THE DISTRIBUTION OF THE RATIOS OF CERTAIN QUADRATIC FORMS IN TIME SERIES[1]

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1. Introduction. In testing the hypothesis that successive members of a series of observations are serially correlated a number of statistics have been proposed. Durbin and Watson [4] gave the exact distribution of several of these statistics when they are slightly modified. We shall extend the work of Durbin and Watson for a non-null case of two of their modified statistics and also find a simple expression for the moments of another of their statistics.

2. The Double root result. Assume that $X' = (x_1, x_2, \cdots, x_n)$ has probability density

$$f(X) = |\Lambda|^{1/2}(2\pi)^{-n/2} \exp[-X'\Lambda X/2], \tag{2.1}$$

where Λ is a positive definite matrix and $n = 2m$. Let

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}; \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_1 \end{pmatrix}; \quad \Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_1 \end{pmatrix}; \tag{2.2}$$

where B_1 is positive definite or positive semi-definite and of rank $m - q$ which is $\geqq$ the rank of A_1, a real symmetric matrix. Further assume that A, B, and Λ commute pairwise, and that the characteristic roots a_j of A and the characteristic roots b_j of B are so numbered that if $a_j \neq 0, b_j > 0$ and $a_j/b_j \geqq a_{j+1}/b_{j+1}$ for all a_j and a_{j+1} which are $\neq 0$.

Now

$$G(z) = P\left[\frac{X'AX}{X'BX} \leqq z\right] = P[X'(A - zB)X \leqq 0], \tag{2.3}$$

where X is $N(0, \Lambda^{-1})$. Making an orthogonal transformation $X = PY$ where $P'AP = D_a$, $P'BP = D_b$, $P'\Lambda P = D_\lambda$ are diagonal matrices with elements $a_j = a_{m+j}$, $b_j = b_{m+j}$ and $\lambda_j = \lambda_{m+j}$, we get

$$G(z) = P[Y'(D_a - zD_b)Y \leqq 0], \tag{2.4}$$

where Y is $N(0, D_\lambda^{-1})$. Now let $Y = D_\lambda^{-1/2}W$ so that

$$G(z) = P[W'(D_a - zD_b)D_\lambda^{-1}W \leqq 0], \tag{2.5}$$

Received June 28, 1955; revised January 3, 1957.

[1] Work under contract with the Office of Naval Research NR 042 031, for investigations in statistics and probability at Chapel Hill. Reproduction in whole or in part is permitted for any purpose of the United States Government.

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where W is $N(0, \mathrm{I})$. Hence by the duplication of roots a_j, b_j and λ_j we get

$$G(z) = P\left[\sum_{j=1}^{m-q} \lambda_j^{-1}(a_j - zb_j)w_j^2 \leqq 0\right] \tag{2.6}$$

where w_j^2 are independent and each is a χ^2 variable with 2 degrees of freedom.

Using a result by R. L. Anderson [1] which in general terms states that if the c_j are all different then

$$P\left[\sum_{j=1}^{p} c_j w_j^2 \leqq 0\right] = 1 - \sum_{j\varepsilon s} c_j^{p-1} \prod_{\substack{j \neq k\\ j=1}}^{p} (c_k - c_j)^{-1} \tag{2.7}$$

where $S = \{j/c_j > 0\}$, we find that

$$G(z) = 1 - \prod_{j=1}^{m-q} \lambda_j \sum_{k=1}^{L} (a_k - b_k z)^{m-q-1}\lambda_k^{-1} \cdot \prod_{\substack{j=1\\ j\neq k}}^{m-q} [\lambda_j(a_k - b_k z) - \lambda_k(a_j - b_j z)]^{-1}, \tag{2.8}$$

where

$$\frac{a_{L+1}}{b_{L+1}} \leqq z \leqq \frac{a_L}{b_L}$$

for $L = 1, \cdots, m - q$. This result could also have been gotten by contour integration through the results of Gurland [6] or by Madow's generalization of Anderson's result.

3. The distribution of the Durbin and Watson Statistic in the non-null case. Using the result (2.8) and letting

$$R = \sum_{\substack{i=1\\ i\neq m}}^{2m-1} x_{i+1} x_i \Big/ \sum_{i=1}^{2m} x_i^2, \tag{3.1}$$

where

$$\Lambda_1 = \begin{pmatrix} 1+\rho^2 & -\rho & & & \\ -\rho & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & -\rho \\ & & & -\rho & 1+\rho^2 \end{pmatrix}, \tag{3.2}$$

we may find the distribution of R. For $a_j = \cos(j\pi)/(m+1)$, $\lambda_j = 1 + \rho^2 - 2\rho a_j$, $b_j = 1$. Now $\prod_{j=1}^{m} \lambda_j = (1 - \rho^{2m+2})(1 - \rho^2)^{-1}$ and $\lambda_k - \lambda_j = (a_k - a_j) \cdot (1 + \rho^2 - 2\rho z)$ and by Geisser [5]

$$\prod_{\substack{j=1\\ j\neq k}}^{m} (a_k - a_j) = (m+1)(-1)^{k+1}2^{-m} \csc^2 \frac{k\pi}{m+1}. \tag{3.3}$$

Therefore

$$G(z;\rho) = 1 - (1-\rho^{2m+2})2^m(m+1)^{-1}(1-\rho^2)^{-1}(1+\rho^2-2\rho z)^{1-m} \tag{3.4}$$
$$\cdot \sum_{k=1}^{L} (-1)^{k+1}\left(\cos\frac{k\pi}{m+1} - z\right)^{m-1} \sin^2\frac{k\pi}{m+1}\left(1+\rho^2 - {}^2\rho\cos\frac{k\pi}{m+1}\right)^{-1}$$

and

$$G'(z;\rho) = g(R;\rho)$$
$$= (1-\rho^{2m+2})2^m(m-1)(m+1)^{-1}(1-\rho^2)^{-1}(1+\rho^2-2\rho R)^{-m} \tag{3.5}$$
$$\cdot \sum_{k=1}^{L} (-1)^{k+1}\left(\cos\frac{k\pi}{m+1} - R\right)^{m-2}\sin^2\frac{k\pi}{m+1},$$

where

$$\cos\frac{(L+1)\pi}{m+1} \leqq R \leqq \cos\frac{L\pi}{m+1}.$$

For $\rho = 0$ it is clear that

$$g(R) = 2^m(m-1)(m+1)^{-1}\sum_{k=1}^{L}(-1)^{k+1} \tag{3.6}$$
$$\cdot\left(\cos\frac{k\pi}{m+1} - R\right)^{m-2}\sin^2\frac{k\pi}{m+1}$$

and hence

$$g(R;\rho) = (1-\rho^{2m+2})(1-\rho^2)^{-1}(1+\rho^2-2\rho R)^{-m}g(R). \tag{3.7}$$

4. Approximations. In a paper by T. W. Anderson and R. L. Anderson [3] in which the circular serial correlation coefficient is discussed for fitted trigonometric series for the mean, they have fitted the trigonometric series for semi-annual data to correct for variation of period two and get a quadratic form

$$q = X'CX \,/\, X'BX \tag{4.1}$$

for $n = 2m$.

They reduce q to the form

$$\sum_{j=1}^{2m-2} c_j y_j^2 \Big/ \sum_{j=1}^{2m-2} y_j^2, \tag{4.2}$$

where the c_j are identical with the a_j of the previous section.

Therefore the distribution in this particular case for $2m$ observations is exactly the same as that for the non-circular case of Durbin and Watson for $2m - 2$ observations when $\rho = 0$. They also give the approximate distribution of their circular statistic as a beta distribution, and if we put $2m - 2$ in place of $2m$ we get the approximate distribution density of R for $2m - 2$ observations when $\rho = 0$ to be

$$g(R) \sim K(1 - R^2)^p, \tag{4.3}$$

where $p = (m^2 + m)(m - 1)^{-1} - 3/2$.

5. Moments of a ratio. The previous work was based on the assumption that $\mathcal{E}x_i = \mu$ was known. However, if μ is unknown, one of the statistics used is the ratio of the mean square successive difference to the variance;

$$\eta = \frac{\delta^2}{s^2} \tag{5.1}$$

The distribution of η is for the present too difficult for explicit evaluation. The moments of η have been found by Williams [10] and much light shed on the distribution by Von Neumann [8]. However, the expression for the rth moment given by Williams is in terms of the rth derivative of a function.

Durbin and Watson [4] suggested a modified statistic in this case for $n = 2m$. Let

$$R = X'AX \,/\, X'BX \quad \text{or} \quad R = 4(m - 1)\delta_0^2/[(2m - 1)s^2], \tag{5.2}$$

where

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix},$$

$$A_1 = \begin{Bmatrix} 1 & -1 & & & & \\ -1 & 2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & 2 & -1 \\ & & & & -1 & 1 \end{Bmatrix},$$

with latent roots

$$a_j = 4 \sin^2 \frac{(m - j)\pi}{2m} = 4 \cos^2 \frac{j\pi}{2m},$$

and

$$B = \frac{1}{2m} \begin{bmatrix} 2m - 1 & -1 \cdots & -1 \\ -1 & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & -1 \\ -1 & \cdot \cdots & 2m - 1 \end{bmatrix}.$$

The distribution of R is given to be

$$P(R > R') = \sum_{k=1}^{L} (a_k - R')^{m-3/2} a_k^{-\frac{1}{2}} \prod_{\substack{j=1 \\ j \neq k}}^{m-1} (a_k - a_j)^{-1} \tag{5.3}$$

for $a_{k+1} \leqq R \leqq a_k$. This result is based on a result of Anderson's where in addition to double roots there is a single root which is less than all of the double roots. By simplification it can be reduced to

$$P(R > R') = 4m^{-1} \sum_{k=1}^{L} (-1)^{k+1} (a_k - R')^{m-3/2} \sin^2 \frac{k\pi}{2m} \cos \frac{k\pi}{2m} \tag{5.4}$$

for $a_{k+1} \leqq R \leqq a_k$. The moments of this ratio can be easily found for $r < 3m - 1$' since we already have evaluated the moments of the numerator in a previous paper (Geisser [5]) and the moments of the denominator are well known. When the successive observations are independent, the moments of this ratio are the ratio of the moments [9]. Therefore

$$\mathcal{E}R^r = \mu_r = \frac{2^{r+1}(2m + 2r - 2)!(2m^2 - m - r)}{[(2m + r)!(2m - 1)(2m + 1) \cdots (2m + 2r - 3)]}. \tag{5.5}$$

6. The distribution of the modified von Neumann ratio. If we consider the ratio

$$\eta_0 = \frac{2\delta_0^2}{s_0^2} = \frac{\sum_{\substack{i=1 \\ i \neq m}}^{2m-1} (x_{i+1} - x_i)^2}{\sum_{1}^{m} (x_i - \bar{x}_1)^2 + \sum_{m+1}^{2m} (x_i - \bar{x}_2)^2}, \tag{6.1}$$

i.e., twice the ratio of the modified mean square successive difference to the pooled variance, we are able to use the Double Root Result and find the distribution of η_0 for the non-null alternative given by T. W. Anderson [2] if we further consider the model to be made up of two independent sets and

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_1 \end{pmatrix}, \tag{6.2}$$

$$\Lambda_1 = \sigma^{-2} \begin{bmatrix} 1 + \rho^2 - \rho & -\rho & & & \\ -\rho & 1 + \rho^2 & & & \\ & & \ddots & & \\ & & & 1 + \rho^2 & -\rho \\ & & & -\rho & 1 + \rho^2 - \rho \end{bmatrix}. \tag{6.3}$$

As was shown previously [2], η_0 provides a uniformly most powerful one-sided test. By the double root result we get, letting $\lambda_j' = \sigma^2 \lambda_j$,

$$G(\eta_0;\rho) = 1 - 2m^{-1}[1+\rho^2-2\rho+\rho\eta_0]^{2-m}\prod_{j=1}^{m-1}\lambda_j' \cdot \sum_{k=1}^{L}(-1)^{k+1}(a_k-\eta_0)^{m-2}\lambda_k^{-1}\sin^2\frac{k\pi}{m} \tag{6.4}$$

and

$$G'(\eta_0;\rho) = g(\eta_0;\rho) = 2m^{-1}(m-2)[1+\rho^2-2\rho+\rho\eta_0]^{2-m}\prod_{j=1}^{m-1}\lambda_j' \cdot \sum_{k=1}^{L}(-1)^{k+1}(a_k-\eta_0)^{m-3}\sin^2\frac{k\pi}{m}, \tag{6.5}$$

where $a_{L+1} \leqq \eta_0 \leqq a_L$. For $\rho = 0$,

$$G(\eta_0) = 2(m-2)m^{-1}\sum_{k=1}^{L}(a_k-\eta_0)^{m-2}(-1)^{k+1}\sin^2\frac{k\pi}{m} \tag{6.6}$$

and

$$G'(\eta_0) = g(\eta_0) = 2(m-2)m^{-1}\sum_{k=1}^{L}(-1)^{k+1}(a_k-\eta_0)^{m-3}\sin^2\frac{k\pi}{m} \tag{6.7}$$

for $a_{L+1} \leqq \eta_0 \leqq a_L$. Hence

$$G'(\eta_0;\rho) = g(\eta_0;\rho) = (1+\rho^2-2\rho+\rho\eta_0)^{1-m}\left(\prod_{j=1}^{m-1}\lambda_j'\right)g(\eta_0); \tag{6.8}$$

and since

$$\prod_{j=1}^{m-1}\lambda_j' = \frac{(1-\rho)(1-\rho^{2m})}{(1+\rho)(1-\rho)^2} = \frac{1-\rho^{2m}}{1-\rho^2},$$

$$G'(\eta_0;\rho) = (1+\rho^2-2\rho+\rho\eta_0)^{1-m}(1-\rho^{2m})(1-\rho^2)^{-1}g(\eta_0). \tag{6.9}$$

It is also quite easy to find the moments $\mathcal{E}\eta_0^r$ when $\rho = 0$ since we have already given the moments of the numerator for $r < 3m - 1$ and the moments of the denominator are well known. Hence for $r < 3m - 1$

$$\mathcal{E}\eta_0^r = \frac{2^{1-r}(2m^2-m-r)(2m+2r-2)!(m-2)!}{(m+r-2)!(2m+r)!}. \tag{6.10}$$

7. Acknowledgements. I am grateful to Dr. Harold Hotelling for his guidance and to the referee for his helpful comments.

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