[3] J. M. HAMMERSLEY, "Percolation processes. II. The connective constant," (To appear, *Proc. Camb. Phil. Soc.*)
[4] J. M. HAMMERSLEY, "Percolation processes, III. Gravity crystals," unpublished.
[5] J. M. HAMMERSLEY, "Percolation processes. V. Upper bounds for the critical probability," unpublished.
[6] E. T. WHITTAKER AND G. N. WATSON, *Modern Analysis*, 4th ed., Cambridge University Press, 1950.

NON-PARAMETRIC UP-AND-DOWN EXPERIMENTATION[1]

BY CYRUS DERMAN

Columbia University

1. Introduction. Let $Y(x)$ be a random variable such that $P(Y(x) = 1) = F(x)$ and $P(Y(x) = 0) = 1 - F(x)$ where $F(x)$ is a distribution function. It is sometimes of interest, as in sensitivity experiments, to estimate a given quantile of $F(x)$ with observations distributed like $Y(x)$ where the choice of x is under control. A procedure for estimating the median was suggested by Dixon and Mood [2]. The validity of their procedure depends on the assumption that $F(x)$ is normal. Robbins and Monro [6] suggested a general scheme which can be used for estimating any quantile and which imposes no parametric assumptions on $F(x)$. Their method does assume, however, that the range of possible experimental values of x is the real line. In practice, this will not be the case. Limitations on the precision of measuring instruments, or natural limitations such as when x is obtained by a counting procedure, will usually restrict the experimental range of x to a set of numbers of the form

$$a + hn(-\infty < a < \infty, h > 0, n = 0, \pm 1, \cdots).$$

In this note we suggest a non-parametric procedure for estimating any quantile of $F(x)$ on the basis of quantal response data when, experimentally, x is restricted to the form $a + hn$.

For convenience we assume $a = 0$, $h = 1$. Suppose we wish to estimate that value of $x = \theta$ such that $F(\theta - 0) \leqq \alpha \leqq F(\theta)$, $\frac{1}{2} \leqq \alpha < 1$. If $0 < \alpha \leqq \frac{1}{2}$ or $a \neq 0$ or $h \neq 1$ the necessary modifications will be apparent. The experimental procedure is as follows: choose x_1 arbitrarily. Recursively, let

$$\begin{aligned} x_n &= x_{n-1} - 1, && \text{with probability } \frac{1}{2\alpha} \text{ if } y_{n-1} = 1, \\ &= x_{n-1} + 1, && \text{with probability } 1 - \frac{1}{2\alpha} \text{ if } y_{n-1} = 1, \\ &= x_{n-1} + 1, && \text{with probability 1 if } y_{n-1} = 0. \end{aligned} \tag{1}$$

Received May 28, 1956; Revised January 29, 1957.

[1] Research supported by the United States Air Force through the Office of Scientific Research of the Air Research and Development Command.

where y_k denotes the zero-or-one response at x_k. The estimate θ_n of θ based on n observations is the *most frequent* value of x, if unique, or the *arithmetic average of the most frequent levels*, if not unique.

We shall prove the following

THEOREM. *If $F(x)$ is strictly increasing for $\theta - 1 \leqq x \leqq \theta + 1$, then*

$$P(\max(| \limsup_{n\to\infty} \theta_n - \theta |, | \liminf_{n\to\infty} \theta_n - \theta |) < 1) = 1.$$

2. Two lemmas.

Let $\{X_n\}(n = 0, 1, \cdots)$ be an irreducible Markov chain with recurrent non-null states and stationary transition probabilities $\{p_{ij}\}$ (see Feller [3] for definitions of terms) such that

$$p_{i,i+1} + p_{i,i-1} = 1 \qquad (i = 0, \pm 1, \cdots). \tag{2}$$

Let v_i $(i = 0, \pm 1, \cdots)$ be the unique solution of the equations

$$\begin{cases} \sum_{i=-\infty}^{\infty} v_i p_{ij} = v_j & (j = 0, \pm 1, \cdots), \\ v_i > 0, & \text{for all } i, \\ \sum_{i=-\infty}^{\infty} v_i = 1. \end{cases} \tag{3}$$

Since $\{X_n\}$ is irreducible and the states are recurrent non-null, the system (3) has such a unique solution. The v_i's play the role of stationary absolute probabilities; i.e., if $P(X_0 = i) = v_j$, then $P(X_n = i) = v_i$ for every n.

LEMMA 1. *If for some $i = b$, $p_{b,b+1} \leqq p_{b,b-1}$, $p_{b,b+1} > p_{b+1,b+2}$ and $p_{i,i+1}$ is non-increasing in i for $i \geqq b + 1$, then $v_b > b_{b+1}$ and v_i is non-increasing in i for $i \geqq b + 1$. Similarly, if for some $i = c$, $p_{c,c-1} \leqq p_{c,c+1}$, $p_{c,c-1} > p_{c-1,c-2}$, and $p_{i,i+1}$ is non-decreasing in i for $i \leqq c - 1$, then $v_c > v_{c-1}$ and v_i is non-decreasing in i for $i \leqq c - 1$.*

Proof. Let $\pi_{ij} = P(X_n = j \text{ for some } n \geqq 1, X_r \neq i \text{ or } j \text{ for } r < n \mid X_0 = i)$. From a result of Harris [5] we know that

$$\frac{v_{i+1}}{v_i} = \frac{\pi_{i,i+1}}{\pi_{i+1,i}}. \tag{4}$$

It is clear however that $\pi_{i,i+1} = p_{i,i+1}$ and $\pi_{i+1,i} = p_{i+1,i}$. Hence, from (4) and by the hypothesis

$$\frac{v_{b+1}}{v_b} = \frac{p_{b,b+1}}{p_{b+1,b}} = \frac{p_{b,b+1}}{1 - p_{b+1,b+2}} < \frac{p_{b,b+1}}{1 - p_{b,b+1}} \leqq 1$$

and thus $v_{b+1} < v_b$. The remainder of the proof follows in the same manner.

Let $N_n(i)$ denote the number of r such that $X_r = i$ for $r \leqq n$. For the truth of the following lemma we need not impose the condition (2).

LEMMA 2. *Let B be the set of states such that $v_{i'} = \max_i \{v_i\}$ for $i' \in B$. Then for every $i' \in B$.*

$$P\left(\lim_{n\to\infty} \frac{N_n(i')}{n} = v_{i'} > \lim_{n\to\infty} \max_{i \notin B} \left\{\frac{N_n(i)}{n}\right\}\right) = 1.$$

Proof. Since $\sum_{i=-\infty}^{\infty} v_i = 1$, there exists a finite set A of states with $B \subset A$ such that $\sum_{i \notin A} v_i < v_{i'}$. From the strong law of large numbers for Markov chains [1], it follows that $P(\lim_{n\to\infty} (N_n(i)/n = v_i) = 1$ for every i and more generally $P(\lim_{n\to\infty} \sum_{i\notin A} (N_n(i)/n = \sum_{i\notin A} v_i) = 1$. Let ϵ be any number such that $0 < \epsilon < v_{i'} - \max (\max_{i\epsilon A-B} \{v_i\}, \sum_{i\notin A} v_i)$ and let E_N denote the event that $(N_n(i') / n > v_{i'} - \epsilon$ for all $n > N$. By the previous remark and since $\{E_N\}$ is a monotone sequence, $\lim_{N\to\infty} P(E_N) = P(\lim_{N\to\infty} E_N) = 1$. Therefore there exists an N_1 such that $P(N_n(i') / n > v_{i'} - \epsilon$ for all $n > N_1) > 1 - \epsilon/3$. Similarly, since A is finite, there exists an N_2 such that $P(\max_{i\epsilon A-B} \{N_n(i) / n\} < v_{i'} - \epsilon$ for all $n > N_2) > 1 - \epsilon/3$ and an N_3 such that $P(\sum_{i\notin A} N_n(i) / n < v_{i'} - \epsilon$ for all $n > N_3) > 1 - \epsilon/3$. Let $N_0 = \max (N_1, N_2, N_3)$. Then it follows that

$$P(N_n(i') / n > v_{i'} - \epsilon$$
$$> \max (\max_{i\epsilon A-B} \{N_n(i) / n\}, \textstyle\sum_{i\notin A} N_n(i) / n) \text{ for all } n > N_0) > 1 - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$P(\lim_{n\to\infty} N_n(i') / n = v_{i'} > \limsup_{n\to\infty} \max_{i\notin B} \{N_n(i) / n\}) = 1.$$

The last assertion implies that $\lim_{n\to\infty} \max_i \{N_n(i) / n\}$ exists. By a similar argument applied to the finite set B_1 of states which have the second largest v_i's it follows that $\limsup_{n\to\infty} \max_{i\notin B} \{N_n(i) / n\}$ can be replaced by $\lim_{n\to\infty} \max_{i\notin B} \{N_n(i)/n\}$. The lemma is proved.

3. Application of lemmas.

Let $\{X_n\}$ be the Markov chain defined by (1); i.e. let $X_n = i$ if $x_n = i$. The transition probabilities are of the form

$$p_{i,i+1} = 1 - \frac{F(i)}{2\alpha},$$

$$p_{i,i-1} = \frac{F(i)}{2\alpha}.$$

The chain is clearly irreducible and the states can be easily shown to be recurrent non-null using a theorem of Harris [5] or a modified version of a theorem of Foster [4]. The numbers $[\theta] + 1$ and $[\theta]$, where $[\theta]$ denotes the largest integer less than or equal to θ, can be taken as b and c of Lemma 1. From Lemma 1 and the condition of strict monotonicity of $F(x)$ for $\theta - 1 \leqq x \leqq \theta + 1$, it is clear that $[\theta]$ or $[\theta] + 1$ or both but no other states belong to B of Lemma 2. Thus, according to Lemma 2, the most frequent state, for n large enough, will be $[\theta] + 1$, $[\theta]$ or both with probability 1. In any case, the difference between θ and $[\theta] + 1$ or $[\theta]$ or the arithmetic average of the two is less than 1. The theorem is therefore proved.

REFERENCES

[1] K. L. Chung, "Contributions to the theory of Markov chains," *Trans. Amer. Math. Soc.*, Vol. 76, No. 3 (1954), pp. 397–419.

[2] W. J. DIXON AND A. M. MOOD, "A method for obtaining and analyzing sensitivity data," *J. Amer. Stat. Assn.*, Vol. 43 (1948), pp. 109–126.
[3] W. FELLER, *Probability Theory and Its Applications*, John Wiley & Sons, Inc., New York, 1950.
[4] F. G. FOSTER, "Markoff chains with an enumerable number of states and a class of cascade processes," *Proc. Cambridge Philos. Soc.*, Vol. 47 (1951), pp. 77–85.
[5] T. E. HARRIS, "First passage and recurrence distributions," *Trans. Amer. Math. Soc.*, Vol. 73, No. 3 (1952), pp. 471–486.
[6] H. ROBBINS AND S. MONRO, "A stochastic approximation method," *Ann. Math. Stat.*, Vol. 22 (1951), pp. 400–407.

APPROXIMATE MOMENTS FOR THE SERIAL CORRELATION COEFFICIENT

BY JOHN S. WHITE[1]

Ball Brothers Co.

1. Introduction and summary. The first order Gaussian auto-regressive process (x_t) may be defined by the stochastic difference equation

$$x_t = \rho x_{t-1} + u_t , \tag{1}$$

where the u's are NID(0, 1) and ρ is an unknown parameter. The choice of a statistic as an estimator for ρ depends on the initial conditions imposed on the difference equation (1). The so-called "circular" model is obtained by considering a sample of size N and then assuming that $x_{N+1} = x_1$. An appropriate estimator for ρ in this case is the circular serial correlation coefficient

$$r = \frac{\sum_{t=1}^{N} x_t x_{t+1}}{\sum_{t=1}^{N} x_t^2} \qquad (x_{N+1} = x_1). \tag{2}$$

Leipnik [1] has derived an approximate density function

$$f(t) = \frac{\Gamma\left(\frac{N+2}{2}\right)}{\Gamma\left(\frac{N+1}{2}\right)\Gamma\left(\frac{1}{2}\right)} (1 - 2t\rho + \rho^2)^{-N/2}(1 - t^2)^{(N-1)/2} \tag{3}$$

for the estimator r. Leipnik also evaluated the first two moments of this distribution. In this paper a formula is obtained which gives $E(r^k)$ as a polynomial of degree k in ρ.

2. The general formula for $E(r^k)$. To calculate the moments of r we must evaluate the integral

$$E(r^k) = \int_{-1}^{1} t^k f(t)\, dt. \tag{4}$$

Received August 13, 1956; revised February 24, 1957.

[1] Now with Aero Division, Minneapolis Honeywell Regulator Company, Minneapolis.