it is known that $E\{[(Q' - EQ') / \sigma_{Q'}]^k\}$ approaches $\mu_k$, the $k$th moment of a standard normal chance variable, for any positive integral $k$. From the discussion above, one might expect the same to hold for $E\{[Q - EQ) / \sigma_Q]^k\}$, and a detailed examination shows that this is so. It is also so for $E\{[(W - EW) / \sigma_W]^k\}$, since the terms in this not given by the corresponding terms with $W$ replaced by $Q$ approach zero in the limit, due to the properties of $\gamma_i$ defined above. This completes the proof.

**4. The asymptotic power of certain tests of fit.** To test the hypothesis that $f(x) = 1$ for $0 \leqq x \leqq 1$, the test that rejects when $V(n) \geqq C_n(\alpha)$ has been suggested, where $C_n(\alpha)$ is a constant depending on the sample size $n$ and on the desired level of significance $\alpha$. Denote $(1/\sqrt{2\pi})\int_v^\infty e^{-(t^2/2)}\, dt$ by $\phi(v)$, and let $k(\alpha)$ denote the value such that $\phi(k(\alpha)) = \alpha$. Then Theorem $A$ shows that for large $n$, $C_n(\alpha)$ is approximately equal to

$$n^{-r+1/2}[\sqrt{n}\Gamma(r + 1) + k(\alpha)\sqrt{\Gamma(2r + 1) - (r^2 + 1)\Gamma^2(r + 1)}],$$

while if the true common density is $f(x)$, then the large-sample power of the test is approximately equal to

$$\phi\left(\frac{n^{r-1/2}C_n(\alpha) - \sqrt{n}\Gamma(r + 1)\displaystyle\int_0^1 f^{1-r}(x)\, dx}{\sqrt{[\Gamma(2r + 1) - 2r\Gamma^2(r + 1)]\displaystyle\int_0^1 f^{1-2r}(x)\, dx - \left[(r - 1)\Gamma(r + 1)\displaystyle\int_0^1 f^{1-r}(x)\, dx\right]^2}}\right).$$

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# THE DISTRIBUTION OF THE NUMBER OF LOCALLY MAXIMAL ELEMENTS IN A RANDOM SAMPLE

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**0. Summary.** The distribution of the number of different locally maximal elements in a random sample is found, where the sampling is from a continuous population of real numbers. This distribution has application in certain nonparametric tests; the problem of finding the distribution may be regarded as identical with the enumeration of permutations according to the number of distinct locally maximal elements.

**1. Introduction.** An ordered sample of $n$ real numbers is drawn at random from a population having a continuous distribution. For a given integer $k$, an element of the sample is called locally maximal if it is the largest of some $k$ consecutive elements of the sample. The distribution of the number of different

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locally maximal elements in a random sample is discussed in the following; this distribution can be used as a basis for certain nonparametric tests in a manner analogous to other order statistics.

Although the problem arose in just such a context, it can, as will be indicated below, be treated as a purely combinatorial problem. The problem is then to enumerate the permutations on $n$ objects according to the number of different locally maximal elements and so belongs to a class of problems similar to those studied by Riordan [1], Sprague [2], Sade [3], e.g., classifications of permutations according to various characteristics, such as rising sequences, falling sequences, readings [cf. Riordan [1]], etc.

**2. Locally maximal elements.** Let $\Omega$ be a population of real numbers with continuous distribution function, and let $0_n = (x_1, x_2, \cdots, x_n)$ be an ordered sample of size $n$ drawn at random from $\Omega$. For a given $k$ let

$$y_i = \max(x_{i+1}, x_{i+2}, \cdots, x_{i+k})\ i = 0, 1, 2, \cdots, n - k.$$

Let

$$z_j = \begin{cases} 1 & \text{if } y_i = x_j \text{ for at least one } i, \\ 0 & \text{otherwise} \end{cases} \qquad j = 1, 2, \cdots, n.$$

If $z_j = 1$, then $x_j$ will be called a *k-maximal element*, or for brevity, a *maximal element*. Note that the probability of a tie, i.e., the event $x_l = x_m$ for $l \neq m$, is zero, and therefore there is no essential ambiguity in the definition of $z_j$. Clearly $z_j$ is itself a random variable, being a function of a random sample. Thus the sequence $z_1, z_2, \cdots, z_n$ is a sequence of random variables (which are neither independent nor identically distributed) associated with the sample $0_n$. Now let $S_n = \sum_{j=1}^{n} z_j$. The problem is to find the distribution of the random variable $S_n$, i.e., the set of numbers $\{p_s\}$,

$$p_s = P[S_n = s], \qquad s = 0, 1, 2, \cdots, n. \tag{1}$$

It is easily seen that the distribution of $S_n$ is independent of the underlying distribution of $\Omega$, and depends only on the order relationships among the numbers $x_1, \cdots, x_n$. It is convenient, therefore, to replace these numbers by the proper permutation of the integers $1, 2, \cdots, n$, i.e., that permutation having the same order relationships as $x_1, x_2, \cdots, x_n$. By symmetry, all permutations of $1, 2, \cdots, n$ have an equal probability of occurrence, so the distribution of $S_n$ may be obtained by finding the number $f_k(n, i)$ of permutations of the first $n$ integers which have exactly $i$ different maximal elements.

**3. Recurrence relationship and generating function for the numbers $f_k(n, i)$.** A recurrence relation for the numbers $\{f_k(n, i)\}$ can be found in the following manner:

Consider all permutations of the first $n + 1$ integers in which the largest element is in the $(m + 1)$st position, i.e., those permutations of the form $a_1, a_2, \cdots, a_m, n + 1, a_{m+2}, \cdots, a_{n+1}$. Certain of these have exactly $i + 1$

different maximal elements. To enumerate such permutations, note that the element $(n + 1)$ is necessarily maximal so that the permutations $(a_1, a_2, \cdots, a_m)$ and $(a_{m+2}, a_{m+3}, \cdots, a_{n+1})$ must between them contribute $i$ different maximal elements. The $m$ integers $a_1, \cdots, a_m$ which appear to the left of $(n + 1)$ can be selected in $\binom{n}{m}$ ways; for any of these choices there are $f_k(m, r)$ permutations of $a_1, a_2, \cdots, a_m$ which have $r$ maximal elements. Similarly, there are $f_k(n - m, i - r)$ permutations of the remaining integers $a_{m+2}, a_{m+3}, \cdots, a_{n+1}$ which have $i - r$ different maximal elements. Thus the total number of permutations of the first $n + 1$ integers which have the largest element in the $(m + 1)$st position and which have $i + 1$ distinct maximal elements is

$$\sum_{r=0}^{i} f_k(m, r) f_k(n - m, i - r) \binom{n}{m}.$$

Summing on $m$, the total number, $f_k(n + 1, i + 1)$, of permutations of the first $n + 1$ integers with $i + 1$ different maximal numbers is given by

$$f_k(n + 1, i + 1) = \sum_{m=0}^{n} \sum_{r=0}^{i} f_k(m, r)\, f_k(n - m, i - r) \binom{n}{m}. \tag{2}$$

with the following boundary conventions:

$$\begin{cases} f_k(n, 0) = n!; & n < k, \\ f_k(n, 0) = 0\ ; & n \geqq k, \\ f_k(n, i) = 0\ ; & i > 0, n < k. \end{cases} \tag{3}$$

Note that, with these conventions, (2) holds for all $n$ whenever $i > 0$ and for values of $n \geqq k - 1$ when $i = 0$; (3) must be used to determine $f_k(n, i)$ for other values of $n$.

Using (2) and (3) the numbers $f_k(n, i)$ can be calculated recursively, and thus the desired distribution in (1) can be found for any fixed $n$ since

$$p_s = P[S_n = s] = \frac{f_k(n, s)}{n!}. \tag{4}$$

Another way of generating the distribution in (4) arises from considering the generating function $v_k(x, y)$ of the numbers $f_k(n, i) / n!$. Let

$$v_k(x, y) = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \frac{f_k(\alpha, \beta)}{\alpha!} x^\alpha y^\beta. \tag{5}$$

From (5), it is obvious that

$$\left.\frac{\partial^n v_k}{\partial x^n}\right|_{x=0} = \sum_{\beta=0}^{\infty} f_k(n, \beta) y^\beta. \tag{6}$$

Equation (6) thus gives the generating function for the numbers $f_k(n, i)$ for fixed $n$, and hence the generating function for the entire distribution in (4) is just

$$\frac{1}{n!}\frac{\partial^n v_k}{\partial x^n}\bigg|_{x=0}.$$

Furthermore, it can be shown from (2), (3), and (5) that $v_k(x, y)$ satisfies the differential equation

$$(7)\qquad \frac{\partial v}{\partial x} = yv^2 + (1 - y)(1 + 2x + 3x^2 + \cdots + (k - 1)x^{k-2}).$$

Also, note that from (2), $v_k(0, y) \equiv 1$, and so

$$\frac{\partial v_k}{\partial x}\bigg|_{x=0} = 1.$$

The generating function in (6) can therefore be found by repeated differentiation of (7). As a check, it can be shown by induction, using (3) and (6) for $y = 1$, that $\sum_{\beta=0}^{\infty} f_k(n, \beta) = n!$, as is of course necessary. Similarly one may show that for $n \geqq k$ the mean of $S_n$ is given by

$$E(S_n) = \sum_{\beta=0}^{\infty} \frac{\beta f_k(n, \beta)}{n!} = \frac{1}{n!}\left[\frac{\partial}{\partial y}\left(\frac{\partial^n v_k}{\partial x^n}\right)\right]_{\substack{x=0\\y=1}} = \frac{2n - k + 1}{k + 1}.$$

These relations may also be derived, though less easily, by induction from (2) and (3).

**4. Numerical examples.** As an example, the first few values of $(\partial^n v_k / \partial x^n) \mid_{x=0}$ for $k = 3$, as found from (7), are given below:

$$(8)\qquad \begin{cases} \dfrac{\partial v}{\partial x}\bigg|_{x=0} = 1, \\ \dfrac{\partial^2 v}{\partial x^2}\bigg|_{x=0} = 2, \\ \dfrac{\partial^3 v}{\partial x^3}\bigg|_{x=0} = 6y, \\ \dfrac{\partial^4 v}{\partial x^4}\bigg|_{x=0} = 12y + 12y^2, \\ \dfrac{\partial^5 v}{\partial x^5}\bigg|_{x=0} = 24y + 72y^2 + 24y^3, \\ \dfrac{\partial^6 v}{\partial x^6}\bigg|_{x=0} = 408y^2 + 264y^3 + 48y^4, \\ \dfrac{\partial^7 v}{\partial x^7}\bigg|_{x=0} = 1008y^2 + 3120y^3 + 816y^4 + 96y^5, \\ \dfrac{\partial^8 v}{\partial x^8}\bigg|_{x=0} = 2016y^2 + 18624y^3 + 17376y^4 + 2112y^5 + 192y^6. \end{cases}$$

The coefficients $f_k(n, \beta)$ of $y^\beta$ in Eq. (8) can also be computed directly from (2).

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# PERCOLATION PROCESSES: LOWER BOUNDS FOR THE CRITICAL PROBABILITY


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**1. Introduction.** A percolation process is the spread of a fluid through a medium under the influence of a random mechanism associated with the medium. This contrasts with a diffusion process, where the random mechanism is associated with the fluid. Broadbent and Hammersley [1] gave examples illustrating the distinction.

Here we shall consider a *medium* consisting of an infinite set of *atoms* and *bonds*. A bond is a path between two atoms: it may be *undirected* (in which case it will allow passage from either atom to the other) or it may be *directed* (in which case it will allow passage from one atom to the other but not vice versa). Two atoms may be linked by several bonds, some directed and some undirected. Broadbent and Hammersley [1] dealt with *crystals*, i.e., media in which the atoms and bonds satisfied three postulates denoted by $P1$, $P2$, and $P3$. Here, however we shall dispense with $P1$ and a part of $P3$; and our surviving assumptions are:

$P2$. The number of bonds *from* (but not necessarily *to*) any atom is finite.

$P3$(a). Any finite subset of atoms contains an atom *from* which a bond leads to some atom not in the subset.

With this medium we associate the following *random mechanism*: each bond has an independent probability $p$ of being *undammed* and $q = 1 - p$ of being *dammed*. *Fluid*, supplied to the medium at a set of *source atoms*, spreads along undammed bonds only (and in the permitted direction only for undammed directed bonds) and thereby *wets* the atoms it reaches. Associated with each atom $A$, there is a *critical probability* $p_d(A)$, defined as the supremum of all values of $p$ such that, when $A$ is the only source atom, $A$ wets only finitely many atoms with probability one. We seek lower bounds for $p_d$.

An *n-stepped walk* is an ordered connected path along $n$ bonds, each step being in a permitted direction along its bond and starting from the atom reached by the previous step. Walks (as opposed to fluid) may traverse dammed bonds: a walk is dammed or undammed according as it traverses at least one or no

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