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APPROXIMATE MOMENTS FOR THE SERIAL CORRELATION COEFFICIENT

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1. Introduction and summary. The first order Gaussian auto-regressive process (x_t) may be defined by the stochastic difference equation

$$x_t = \rho x_{t-1} + u_t , \tag{1}$$

where the u's are NID(0, 1) and ρ is an unknown parameter. The choice of a statistic as an estimator for ρ depends on the initial conditions imposed on the difference equation (1). The so-called "circular" model is obtained by considering a sample of size N and then assuming that $x_{N+1} = x_1$. An appropriate estimator for ρ in this case is the circular serial correlation coefficient

$$r = \frac{\sum_{t=1}^{N} x_t x_{t+1}}{\sum_{t=1}^{N} x_t^2} \qquad (x_{N+1} = x_1). \tag{2}$$

Leipnik [1] has derived an approximate density function

$$f(t) = \frac{\Gamma\left(\frac{N+2}{2}\right)}{\Gamma\left(\frac{N+1}{2}\right)\Gamma\left(\frac{1}{2}\right)} (1 - 2t\rho + \rho^2)^{-N/2}(1 - t^2)^{(N-1)/2} \tag{3}$$

for the estimator r. Leipnik also evaluated the first two moments of this distribution. In this paper a formula is obtained which gives $E(r^k)$ as a polynomial of degree k in ρ.

2. The general formula for $E(r^k)$. To calculate the moments of r we must evaluate the integral

$$E(r^k) = \int_{-1}^{1} t^k f(t)\, dt. \tag{4}$$

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The direct integration of this function is not obvious; however, it can be evaluated quite easily by means of the Gegenbauer polynomials.

Gegenbauer's function $C_j^n(t)$ for integral values of j is defined to be the coefficient of ρ^j in the expansion of $(1 - 2t\rho + \rho^2)^{-n}$ in powers of ρ (for this and the following results concerning the Gegenbauer functions see Magnus and Oberhettinger [2] pp. 77 and 78).

$$(1 - 2t\rho + \rho^2)^{-n} = \sum_{j=0}^{\infty} C_j^n(t)\rho^j. \tag{5}$$

The Gegenbauer polynomials are orthogonal over the interval $(-1, 1)$ with weight function $(1 - t^2)^{n-1/2}$ and have the general properties of the classical orthogonal polynomials.

One special result which we shall apply is the following. Let $g(t)$ be a continuous function with j continuous derivatives; then

$$\int_{-1}^{1} g(t)(1 - t^2)^{n-1/2} C_j^n(t)\, dt = K(n, j) \int_{-1}^{1} (1 - t^2)^{j+n-1/2} \frac{d^j g(t)}{dt^j}\, dt, \tag{6}$$

where

$$K(n, j) = \frac{\Gamma(2n + j)\Gamma(n + \frac{1}{2})}{\Gamma(2n)\Gamma(n + j + \frac{1}{2})\Gamma(j + 1)2^j}.$$

This result may be verified by applying the "Rodrigues Formula" for $C_j^n(t)$ (see [2], p. 78, line 2) to the left-hand side of (6) and then integrating by parts j times.

Expanding the denominator of (4) in a series (5) we have

$$E(r^k) = \frac{\Gamma\left(\frac{N + 2}{2}\right)}{\Gamma\left(\frac{N + 1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} t^k(1 - t^2)^{(N-1)/2} \left[\sum_{j=0}^{\infty} C_j^{N/2}(t)\rho^j\right] dt. \tag{7}$$

Since, for this problem, $|\rho| < 1$ and $|t| \leqq 1$, (5) may be written as

$$(1 - 2\rho\cos\theta + \rho^2)^{-n} = (1 - \rho e^{-i\theta})^{-n}(1 - \rho e^{i\theta})^{-n} \tag{8}$$

$$= \sum_{j=0}^{\infty} C_j^n(\cos\theta)\rho^j. \tag{8a}$$

Expanding the right-hand side of (8) in powers of h as the product of two binomial series and comparing coefficients of h^j with those in (8a) we have

$$|C_j^n(\cos\theta)| \leqq \binom{-2n}{j}. \tag{9}$$

Hence, by the Weierstrass M-test, the series $\sum C_j^n(\cos\theta)\rho^j = \sum C_j^n(t)\rho^j$ converges uniformly in t.

Since the series converges uniformly we may invert the order of integration and summation in (7). Applying (6) to (7) with $g(t) = t^k$ and $n = N/2$, we have

$$E(r^k) = \sum_{j=0}^{k} K(N/2, j) \frac{\Gamma\left(\frac{N+2}{2}\right)}{\Gamma\left(\frac{N+1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \cdot \int_{-1}^{1} k(k-1)\cdots(k-j+1)t^{k-j}(1-t^2)^{j+(N-1)/2}\,dt. \tag{10}$$

We note that

$$\int_{-1}^{1} t^p(1-t^2)^q\,dt = 0 \qquad \text{for } p \text{ odd,}$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma(q+1)}{\Gamma\left(\frac{p+1}{2}+q+1\right)} \qquad \text{for } p \text{ even.}$$

If we now let $2i = k - j$ ($i = 0, 1, \cdots, [k/2]$), (10) becomes

$$E(r^k) = \sum_{i=0}^{[k/2]} \frac{\Gamma(N+k-2i)\Gamma(k+1)\Gamma(i+\frac{1}{2})\Gamma([N+2]/2)\rho^{k-2i}}{\Gamma(N)\Gamma(k-2i+1)\Gamma(2i+1)\Gamma(\frac{1}{2})\Gamma\left(\frac{N+2}{2}+k-i\right)2^{k-2i}}. \tag{11}$$

Applying the multiplication theorem for the gamma function

$$2^{2p}\Gamma(p+\tfrac{1}{2})\Gamma(p+1) = \Gamma(2p+1)\Gamma(1/2),$$

we find

$$E(r^k) = \sum_{i=0}^{[k/2]} \frac{\Gamma(N+k-2i)\Gamma(k+1)\Gamma\frac{(N+2)}{2}\rho^{k-2i}}{\Gamma(N)\Gamma(k-2i+1)\Gamma\left(\frac{N+2}{2}+k-i\right)2^k\Gamma(i+1)}. \tag{12}$$

The above formula may be simplified by considering separately the cases k even and k odd. Setting $2j = k$, (10) becomes

$$E(r^{2j}) = \sum_{i=0}^{j} \frac{\Gamma(N+2j-2i)\Gamma(2j+1)\Gamma\left(\frac{N+2}{2}\right)\rho^{2j-2i}}{\Gamma(N)\Gamma(2j-2i+1)\Gamma\left(\frac{N+2}{2}+2j-i\right)2^{2j}\Gamma(i+1)}. \tag{13}$$

Setting $p = j - i$ and applying the multiplication theorem again, (13) may be written as[2]

$$E(r^{2j}) = \sum_{p=0}^{j} \frac{\Gamma\left(p+\frac{N}{2}\right)\Gamma\left(p+\frac{N+1}{2}\right)\Gamma(j+\frac{1}{2})\Gamma(j+1)\Gamma\frac{(N+2)}{2}\rho^{2p}}{\Gamma\frac{(N)}{2}\Gamma\frac{(N+1)}{2}\Gamma(p+\frac{1}{2})\Gamma(p+1)\Gamma\left(\frac{N+2}{2}+j+p\right)}, \tag{14}$$

[2] For $p = 0$ the expression in the braces $\{\cdots\}$ in (15) is to be taken as 1.

or

$$E(r^{2j}) = \sum_{p=0}^{j} \frac{(2j)!\{N(N+1)(N+2)\cdots(N+2p-1)\}\rho^{2p}}{2^{j-p}(2p)!(j-p)!(N+2)(N+4)\cdots(N+2j+2p)}. \tag{15}$$

The corresponding results for k odd, $k = 2j + 1$, are

$$E(r^{2j+1}) = \sum_{p=0}^{j} \frac{\Gamma\left(p+\frac{N+1}{2}\right)\Gamma\left(p+\frac{N+2}{2}\right)\Gamma\left(j+\frac{3}{2}\right)\Gamma(j+1)\left(\frac{N+2}{2}\right)\rho^{2p+1}}{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N+1}{2}\right)\Gamma(p+1)\Gamma\left(p+\frac{3}{2}\right)\Gamma\left(\frac{N+2}{2}+p+j+1\right)\Gamma(j-p+1)}, \tag{16}$$

$$E(r^{2j+1}) = \sum_{p=0}^{j} \frac{(2j+1)!N(N+1)(N+2)\cdots(N+2p)\rho^{2p+1}}{2^{j-p}(2p+1)!(j-p)!(N+2)(N+4)\cdots(N+2j+2p+2)}. \tag{17}$$

From (15) and (17) we see that

$$\operatorname*{Lim}_{N=\infty} E(r^k) = \rho^k, \quad \text{for all } k. \tag{18}$$

3. Specific moments of r. Direct substitution in (15) and (17) yields the following:

$$\begin{aligned} E(r) &= \frac{N\rho}{N+2} = \mu, \\ E(r^2) &= \frac{1}{N+2} + \frac{N(N+1)\rho^2}{(N+2)(N+4)}, \\ E(r^3) &= \frac{3N\rho}{(N+2)(N+4)} + \frac{N(N+1)(N+2)\rho^3}{(N+2)(N+4)(N+6)}, \\ E(r^4) &= \frac{3}{(N+2)(N+4)} + \frac{6N(N+1)\rho^2}{(N+2)(N+4)(N+6)} \\ &\quad + \frac{N(N+1)(N+2)(N+3)\rho^4}{(N+2)(N+4)(N+6)(N+8)}. \end{aligned} \tag{19}$$

The first two moments agree with those obtained by Leipnik, who evaluated them by another method

The central moments of r are

$$\begin{aligned} E(r-\mu)^2 &= \frac{1}{N+2} - \frac{N(N-2)\rho^2}{(N+2)(N+4)(N+2)} = \sigma^2, \\ E(r-\mu)^3 &= \frac{1}{(N+2)^2}\left(\frac{-6N\rho}{N+4} + \frac{2N(N-2)(3N-2)\rho^3}{(N+2)(N+4)(N+6)}\right) = \mu_3, \\ E(r-\mu)^4 &= \frac{3}{(N+2)(N+4)}\left[1 - \frac{2N(N^2-8N-4)\rho^3}{(N+2)^2(N+6)}\right. \\ &\quad \left. + \frac{N(N^4-16N^3+40N^2-32N+16)\theta^4}{(N+2)^3(N+6)(N+8)}\right] = \mu_4. \end{aligned} \tag{20}$$

For large values of N the variance, skewness and kurtosis of r are

$$(21)\qquad \begin{aligned} \sigma^2 &= \frac{1-\rho^2}{N+2} + O(N^{-2}), \\ \sqrt{\beta_1} &= \mu_3/\sigma^3 = o(N^{-1}), \\ \beta_2 &= \mu_4/\sigma^4 = 3 + o(N^{-1}). \end{aligned}$$

These last results are to be expected since it is well known that r has an asymptotic normal distribution.

4. Final remarks. The above results should be adequate, as Leipnik has suggested, for serial correlation problems when $N \geqq 20$. In particular the expressions for the moments of r will be of assistance in evaluating the moments of functions of r; for example, the variance stabilizing transformation $z = \sin^{-1} r$, which will be treated in a future paper.

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ON A DECISION PROCEDURE BASED ON THE TUKEY STATISTIC

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1. Summary. In this paper a decision procedure based on the Tukey Studentized range ([5], [6], [8]) has been shown to be an optimum procedure for a particular type of slippage of means of univariate normal populations based on a common but unknown variance. The method given here is similar to that used by Paulson [2] and Truax [7].

2. Introduction. Let $x_{ij}(i = 1, 2, \cdots, k; j = 1, 2, \cdots, n)$ be the elements of k independent samples of size n from normal populations with means μ_i and variance $\sigma^2(i = 1, 2, \cdots, k)$. Let

$$\bar{x}_i = \textstyle\sum_{j=1}^{n} (x_{ij}/n), \qquad s^2 = \sum_{i=1}^{k} \sum_{j=1}^{n} (x_{ij} - \bar{x}_i)^2/k(n-1),$$

$\bar{x}_{\max} = \max(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k)$ and $\bar{x}_{\min} = \min(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k)$. Let D_{00} denote the decision that the k means are all equal, and let

$$D_{ij}(i \neq j; i, j = 1, 2, \cdots, k)$$

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