdots = \mu_k = \mu$ (say) and $\mu_i = \mu - \Delta$ and $\mu_j = \mu + \Delta$. The first formulation of the problem is the following: to find a statistical procedure for selecting one of the decisions $(D_{00}, D_{ij})(i \neq j; i, j = 1, 2, \cdots, k)$ which will maximize the probability of making the correct decision when some pair slipped subject to the following restriction (a) when all the means are equal, D_{00} should be selected with probability $1 - \alpha$ (where α is some small positive quantity fixed in advance of the experiment).

Since the class of possible decision procedures seems to be too large to admit an optimum solution we will impose the following restrictions, (b) the decision procedure must be invariant under location and scale transformations of the variates (c) the decision procedure must be symmetric in the sense that the probability of making the correct decision when the pair (μ_i, μ_j) has slipped by an amount Δ must be the same for all $i, j = 1, 2, \cdots, k$; $i \neq j$. These additional restrictions are rather weak and seem to be reasonable requirements to impose in many practical problems. We will now reformulate the problem as follows: we want a statistical procedure for selecting one of the decisions

$$(D_{00}, D_{ij})(i \neq j; i, j = 1, 2, \cdots, k)$$

which, subject to conditions (a), (b) and (c), will maximize the probability of making the correct decision when one of the pairs has slipped. We shall prove that the optimum solution is the following: if

$$(1) \qquad \bar{x}_i = \bar{x}_{\min}, \qquad \bar{x}_j = \bar{x}_{\max}, \qquad \text{and} \qquad \frac{n(\bar{x}_j - \bar{x}_i)}{[(nk-1)s_0^2]^{1/2}} > q_\alpha,$$

select D_{ij}; if

$$\frac{n(\bar{x}_j - \bar{x}_i)}{[(nk-1)s_0^2]^{1/2}} \leqq q_\alpha,$$

select D_{00}, where q_α is a constant whose value is determined by restriction (a), and $(nk - 1)s_0^2 = \sum_{i=1}^{k} \sum_{j=1}^{n} (x_{ij} - \bar{x})^2$. This statistic has been suggested, on intuitive grounds, by Tukey [8]. Roy and Bose [6] have shown that the statistic can be derived by the union-intersection principle of test construction. Tables of the distribution of q for different values of α, n, and k are available ([1], [3], [4]).

3. Derivation of the optimum procedure. Since $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k, s^2)$ constitute a set of sufficient statistics for the unknown parameters $(\mu_1, \mu_2, \cdots, \mu_k, \sigma^2)$ there is no loss in considering only procedures which depend on this set of statistics. Making use of this in connection with restriction (b) it is easy to see that any allowable decision procedure will depend only on the $k - 1$ statistics $(\bar{x}_i - \bar{x}_k)/s$, $(i = 1, 2, \cdots, k - 1)$. Let $w_i = (\bar{x}_i - \bar{x}_k)/s$, $(i = 1, 2, \cdots, k - 1)$ and let $\lambda_i = (\mu_i - \mu_k)/\sigma$, $(i = 1, 2, \cdots, k - 1)$. The joint distribution of the set $(w_1, w_2, \cdots, w_{k-1})$ depends only on the parameters $(\lambda_1, \lambda_2, \cdots, \lambda_{k-1})$. Let $\bar{D}_{00}$ be the decision that $\lambda_1 = \lambda_2 = \cdots = \lambda_{k-1} = 0$ and let $\bar{D}_{ij}$ be the decision that $\lambda_1 = \lambda_2 = \cdots = \lambda_{i-1} = \lambda_{i+1} = \cdots = \lambda_{j-1} = \lambda_{j+1} = \cdots = \lambda_{k-1} = 0$, $\lambda_i = -\Delta/\sigma$ and $\lambda_j = \Delta/\sigma$, $(i \neq j;\ i, j = 1, 2, \cdots, k-1)$, while $\bar{D}_{ik}$ denotes the decision that $\lambda_1 = \lambda_2 = \cdots = \lambda_{i-1} = \lambda_{i+1} = \cdots = \lambda_{k-1} = -\Delta/\sigma$, $\lambda_i = -2\Delta/\sigma$, $(i = 1, 2, \cdots, k-1)$ and $\bar{D}_{ki}$ denotes the decision that $\lambda_1 = \lambda_2 = \cdots = \lambda_{i-1} = \lambda_{i+1} = \cdots = \lambda_{k-1} = \Delta/\sigma$, $\lambda_i = 2\Delta/\sigma$ $(i = 1, 2, \cdots, k-1)$. Since any allowable decision procedure for selecting one of the set (D_{00}, D_{ij}) $(i \neq j; i, j = 1, 2, \cdots, k)$ must be a function only of $(w_1, w_2, \cdots, w_{k-1})$, it can be transformed into a procedure for selecting one of the set $(\bar{D}_{00}, \bar{D}_{ij})$, $(i \neq j, i, j = 1, 2, \cdots, k)$ by making D_{ij} correspond to $\bar{D}_{ij}$, $(i, j = 0, 1, 2, \cdots, k; i \neq j$ if $i, j > 0)$, i.e. whenever the original decision procedure selects D_{ij}, the transformed decision procedure is to select $\bar{D}_{ij}$. Because of restriction (a), the probability that any transformed allowable decision procedure will select D_{00} when $\lambda_1 = \lambda_2 = \cdots = \lambda_{k-1} = 0$ will be equal to $1 - \alpha$, in addition the probability that any allowable decision procedure will select D_{ij} when the pair (μ_i, μ_j) has slipped is equal to the probability that the transformed procedure select $\bar{D}_{ij}$ when $\bar{D}_{ij}$ is the correct decision, and this last probability must be the same for each (i, j) because of restriction (c).

The proof that (1) is the optimum solution consists mainly in showing that for any Δ and σ there exist a set of nonzero a priori probabilities g_{00}, g_{ij}, $(i \neq j; i, j = 1, 2, \cdots, k)$ which are functions of Δ and σ so that when (1) is transformed in the manner indicated above into a decision procedure for selecting one of $(\bar{D}_{00}, \bar{D}_{ij})$, $(i \neq j; i, j = 1, 2, \cdots, k)$, it will maximize the probability of making the correct decision among the set $(\bar{D}_{00}, \bar{D}_{ij})$, $(i \neq j; i, j = 1, 2, \cdots, k)$ when g_{ij} is the probability that $\bar{D}_{ij}$ is the correct decision $(i, j = 0, 1, 2, \cdots, k, i \neq j$ if $i, j > 0)$.

Assuming this has been demonstrated, it follows easily that (1) must be the optimum solution. For suppose there existed an allowable decision procedure D^*, which for some Δ and σ had a greater probability than (1) of making the correct decision when some pair had slipped. Then D^*, which must only be a function of $(w_1, w_2, \cdots, w_{k-1})$ when transformed in the indicated manner into a decision procedure for selecting one of $(\bar{D}_{00}, \bar{D}_{ij})$, $(i \neq j, i, j = 1, 2, \cdots, k)$ will have greater probability than (1) of making the correct decision among $(\bar{D}_{00}, \bar{D}_{ij})$, $(i \neq j; i, j = 1, 2, \cdots, k)$ with respect to any set of nonzero a priori probabilities, which would be a contradiction.

To show that the required a priori distribution exists, first let $u_i = (\bar{x}_i - \bar{x}_k)/\sigma$, $(i = 1, 2, \cdots, k-1)$ so that $w_i = u_i\sigma/s$.

The joint probability density function $f(w_1, w_2, \cdots, w_{k-1})$ of $w_1, w_2, \cdots, w_{k-1}$ is easily found to be given by

$$f(w_1, w_2, \cdots, w_{k-1}) = C \int_0^\infty t^{n'+k-2} \exp - \frac{1}{2}\left[n't^2 + A \sum_{i=1}^{k-1} (w_i t - \lambda_i)^2 + B \sum_{i \neq j} (w_i t - \lambda_i)(w_j t - \lambda_j) \right] dt, \tag{2}$$

where $n' = k(n - 1)$, $A = n(k - 1/k)$, $B = -n/k$, and C is a constant whose precise value is not needed.

Let $f_{ij} = f(w_1, w_2, \cdots, w_{k-1} \mid \bar{D}_{ij})$, $(i, j = 0, 1, 2, \cdots, k; i \neq j \text{ if } i, j > 0)$ be the joint probability density function of $w_1, w_2, \cdots, w_{k-1}$ when $\bar{D}_{ij}$ is the correct decision. The decision procedure which will maximize the probability of making the correct decision among the set $(\bar{D}_{00}, \bar{D}_{ij})$, $(i \neq j; i, j = 1, 2, \cdots, k)$ when the a priori probability distribution is $(p_{00}, p_{12}, \cdots, p_{k-1k})$, i.e. the Bayes solution with respect to $(p_{00}, p_{12}, \cdots, p_{k-1k})$, is known [9] to be given by the rule: for each i, j $(i, j = 0, 1, 2, \cdots, k; i \neq j \text{ if } i, j > 0)$ select $\bar{D}_{ij}$ for all points in the w space where $p_{ij}f_{ij} = \max (p_{00}f_{00}, p_{12}f_{12}, \cdots, p_{k-1k}f_{k-1k})$. For the problem at hand, this is the unique Bayes solution except possibly for a set of measure zero according to all f_{ij}. Using (2) it is easy to calculate for each i, j the region where $\bar{D}_{ij}$ is selected for the special a priori distribution $p_{00} = 1 - k(k - 1)p$, $p_{12} = \cdots = p_{k-1k} = p$.

It can be easily checked that the Bayes solution is the following procedure: Select $\bar{D}_{ij}(i, j = 1, 2, \cdots, k - 1, i \neq j)$ if

$$(3)\quad \begin{aligned} &(w_j - w_i) > (w_{j'} - w_{i'}) \\ &\quad\cdot (i', j' = 1, 2, \cdots, k - 1, i' \neq j' \quad \text{and} \quad i \neq i', j \neq j' \quad \text{simultaneously}) \\ &(w_j - w_i) > |w_{i'}| \quad \text{and} \quad \frac{(w_j - w_i)(A - B)}{\sqrt{n' + A \sum_{r=1}^{k-1} w_r^2 + B \sum_{r \neq s=1}^{k=1} w_r w_s}} > q; \end{aligned}$$

Select $\bar{D}_{ik}$ $(i = 1, 2, \cdots, k - 1)$ if

$$(4)\quad \begin{aligned} &-w_i > (w_{j'} - w_{i'})(i', j' = 1, 2, \cdots k - 1, i' \neq j') \\ &\quad \text{and} \quad \frac{-w_i(A - B)}{\sqrt{n' + A \sum_{r=1}^{k-1} w_r^2 + B \sum_{r \neq s=1}^{k-1} w_r w_s}} > q; \end{aligned}$$

Select $\bar{D}_{ki}$ $(i = 1, 2, \cdots, k - 1)$ if

$$(5)\quad \begin{aligned} &w_i > (w_{j'} - w_{i'})(i', j' = 1, 2, \cdots, k - 1, i' \neq j') \\ &\quad \text{and} \quad \frac{w_i(A - B)}{\sqrt{n' + A \sum_{r=1}^{k-1} w_r^2 + B \sum_{r \neq s=1}^{k-1} w_r w_s}} > q; \end{aligned}$$

and select $\bar{D}_{00}$ otherwise.

Define the function $F(p)$ by the equation

$$(6)\quad \begin{aligned} F(p) = \int_0^\infty &y^{n'+k-2} \exp\left(\frac{-y^2}{2}\right) \\ &\left\{p \exp\left(\frac{-n\Delta^2}{2}\right) \exp\left(\frac{\Delta y q_\alpha}{\sigma}\right) - [1 - k(k - 1)p]\right\} dy, \end{aligned}$$

where q_α is the constant used in (1).

It is obvious that $F(p)$ is a continuous function of p with $F(0) < 0$ and $F(1/k(k-1)) > 0$. Hence there exists a p^* with $0 < p^* < 1/k(k-1)$ which is a function of Δ/σ so that $F(p^*) = 0$. Once the Bayes solution relative to $[1 - k(k-1)p, p, \cdots, p]$ has been worked out, it is obvious that to get the Bayes solution relative to $[1 - k(k-1)p^*, p^*, \cdots, p^*]$ it is only necessary to replace q by q_α. If we now substitute $w_i = (\bar{x}_i - \bar{x}_k)/s$ and replace A and B by their values, we find after some simplifications that the Baye's solution relative to $[1 - k(k-1)p^*, p^*, \cdots, p^*]$ reduces to (1) when $\bar{D}_{ij}$ is made to correspond to D_{ij}, $(i, j = 0, 1, 2, \cdots, k; i \neq j \text{ if } i, j > 0)$. Since (1) is an allowable procedure, this proves that it is an optimum one.

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ESTIMATES OF THE MEAN AND STANDARD DEVIATION OF A NORMAL POPULATION[1]

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0. Summary. Several simple estimates of the mean and standard deviation of a normal population are discussed. The efficiencies of these estimates are compared to the sample mean and sample standard deviation and to the best linear unbiased estimates. Little efficiency is lost when simple rather than optimum weights are used.

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