TABLE IV

Power of β-expectation tolerance regions,
$(\xi - \bar{\omega})A^{-1}(\xi - \bar{\omega})' \leqq c_\beta$

	Measure of Desirability = .99			
α	.88927	.79697	.69432	.53403
n \ β	.925	.95	.90	.75
3	.9755	.9531	.9105	.7810
4	.9770	.9598	.9318	.8502
5	.9784	.9661	.9522	.9022
7	.9809	.9752	.9606	.9291
11	.9838	.9794	.9751	.9578
21	.9869	.9845	.9818	.9770
30	.9880	.9865	.9847	.9812
31	.9881	.9866	.9849	.9815
32	.9882	.9868	.9851	.9818

In [1], it was shown that $c_\beta = (1 + n^{-1})\cdot(n - 1)\cdot(k / n - k)\cdot F_{1-\beta}$, where $F_{1-\beta}$ is the point exceeded with probability $1 - \beta$ using the F distribution with k, $n - k$ degrees of freedom. Hence the regions (3.3) have power given by

$$\text{Power} = P\left(F \leqq \frac{1 + n^{-1}}{\alpha^2 + n^{-1}} F_{1-\beta}\right). \tag{3.5}$$

Values of the power function (3.5) are given for the case of sampling from the bi-variate normal distribution ($k = 2$), when the correlation coefficient ρ is zero, and desirability of the centre $100\beta\%$ sets is .99, in Table IV.

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THE CONVERGENCE OF CERTAIN FUNCTIONS OF SAMPLE SPACINGS[1]

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1. Introduction and summary. Suppose $g(u_1, \cdots, u_k)$ is a continuous function of its arguments, homogeneous of order r, monotonic nondecreasing in each of its

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arguments, which is positive whenever each of its arguments is positive, and is such that for any given $K(0 < K < \infty)$, there is a number $R(K)(0 < R(K) < \infty)$ such that $g(u_1, \cdots, u_k) < K$ and $u_1 \geqq 0, \cdots, u_k \geqq 0$ imply that $u_1 + \cdots + u_k < R(K)$.

Let $U_1, \cdots, U_k$ be chance variables with joint density $e^{-(u_1+\cdots+u_k)}$ for $u_1 \geqq 0, \cdots, u_k \geqq 0$, and zero elsewhere. For any t, define $U(t)$ as $P[g(U_1, \cdots, U_k) \leqq t]$. We note that $U(t)$ is a continuous distribution function, with $U(0) = 0$.

Let $\rho(v)$ be a bounded nonnegative function with a finite number of discontinuities, defined for $0 \leqq v \leqq 1$. Suppose $X_1, X_2, \cdots, X_n$ are independently and identically distributed chance variables, each with density $f(x)$, $f(x)$ being bounded, and having a finite number of discontinuities and oscillations. $F(x)$ denotes $\int_{-\infty}^{x} f(x)\,dx$. Define $Y_1 \leqq Y_2 \leqq \cdots \leqq Y_n$ as the ordered values of $X_1, \cdots, X_n$, and define T_i as $Y_{i+1} - Y_i (i = 1, \cdots, n-1)$. Let $R_n(t)$ denote the proportion of the values

$$\rho\left(\frac{1}{n}\right) g(T_1, \cdots, T_k), \qquad \rho\left(\frac{2}{n}\right) g(T_2, \cdots, T_{k+1}), \cdots,$$
$$\rho\left(\frac{n-k}{n}\right) g(T_{n-k}, \cdots, T_{n-1})$$

which are less than or equal to t/n^r.

Let $\bar{U}[[tf^r(x)] / \{\rho[F(x)]\}]$ be defined as follows. If $f(x) = 0$,

$$\bar{U}[[tf^r(x)] / \{\rho[F(x)]\}] = 0$$

regardless of the value of t. If x is such that $f(x) > 0$ and $\rho[F(x)] = 0$, then $\bar{U}[[tf^r(x)] / \{\rho[F(x)]\}] = 1$ regardless of the value of t. If $f(x) > 0$ and $\rho[F(x)] > 0$, then $\bar{U}[[tf^r(x)] / \{\rho[F(x)]\}] = U[[tf^r(x)] / \{\rho[F(x)]\}]$. Let $S(t)$ denote

$$\int_{-\infty}^{\infty} \bar{U}[[t \cdot f^r(x)] / \{\rho[F(x)]\}] f(x)\,dx,$$

and let $V(n)$ denote $\sup_{t \geqq 0} |R_n(t) - S(t)|$. Then $V(n)$ converges to zero stochastically as n increases. This generalizes the result of [1], where $k = 1$, $g(u_1) = u_1$, $\rho(v) = 1$. The present result may be used to construct tests of fit in the presence of unknown location and scale parameters.

2. Proof of the convergence of $V(n)$.

LEMMA 1. *If for each given positive t, $R_n(t)$ converges to $S(t)$ stochastically as n increases, then $V(n)$ converges to zero stochastically as n increases.*

Proof. $S(t)$ is continuous for all $t > 0$, and is continuous on the right at $t = 0$. $S(0+) = \int_{\rho[F(x)]=0} f(x)\,dx$. It is easily seen that $R_n(0)$ converges to $\int_{\rho[F(x)]=0} f(x)\,dx$ with probability one as n increases. The rest of the proof of the lemma is almost exactly the same as the proof of Lemma 1 of [1].

LEMMA 2. *Let $X_1, X_2, \cdots, X_n$ be independent chance variables, each with a uniform distribution on* [0, 1]. *Let M denote the number of these variables falling in the closed interval $[C, D]$, where $0 \leqq C < D \leqq 1$, and let $Y_1 \leqq Y_2 \leqq \cdots \leqq Y_M$ denote the ordered values of the variables in $[C, D]$. Define $W_1 = Y_2 - Y_1, \cdots,$*

$W_{M-1} = Y_M - Y_{M-1}$. *For a given positive t, define* $L(n, t)$ *as the total number of values of* $g(W_1, \cdots, W_k), g(W_2, \cdots, W_{k+1}), \cdots, g(W_{M-k}, \cdots, W_{M-1})$ *which are not greater than* t/n^r. *Then* $[L(n, t)] / n$ *converges to* $(D - C)U(t)$ *stochastically as n increases.*

Proof. Define Z_i to be one if $g(W_i, \cdots, W_{k-1+i}) \leqq t/n^r$, and zero otherwise. M/n converges to $(D - C)$ with probability one as n increases. The conditional distribution given M of $Q_1 = MW_{i_1}$,

$$Q_2 = MW_{i_2}, \cdots, Q_L = MW_{i_L}(1 \leqq i_1 < i_2 < \cdots < i_L \leqq M - 1)$$

is easily verified to be

$$\left[D - C - \frac{(q_1 + \cdots + q_L)}{M}\right]^{M-L} \cdot \frac{M!}{M^L(D - C)^M(M - L)!}$$

for $q_1 + \cdots + q_L \leqq M(D - C)$, and zero elsewhere. As M increases, this density approaches $[1/(D - C)^L] \exp \{-(q_1 + \cdots + q_L)/(D - C)\}$ uniformly in any region where $q_1 + \cdots + q_L < K < \infty$. We note that under this limiting density, $Q_1, \cdots, Q_L$ are independent. To say that $g(W_i, \cdots, W_{k-1+i}) \leqq t/n^r$ is the same as saying that $g((n / M)MW_i, \cdots, (n / M)MW_{k-1+i}) \leqq t$, and as n increases the probability of this last occurrence approaches the probability that $g[(MW_i) / (D - C), \cdots, (MW_{k-1+i}) / (D - C)] \leqq t$. Since M approaches infinity with probability one as n increases, and from the restrictions on $g(u_1, \cdots, u_k)$ given in Sec. 1, we can use the limiting distribution of $MW_i, \cdots, MW_{k-1+i}$ to compute the limiting

$$P\{g[(MW_i) / (D - C), \cdots, (MW_{k-1+i}) / (D - C)] \leqq t\},$$

and we get $U(t)$ as this limiting probability.

$$\frac{L(n, t)}{n} = \frac{Z_1 + \cdots + Z_{M-k}}{n} = \frac{M}{n}\left[\frac{Z_1 + \cdots + Z_{M-k}}{M}\right],$$

and from the considerations above, it is easily seen that $E\{L(n, t)/n\}$ approaches $(D - C)U(t)$ as n increases.

Next we show that

$$E\left\{\left[\frac{L(n, t)}{n} - \frac{EL(n, t)}{n}\right]^2\right\}$$

approaches zero as n increases, which will complete the proof of Lemma 2. The expectation in question is equal to

$$(1 / n^2)E\{\textstyle\sum_1^{M-k} (Z_i - EZ_i)^2\} + (1 / n^2)E\{\textstyle\sum\sum_{i \neq j} (Z_i - EZ_i)(Z_j - EZ_j)\}.$$

$\{Z_i\}$ are uniformly bounded variables, and $M - k < n$, therefore the first term in this last expression certainly approaches zero as n increases. As for $\sum\sum_{i \neq j}(Z_i - EZ_i)(Z_j - EZ_j)$, any such term with $|i - j| > k$ has Z_i and Z_j defined in terms of entirely different and nonoverlapping sets of W's, and by the result on the independence of Q's derived above, if $|i - j| > k$, $E(Z_i - EZ_i)\cdot$

$(Z_j - EZ_j)$ must approach zero as n increases. But the number of terms $E(Z_i - EZ_i)(Z_j - EZ_j)$ with $|i - j| \leqq k$ is less than $2kn$. From these considerations, it follows easily that

$$E\left\{\left[\frac{L(n, t)}{n} - \frac{EL(n, t)}{n}\right]^2\right\}$$

approaches zero as n increases.

Now we turn to the proof of the stochastic convergence of $V(n)$. For simplicity, we assume that both $f(x)$ and $\rho(v)$ are continuous, for the time being. Given any positive ϵ, we can find H intervals $I_1 = (-\infty, z_1)$, $I_2 = (z_1, z_2)$, $I_3 = (z_2, z_3), \cdots, I_H = (z_{H-1}, \infty)$, such that the variation of $f(x)$ and of $\rho[F(x)]$ in each of these intervals is less than ϵ. Denote $\inf_{x \text{ in } I_i} \{f(x)\}$ by g_i, $\sup_{x \text{ in } I_i} \{f(x)\}$ by G_i, $\inf_{x \text{ in } I_i} \{\rho[F(x)]\}$ by h_i, $\sup_{x \text{ in } I_i} \{\rho[F(x)]\}$ by H_i. Let M_i denote the number of variables $X_1, X_2, \cdots, X_n$ that fall in I_i. Define $L_i(n, t)$ in terms of the M_i variables falling in I_i just as $L(n, t)$ was defined in terms of the variables falling in $[C, D]$ in Lemma 2. Define $L_i'(n, t)$ in the same way, except that each variable X_i is replaced by $F(X_i)$. Since $F(X_i)$ has a uniform distribution, Lemma 2 states that $[L_i'(n, t)]/n$ converges stochastically to $[F(z_i) - F(z_{i-1})]U(t)$ as n increases, where z_0 denotes $-\infty$, z_H denotes ∞. Also, since $F(Y_{i+1}) - F(Y_i) = f(\theta)[Y_{i+1} - Y_i]$, $Y_i \leqq \theta \leqq Y_{i+1}$, and from the assumptions about $g(u_1, \cdots, u_k)$, we have $L_i'(n, g_i^r t) \leqq L_i(n, t) \leqq L_i'(n, G_i^r t)$. $(M_1 + \cdots + M_i)/n$ converges to $F(z_i)$ with probability one as n increases, therefore the probability approaches one that

$$\frac{\sum_1^H L_i\left(n, \frac{t}{H_i}\right)}{n} - \frac{2H}{n} \leqq R_n(t) \leqq \frac{\sum_1^H L_i\left(n, \frac{t}{h_i}\right)}{n} + \frac{2H}{n}.$$

This implies that the probability approaches one that

$$\sum_1^H [F(z_i) - F(z_{i-1})]U\left(\frac{g_i^r t}{H_i}\right) \leqq R_n(t) \leqq \sum_1^H [F(z_i) - F(z_{i-1})]U\left(\frac{G_i^r t}{h_i}\right).$$

But by taking ϵ small enough (i.e., increasing H properly) the two extremes of this last inequality approach $S(t)$, proving the stochastic convergence of $V(n)$.

In the case where $\rho(v)$ and/or $f(x)$ have discontinuities, we can enclose the points of discontinuity in intervals whose total probability is arbitrarily small, change $\rho[F(x)]$ and $f(x)$ within these intervals to remove the discontinuities, and use the results above. The theorem would follow from a realization that the probability structure would be changed very little. The same device could be used to extend the results to cases where $f(x)$ is unbounded.

3. Application of results to tests of fit. First we prove the following lemma: If $F(x)$ and $G(x)$ are continuous distribution functions with density functions $f(x)$ and $g(x)$ respectively, then $f[F^{-1}(x)] \equiv cg[G^{-1}(x)]$ for some $c > 0$ if and only if $F(x) \equiv G(cx + b)$ for some constant b. To prove this, we note that the fact that $F[F^{-1}(x)] = x$ gives by differentiation that $f[F^{-1}(x)] \cdot (d/dx)F^{-1}(x) = 1$, and $g[G^{-1}(x)]\,(d/dx)\,G^{-1}(x) \equiv 1$. Thus, if $f[F^{-1}(x)] \equiv cg[G^{-1}(x)]$, then $c\,(d/dx)\,F^{-1}(x) \equiv (d/dx)\,G^{-1}(x)$, so $cF^{-1}(x) \equiv G^{-1}(x) + B$, for some constant B. Letting $x = F(y)$, we get $cy = G^{-1}[F(y)] + B$, or $cy - B \equiv G^{-1}[F(y)]$, or $G(cy - B) \equiv F(y)$. Conversely, if $G(cx + b) \equiv F(x)$, then $cg(cx + b) \equiv f(x)$, while $cx + b \equiv G^{-1}[F(x)]$, so that $cg(G^{-1}[F(x)]) \equiv f(x)$, or setting $y = F(x)$, $cg(G^{-1}(y)) \equiv f[F^{-1}(y)]$, completing the proof of the lemma.

Now we examine the theorem of Sec. 2 for the special case $k = 1$, $g(u) = u$ (therefore $r = 1$), and $\rho(v) = (1/\beta)f[F^{-1}(v)]$, where β is a positive constant. Then $U(t) = 1 - e^{-t}$, and $S(t) = \int_{-\infty}^{\infty} [1 - e^{-\beta t}]f(x)\,dx = 1 - e^{-\beta t}$, and thus does not depend on $f(x)$. Suppose we are confronted with the following problem in hypothesis testing: $X_1, X_2, \cdots, X_n$ are known to be independent and identically distributed chance variables, with a continuous distribution, and the hypothesis is that the common distribution function is $F(cx + b)$ for some unknown constants c, $b (c > 0)$, where the form of $F(x)$ is known. Here c is a scale parameter, and b is a location parameter. We are going to examine the properties of the test which rejects the hypothesis when $\inf_{\beta>0} \sup_{t\geqq 0} |R_n(t) - (1 - e^{-\beta t})|$ is "too large." We are going to show that this last expression converges stochastically to zero if and only if the hypothesis is true, so that the test is consistent. Also, when the hypothesis is true, the distribution of the expression is independent of the parameters c, b.

When the hypothesis is true, there is some $\alpha > 0$ such that $\sup_{t\geqq 0} |R_n(t) - (1 - e^{-\alpha t})|$ converges stochastically to zero as n increases. This follows from the lemma at the beginning of this section, and implies that when the hypothesis is true, $\inf_{\beta>0} \sup_{t\geqq 0} |R_n(t) - (1 - e^{-\beta t})|$ converges stochastically to zero as n increases. If the hypothesis is not true, then the true common distribution is $H(x)$, say, with density $h(x)$. Then, defining $S(t)$ as

$$\int_{-\infty}^{\infty} \left[1 - \exp\left\{\frac{-\beta t h(x)}{f[F^{-1}(H(x))]}\right\}\right] h(x)\,dx,$$

$\sup_{t\geqq 0} |R_n(t) - S(t)|$ converges stochastically to zero as n increases. But $S(t)$ will equal $1 - e^{-\alpha t}$ for some positive α if and only if the hypothesis is true, and therefore when the hypothesis is not true, $\inf_{\beta>0} \sup_{t\geqq 0} |R_n(t) - (1 - e^{-\beta t})|$ will not converge stochastically to zero. The fact that the distribution of $\inf_{\beta>0} \sup_{t\geqq 0} |R_n(t) - (1 - e^{-\beta t})|$ is independent of the parameters c, b when the hypothesis is true follows immediately from the fact that if A, B are constants $(A > 0)$, and $\bar{R}_n(t)$ is the expression defined in terms of $AX_1 + B, AX_2 + B, \cdots, AX_n + B$ in exactly the same way as $R_n(t)$ was defined in terms of $X_1, X_2, \cdots, X_n$, then $\inf_{\beta>0} \sup_{t\geqq 0} |\bar{R}_n(t) - (1 - e^{-\beta t})|$ is equal to

$$\inf_{\beta>0} \sup_{t\geqq 0} |R_n(t) - (1 - e^{-\beta t})|.$$

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