RESTRICTION AND SELECTION IN MULTINORMAL DISTRIBUTIONS[1]

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1. Summary. Maximum likelihood estimators of the parameters of a p-dimensional multinormal population are derived in this paper which are applicable when sample selection and observation is restricted with respect to x_1 but otherwise unrestricted with respect to $x_2, \cdots, x_p$. Restrictions imposed may consist of truncation, censoring, or a selection which results in full observation of all sample specimens with respect to x_1, but eliminates certain sample specimens from subsequent observation with respect to $x_2, \cdots, x_p$.

2. Introduction. Samples from a multidimensional universe are often obtained under circumstances such that observation in certain regions of the universe is restricted. For example, in studies of psychological traits, observation is often limited to individuals who have passed certain admission tests or who have been subjected to other screening processes. This situation likewise arises in connection with multivariate studies of physical characteristics in which specimens available for observation have previously undergone some type of sorting procedure. From such samples, it is often necessary to estimate the means, variances and correlation coefficients of the universe. Considering their most general aspects without limitation as to type of distribution, restricted or "screened" samples pose a broad class of estimation problems, some of which are quite involved. The present paper is limited to samples from a p-dimensional multinormal distribution with probability density function

$$f(x_1, x_2, \cdots, x_p) = (2\pi)^{-p/2} |\sigma^{ij}|^{1/2} \exp\left[-\frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} \sigma^{ij}(x_i - m_i)(x_j - m_j) \right], \tag{1}$$

where the symmetric matrix $\|\sigma^{ij}\|$ of the quadratic form in the exponent is the inverse of the variance-covariance matrix $\|\sigma_{ij}\|$, and has the positive determinant $|\sigma^{ij}|$. Maximum likelihood estimators (estimates) for parameters of (1) are obtained from truncated, censored and selected samples, with x_1 designating the restricted variable; that is, the variable on which screening is based. Similar estimators obtained previously ([8], [10]) for restricted samples from a bivariate normal distribution, follow as a special case of results obtained here. Results obtained by Hotelling [12], Tukey [18], Pittman [16], and Chapman [5] guarantee that both the method of moments and the method of maximum likelihood lead to identical estimates in the case of truncated samples from multinormal distributions. Hence for truncated samples we might have employed the method of moments. However, we elected to use the method of maximum likelihood

Received November 26, 1954; revised June 30, 1956.

[1] Sponsored by the Office of Ordnance Research, U. S. Army.

as it permits a uniform treatment of all the various types of restricted samples under consideration and it introduces no unusual algebraic difficulties.

In practical applications such as in studies of psychological traits, the screening variable x_1 might actually be a composite score based on a battery of tests rather than the score achieved on a single test. However, in this paper, we limit our consideration to cases in which each of the component variables, including x_1, has a univariate normal marginal distribution. In some applications, one or more achievement scores might be involved which also are composite scores. For example, x_p could be such a score. Here again, however, the limitation of normality on marginal distributions holds.

Various aspects of some of the basic problems involved in the present study have previously been investigated by Karl Pearson [15], Aitken [1], Wilks [20], Birnbaum Paulson and Andrews [3], Votaw, Rafferty and Deemer [19], Campbell [4], Des Raj [17], and the author [8], [9], [10]. A more complete bibliography of related papers can be found in reference [7].

3. Estimating means, standard deviations, and correlation coefficients. For a random sample of n (fixed) measured observations $(x_{1\alpha}, x_{2\alpha}, \cdots, x_{p\alpha})$, $\alpha = 1, 2, \cdots n$, drawn from a population distributed according to (1), subject to a restriction on observation of variable x_1, the logarithm of the likelihood function is

$$L = -(np/2) \ln 2\pi + (n/2) \ln |\sigma^{ij}| - \tfrac{1}{2}\sum_i \sum_j \sum_\alpha \sigma^{ij}(x_{i\alpha} - m_i)(x_{j\alpha} - m_j) + \ln G(m_1, \sigma_{11}), \tag{2}$$

where $G(m_1, \sigma_{11})$ is a restriction function which depends upon the type of restriction imposed with respect to observation of x_1 by screening or acceptance criteria. When G is to be interpreted with full generality, it is not only a function of m_1 and σ_{11}, but also of x_1. By thus introducing G, much repetition in the derivation of estimators is avoided which otherwise would arise with the various selection criteria to be considered. Specific examples of G are given subsequently in this paper.

For an unrestricted sample, $G(m_1, \sigma_{11}) \equiv 1$, and maximum likelihood estimates of parameters m_i and σ^{ij} are obtained by equating to zero, the partial derivatives of L with respect to these parameters and solving the resulting system of equations. (Cf. for example Mood [14], pp. 186–188.) In the cases involving restricted or screened samples, we follow a similar procedure. However, in order to avoid certain complications which restrictions on x_1 introduce into derivatives with respect to σ^{ij}, we employ derivatives with respect to σ_{11} and ρ_{ij}. According to the notation employed here, $\sigma_{ij} = \sigma_i\sigma_j\rho_{ij}$ and $\sigma_{11} = \sigma_1^2$, where ρ_{ij} is the coefficient of correlation between x_i and x_j.

Considering first the means, we have

$$(3)\quad \begin{aligned} &\text{(a)} \quad \frac{\partial L}{\partial m_1} = n \sum_{i=1}^{p} \sigma^{1i} C_{i0} + \frac{1}{G}\frac{\partial G}{\partial m_1}, \\ &\text{(b)} \quad \frac{\partial L}{\partial m_r} = n \sum_{i=1}^{p} \sigma^{ri} C_{i0}, \qquad r = 2, 3, \cdots, p, \end{aligned}$$

where

$$(4) \quad C_{ij} = \sum_{\alpha} (x_{i\alpha} - m_i)(x_{j\alpha} - m_j)/n \qquad \text{and} \qquad C_{i0} = \sum_{\alpha} (x_{i\alpha} - m_i)/n.$$

On setting $\partial L/\partial m_r = 0$, $(r \geqq 2)$ and dividing by C_{10}, we obtain the system of $p - 1$ equations:

$$(5) \quad \sum_{i=1}^{p} \sigma^{ri} C_{i0}/C_{10} = 0, \qquad r = 2, 3, \cdots, p.$$

On solution, these yield the result

$$(6) \quad C_{i0}/C_{10} = A_{1i}/A_{11},$$

where A_{ij} is the cofactor of $|\sigma^{ij}|$. Since $\|\sigma^{ij}\| = \|\sigma_{ij}\|^{-1}$, then $\sigma_{ij} = A_{ij} / |\sigma^{ij}|$ and $A_{ij} = \sigma_{ij} |\sigma^{ij}|$. On substituting this result into (6), we have

$$(7) \quad C_{i0}/C_{10} = [\sigma_{1i} |\sigma^{ij}|]/\sigma_{11} |\sigma^{ij}|] = \sigma_{1i}/\sigma_{11}.$$

After making the further substitution $\sigma_{1i} = \sigma_1 \sigma_i \rho_{1i}$, it follows from (7) that

$$(8) \quad C_{i0} = \rho_{1i}(\sigma_i / \sigma_1) C_{10}.$$

We turn now to the variances and to the correlation coefficients. Since $C_{ij} = C_{ji}$ and $\sigma^{ij} = \sigma^{ji}$, we need only the derivatives

$$(9)\quad \begin{aligned} &\text{(a)} \quad \frac{\partial L}{\partial \sigma_{11}} = -\frac{n}{2\sigma_{11}} \left\{ 1 - \sum_{j=1}^{p} \sigma^{ij} C_{ij} \right\} + \frac{1}{G}\frac{\partial G}{\partial \sigma_{11}}, \\ &\text{(b)} \quad \frac{\partial L}{\partial \sigma_{ss}} = -\frac{n}{2\sigma_{ss}} \left\{ 1 - \sum_{j=1}^{p} \sigma^{sj} C_{sj} \right\}, \qquad s = 2, 3, \cdots, p, \\ &\text{(c)} \quad \frac{\partial L}{\partial \rho_{rs}} = -n\sigma_r \sigma_s \left\{ \sigma^{rs} - \sum_{i=1}^{p} C_{ii} \sigma^{ir} \sigma^{is} \right. \\ &\qquad \left. - \sum_{\substack{i=1 \\ i<j}}^{p-1} \sum_{j=2}^{p} C_{ij} [\sigma^{rj} \sigma^{ij} + \sigma^{ri} \sigma^{is} (1 - \delta_{ij}^{rs}) + \sigma^{si} \sigma^{jr} \delta_{ij}^{rs}] \right\}, \\ &\qquad r = 1, 2, \cdots, p - 1;\ s = 2, 3, \cdots, p;\ r < s, \end{aligned}$$

where δ_{ij}^{rs} is a generalized form of Kronecker's delta such that it has the value 1 if $\sigma^{ri}\sigma^{js} = \sigma^{rs}\sigma^{ij}$, but otherwise it has the value zero.

We equate to zero, the $(p - 1)$ derivatives $\partial L/\partial \sigma_{ss}$ of (9b) and the $p(p - 1)/2$ derivatives $\partial L/\partial \rho_{rs}$ of (9c) to form a total of $(p - 1)(p + 2)/2$ equations that are linear in $C_{ij}(i \leqq j)$. These we now write as

$$\sum_{i=1}^{p} \sigma^{is} C_{is} - 1 = 0, \qquad s = 2, 3, \cdots, p,$$

(10)
$$\sum_{i=1}^{p} C_{ii}\sigma^{ir}\sigma^{is} + \sum_{i=1}^{p-1}\sum_{j=2}^{p}{}_{i<j} C_{ij}[\sigma^{rs}\sigma^{ij} + \sigma^{ri}\sigma^{js}(1 - \delta_{ij}^{rs}) + \sigma^{si}\sigma^{jr}\delta_{ij}^{rs}]$$
$$- \sigma^{rs} = 0, \quad r = 1, 2, \cdots, p-1; s = 2, \cdots, p; r < s.$$

As a solution of this system of equations, we obtain C_{ij} in terms of C_{11} as

(11)
$$C_{ij} = \sigma_{ij} + (\sigma_{1i}\sigma_{1j}/\sigma_{11})(C_{11}/\sigma_{11} - 1), \qquad i \leqq j,$$

which can be verified by direct substitution back into the equations of (10). As a special case of (11), we have

(11a)
$$C_{1i} = (\sigma_{1i}/\sigma_{11})C_{11}.$$

Returning now to the definitions for C_{ij} and C_{i0} as given in (4), we can write

$$C_{ij} - C_{i0}C_{j0} = \sum_{\alpha}(x_{i\alpha} - m_i)(x_{j\alpha} - m_j)/n$$
$$- [\sum_{\alpha} (x_{i\alpha} - m_i)/n][\sum_{\alpha}(x_{j\alpha} - m_j)/n]$$
$$= [\sum_{\alpha} x_{i\alpha}x_{j\alpha}/n - m_j\bar{x}_i - m_i\bar{x}_j + m_i m_j]$$
$$- [\bar{x}_i\bar{x}_j - m_i\bar{x}_j - m_j\bar{x}_i + m_i m_j]$$

and thus

(12)
$$C_{ij} - C_{i0}C_{j0} = \sum_{\alpha} x_{i\alpha}x_{j\alpha}/n - \bar{x}_i\bar{x}_j, \quad \alpha = 1, 2, \cdots, n,$$

where $\bar{x}_k = \sum_{\alpha} x_{k\alpha}/n$.

With restricted sample standard deviations written as $\bar{s}_i$ and restricted sample correlation coefficients written as $\bar{r}_{ij}$ where

(13)
$$\bar{s}_i = \left[\sum_{\alpha=1}^{n} x_{i\alpha}^2/n - \left(\sum_{\alpha=1}^{n} x_{i\alpha}/n\right)^2\right]^{1/2}$$
$$\bar{r}_{ij} = \left[n\sum_{\alpha=1}^{n} x_{i\alpha}x_{j\alpha} - \sum_{\alpha=1}^{n} x_{i\alpha}\sum_{\alpha=1}^{n} x_{j\alpha}\right] \Big/ n^2\bar{s}_i\bar{s}_j,$$

Eq. (12) becomes

(14)
$$C_{ij} - C_{i0}C_{j0} = \bar{r}_{ij}\bar{s}_i\bar{s}_j.$$

Since $\sigma_{ij} = \sigma_i\sigma_j\rho_{ij}$, Eqs. (8) and (11) permit us to write

(15)
$$C_{ij} - C_{i0}C_{j0} = \sigma_i\sigma_j[\rho_{ij} - \lambda\rho_{1i}\rho_{1j}],$$

where

(16)
$$\lambda = 1 - \bar{s}_1^2/\sigma_1^2.$$

Equating the right side of (14) to the right side of (15), we have

$$\bar{r}_{ij}\bar{s}_i\bar{s}_j = \sigma_i\sigma_j[\rho_{ij} - \lambda\rho_{1i}\rho_{1j}]. \tag{17}$$

We let $i = j$, and since $\bar{r}_{ii} \equiv 1$ and $\rho_{ii} \equiv 1$, it follows from (17) that

$$\sigma_j^2 = \bar{s}_j^2/(1 - \rho_{1j}^2\lambda), \qquad j = 2, 3, \cdots, p. \tag{18}$$

Let $i = 1$, eliminate σ_j between (17) and (18), and we have

$$\rho_{1j} = \bar{r}_{1j}/\sqrt{1 - \lambda(1 - \bar{r}_{1j}^2)}, \qquad j = 2, 3, \cdots, p. \tag{19}$$

Use (18) to write first σ_i and then σ_j. Substitute these results into (17) and simplify to obtain

$$\rho_{ij} = \bar{r}_{ij}\sqrt{(1 - \rho_{1i}^2\lambda)(1 - \rho_{1j}^2\lambda)} + \rho_{1i}\,\rho_{1j}\,\lambda, \tag{20}$$

with $i, j = 2, 3, \cdots, p, i < j$.

Estimates $\hat{m}_1$ and $\hat{\sigma}_1$ are yet to be determined, but Eqs. (8), (18), (19) and (20) enable us to express estimators for the remaining parameters of (1) in terms of these two as

$$\begin{aligned} \hat{m}_j &= \bar{x}_j - \bar{r}_{1j}(\bar{s}_j/\bar{s}_1)(\bar{x}_1 - \hat{m}_1), \\ \hat{\sigma}_j &= \bar{s}_j\sqrt{\frac{1 - \hat{\lambda}(1 - \bar{r}_{1j}^2)}{1 - \hat{\lambda}}}, \\ \hat{\rho}_{ij} &= \frac{\bar{r}_{ij} - \hat{\lambda}(\bar{r}_{ij} - \bar{r}_{ij}\bar{r}_{ij})}{\sqrt{[1 - \hat{\lambda}(1 - \bar{r}_{1i}^2)][1 - \hat{\lambda}(1 - \bar{r}_{1j}^2)]}}, \end{aligned} \tag{21}$$

with $i = 1, 2, \cdots, p - 1, j = 2, 3, \cdots, p, i < j$, and $\hat{\lambda} = 1 - \bar{s}_1^2/\hat{\sigma}_1^2$. Since by definition, $\bar{r}_{ii} \equiv 1$, the last equation of (21), in agreement with (19), simplifies to

$$\hat{\rho}_{1j} = \bar{r}_{1j}/\sqrt{1 - \hat{\lambda}(1 - \bar{r}_{1j}^2)}, \tag{22}$$

when $i = 1$. Here and throughout this paper, the maximum likelihood symbol ($\hat{}$) serves to distinguish estimates from the parameters estimated.

Although the derivations were somewhat more laborious, the above results were given earlier in [9]. Estimators for restricted samples from a bivariate normal population as given in [8] and [10] now follow as a special case of (21) and (22) with $j = p = 2$.

To estimate m_1 and σ_1, we substitute (7) into (3a) and (11a) into (9a), equate to zero, and thereby obtain

$$\begin{aligned} \frac{\sigma_{11}}{n}\frac{\partial L}{\partial m_1} &= C_{10}\sum_1^p \sigma^{1i}\sigma_{1i} + \frac{\sigma_{11}}{nG}\frac{\partial G}{\partial m_1} = 0, \\ \frac{2\sigma_{11}}{n}\frac{\partial L}{\partial \sigma_{11}} &= \frac{C_{11}}{\sigma_{11}}\sum_1^p \sigma^{1i}\sigma_{1i} - 1 + \frac{2\sigma_{11}}{nG}\frac{\partial G}{\partial \sigma_{11}} = 0. \end{aligned} \tag{23}$$

Since $\sum_i \sigma^{im}\sigma_{mj} = \delta_{ij}$ (cf., for example, [14], p. 179), where $\delta_{ij} = 1$, if $i = j$, and $\delta_{ij} = 0$, if $i \neq j$, it follows that $\sum_1^p \sigma^{1i}\sigma_{1i} = 1$, and with the defining relations for C_{10} and C_{11} as given by (4), this result enables us to write

$$
(24) \quad
\begin{aligned}
&\sum_{\alpha=1}^{n} (x_{1\alpha} - m_1)/n + \frac{\sigma_{11}}{nG}\frac{\partial G}{\partial m_1} = 0, \\
&\sum_{\alpha=1}^{n} (x_{1\alpha} - m_1)^2/n - \sigma_{11}\left[1 - \frac{2\sigma_{11}}{nG}\frac{\partial G}{\partial \sigma_{11}}\right] = 0.
\end{aligned}
$$

The required estimates $\hat{m}_1$ and $\hat{\sigma}_1$ are the values found on solving this pair of equations. The restriction function, $G(m_1, \sigma_{11})$ which depends upon the nature of the restrictions imposed on x_1 must be specified before Eqs. (24) are completely determined, but it is to be noted that regardless of G, they involve only the x_1-marginal distribution and are independent of the remaining variables.

Truncated and censored samples. When samples under consideration have been truncated or censored with respect to x_1, estimating Eqs. (24) reduce to forms identical with those obtained previously in reference [7] in connection with various types of truncated and censored samples from univariate normal populations. They can be solved as therein described for the univariate cases. For example, when x_1 is *singly truncated* on the left at a fixed terminal x_{10}, then $G(m_1, \sigma_{11}) = [I_0(\xi)]^{-n}$, where $I_0(\xi) = \int_\xi^\infty \varphi(t)\,dt$, $\varphi(t) = [(2\pi)^{1/2}]^{-1} \exp(-t^2/2)$, and $\xi = (x_{10} - m_1)/\sigma_1$. In this case, estimating Eqs. (24) reduce to

$$
(25) \quad
\begin{aligned}
&\text{(a)} \quad \frac{1}{\hat{Z} - \hat{\xi}}\left[\frac{1}{\hat{Z} - \hat{\xi}} - \hat{\xi}\right] = \frac{n \sum_{\alpha=1}^{n} (x_{1\alpha} - x_{10})^2}{\left[\sum_{\alpha=1}^{n} (x_{1\alpha} - x_{10})\right]^2}, \\
&\text{(b)} \quad \hat{\sigma}_1 = \sum_{\alpha=1}^{n} (x_{1\alpha} - x_{10})/n(\hat{Z} - \hat{\xi}), \\
&\text{(c)} \quad m_1 = x_{10} - \hat{\sigma}_1 \hat{\xi},
\end{aligned}
$$

where

$$
(26) \quad Z(\xi) = \varphi(\xi)/I_0(\xi) = \exp(-\xi^2/2) \Big/ \int_\xi^\infty \exp(-t^2/2)\,dt.
$$

Equation (25a) can be solved for $\hat{\xi}$, so that $\hat{\sigma}_1$ and $\hat{m}_1$ follow in turn from (25b) and (25c). For further details, reference is again made to [7]. Whenever m_1 and σ_1 are known a priori, the remaining parameters can be estimated from (21) with $\hat{\lambda} = 1 - s_1^2/\hat{\sigma}_1^2$ replaced by $\lambda = 1 - s_1^2/\sigma_1^2$ and $\hat{m}_1$ replaced by the known value of m_1.

Selected samples. When sampling procedure is such that a total of N unrestricted observations are made with respect to x_1 although as a result of selection or screening, there may be only $n(< N)$ observations of $x_2, \cdots, x_p$, then

$$
G(m_1, \sigma_{11}) = (\sqrt{2\pi\sigma_{11}})^{n-N} \exp\left[-\sum_{1}^{N-n} (x_{1\alpha} - m_1)^2/2\sigma_{11}\right],
$$

and Eqs. (24) lead to the familiar estimates

$$(27) \qquad m_1 = \sum_1^N x_{1\alpha}/N = \bar{x}_1, \qquad \hat{\sigma}_1^2 = \sum_1^N (x_{1\alpha} - \bar{x}_1)^2/N.$$

Regardless of how the selection which determines subsequent observation with respect to $x_2, \cdots, x_p$ is made, $\hat{m}_1$ and $\hat{\sigma}_1$ are given by (27) while the other estimates are given by (21) where $\bar{x}_j$, $\bar{s}_j$, and $\bar{r}_{ij}$ are computed from observations of the n "selected" members of the sample.

Unrestricted samples. When no sample restrictions are imposed, and no selection is made, then not only is $G \equiv 1$, but $N = n$, $\lambda = 0$, and the required estmates follow from (27) and (21) as

$$(28) \qquad \hat{m}_j = \bar{x}_j, \qquad \hat{\sigma}_j = s_j, \qquad \hat{\rho}_{ij} = r_{ij}, \quad i, j = 1, 2, \cdots p,$$

which, as already mentioned, are well known for this case. The bars ($^-$) are omitted over r_{ij} and s_j in (28) since here the computations are based on the complete rather than a restricted sample.

4. Reliability of estimates. Asymptotic variances and covariances of estimates given in the preceding section can, of course, be obtained from the likelihood information matrices with elements which are expected values of the second partial derivatives of the likelihood function L. These variances and covariances are of the order of $1/n$, but exact expressions for them are too unwieldy to be of much practical value. For parameters of the restricted variable, in this case x_1, asymptotic variances and covariances given in [7] for truncated and censored samples from univariate normal distributions are applicable when restrictions are of these types. When a selection based on x_1 is made which does not restrict observation of x_1 itself, then complete sample variances

$$V(\hat{m}_1) = \sigma_1^2/N,$$
$$V(\hat{\sigma}_1) = \sigma_1^2/2N,$$

are applicable as are various exact small sample results based on the x_1 marginal distribution. If the restrictions involved have not been unduly severe, that is, if only minor portions of the tails of the x_1 distribution have been affected, then asymptotic variances and covariances for complete (unrestricted) samples from a multinormal distribution will afford reasonably satisfactory approximations to the desired values. (Cf. Kendall [13], Vol. 11, third edition, p. 38.)

5. Practical applications. The practical application of estimators obtained in this paper is illustrated with a sample given by Baten [2], and attributed by him to H. C. Carver. The basic sample consists of weight, height, shoulder, chest, waist, and hip measurements on 119 individuals. We designate these variates in the order listed as x_1, x_2, x_3, x_4, x_5, and x_6 respectively. Baten's data include 120 sets of measurements, but it was necessary to eliminate the last one because of a typographical error. As given by Baten, the sample was considered to be complete, but for purposes of the illustrations here, it is arbitrarily truncated with respect to weight (x_1) at 119.5 pounds. Thereby eleven sets of measurements are eliminated. Estimates of the population parameters are then computed considering the sample as truncated with $n = 108$, and as censored with $n = 108$ and $n_1 = 11$. A complete summary of estimates calculated for each of these two cases is included in Table 2 along with corresponding estimates computed from the complete sample. As can be observed from this table, estimates based on the truncated and censored samples are in close agreement with those computed from the complete sample. The computing procedures employed are illustrated below.

Truncated sample—number missing observations unknown. For this case, the sample data are summarized in Table 1.

To estimate parameters of x_1, we may follow the procedure described in [7] and first solve equation (25a), which for this example is

$$\frac{1}{\hat{Z} - \hat{\xi}}\left[\frac{1}{\hat{Z} - \hat{\xi}} - \hat{\xi}\right] = \frac{108(64{,}169.00)}{(2{,}301.0)^2} = 1.308928.$$

Thereby, we obtain $\hat{\xi} = -1.379$, and from (25b) we computed $\hat{\sigma}_1 = 13.7697$, and from (25c) $\hat{m}_1 = 119.5 - (13.7697)(-1.379) = 138.4884$. Tables [11] were employed to reduce the computing effort which otherwise would have been required.

From Eq. (16), we compute $\hat{\lambda} = 1 - \bar{s}_1^2/\hat{\sigma}_1^2 = 1 - (11.8419/13.7697)^2 = 0.2604$, and the remaining estimates are obtained from Eqs. (21). For illustration, specimen computations are given below.

$$\hat{m}_2 = 67.9241 - 0.4701(2.4008/11.8419)(21.3056 - 18.9884) = 67.7033,$$

$$\hat{\sigma}_2 = 2.4008\sqrt{\frac{1 - 0.2604(1 - 0.4701^2)}{1 - 0.2604}} = 2.4923,$$

$$\hat{\rho}_{12} = 0.4701/\sqrt{1 - (0.2604)(1 - 0.4701^2)} = 0.5265,$$

$$\hat{\rho}_{23} = \frac{0.2361 - 0.2604[0.2361 - (0.4701)(0.4326)]}{\sqrt{[1 - 0.2604(1 - 0.4701^2)][1 - 0.2604(1 - 0.4326^2)]}} = 0.2872.$$

TABLE 1

Summary of Sample Data

$n = 108$	Truncation at $x_1 = 119.5$ lbs.		
$\bar{x}_1 = 140.8056$	$\bar{s}_1 = 11.8419$	$\bar{r}_{12} = 0.4701$	$\bar{r}_{25} = -0.1389$
	$\bar{s}_2 = 2.4008$	$\bar{r}_{13} = 0.4326$	$\bar{r}_{26} = 0.3019$
$\bar{x}_2 = 67.9241$	$\bar{s}_3 = 0.7103$	$\bar{r}_{14} = 0.6501$	$\bar{r}_{34} = 0.5904$
$\bar{x}_3 = 16.4500$	$\bar{s}_4 = 1.5373$	$\bar{r}_{15} = 0.4415$	$\bar{r}_{35} = 0.1852$
$\bar{x}_4 = 35.4537$	$\bar{s}_5 = 1.6375$	$\bar{r}_{16} = 0.7873$	$\bar{r}_{36} = 0.4059$
$\bar{x}_5 = 28.1574$	$\bar{s}_6 = 1.3746$	$\bar{r}_{23} = 0.2361$	$\bar{r}_{45} = 0.4931$
$\bar{x}_6 = 35.5898$		$\bar{r}_{24} = 0.1194$	$\bar{r}_{46} = 0.5491$
			$\bar{r}_{56} = 0.4310$
$\sum_1^n(x_{1\alpha} - x_{10}) = 2301.0$		$\sum_1^n(x_{1\alpha} - x_{10})^2 = 64169.00$	

TABLE 2

Summary of Estimates

Parameters	Estimates Based on Complete Sample	Estimates Based on Restricted Sample	
		Truncated	Censored
		Number Missing Observations Unknown	Number Missing Observations Known
ξ		−1.379	−1.342
m_1	138.2353	138.4884	138.2382
m_2	67.6664	67.7033	67.6794
m_3	16.3672	16.3899	16.3834
m_4	35.1899	35.2581	35.2370
m_5	27.9252	28.0159	28.0007
m_6	35.3513	35.3780	35.3549
σ_1	13.9421	13.7697	13.9629
σ_2	2.5330	2.4923	2.5021
σ_3	0.7417	0.7333	0.7358
σ_4	1.7280	1.6477	1.6557
σ_5	1.7857	1.6927	1.6987
σ_6	1.5235	1.5172	1.5318
ρ_{12}	0.5239	0.5265	0.5318
ρ_{13}	0.5446	0.4872	0.4924
ρ_{14}	0.7339	0.7053	0.7037
ρ_{15}	0.5566	0.4966	0.5018
ρ_{16}	0.8369	0.8294	0.8330
ρ_{23}	0.2996	0.2872	0.2992
ρ_{24}	0.2406	0.2040	0.2114
ρ_{25}	−0.0120	−0.0613	−0.0536
ρ_{26}	0.3732	0.3772	0.3842
ρ_{34}	0.6193	0.6229	0.6265
ρ_{35}	0.3193	0.2365	0.2416
ρ_{36}	0.4908	0.4615	0.4667
ρ_{45}	0.5943	0.5362	0.5404
ρ_{46}	0.6569	0.6166	0.6220
ρ_{56}	0.5344	0.4849	0.4902
λ		0.2604	0.2806
Sample size	$n = 119$	$n = 108$	$n = 108$ $n_1 = 11$

Censored sample—number of missing (unmeasured) observations known. The sample data remain unchanged from the previous case except for the additional information that $n_1 = 11$. To estimate parameters of x_1, we determine ξ by solving

$$\frac{1}{\hat{Y}-\xi}\left[\frac{1}{\hat{Y}-\xi}-\xi\right]=\frac{n\sum_{\alpha=1}^{n}(x_{1\alpha}-x_{10})^2}{\left[\sum_{\alpha=1}^{n}(x_{1\alpha}-x_{10})\right]^2}=1.308928,$$

where

$$Y(\xi)=\frac{n_1}{n}\left[\exp(-\xi^2/2)/(\sqrt{2\pi}-\int_{\xi}^{\infty}\exp(-t^2/2)\,dt)\right]=\frac{n_1}{n}Z(-\xi),$$

in the same manner as for the truncated case, and this time find $\hat{\xi} = -1.342$. Subsequently we compute $\hat{\sigma}_1 = \sum_1^n (x_{1\alpha} - x_{10})/n(\hat{Y} - \hat{\xi}), = 13.9629$. We then calculate $\hat{m}_1 = 119.5 - (13.9629)(-1.342) = 138.2382$. Using (16), we have $\hat{\lambda} = 1 - (11.8419/13.9629)^2 = 0.2806$. For further details, reference is again made to [7]. With $\hat{m}_1$ and $\hat{\sigma}_1$ thus determined, these values along with the original sample data are substituted into (21) to obtain estimates of the remaining parameters.

Although not complete in all details, the above calculations serve to indicate the general manner in which results of this paper are applicable in practical problems. To a certain extent, they also serve to indicate the degree of agreement to be expected among corresponding estimates based on truncated, censored and complete (unrestricted) samples.

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