REMARKS CONCERNING CHARACTERISTIC FUNCTIONS

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1. Summary. In the first part of this note, we study functions of characteristic functions which are themselves characteristic functions and discuss also a property of analytic characteristic functions. In the second part, an example is constructed to answer a question raised by D. Dugué [3].

2. Functions of characteristic functions. Let $F(x)$ be a distribution function, that is, a never-decreasing function which is continuous to the right and is such that $F(-\infty) = 0$ while $F(+\infty) = 1$. Its Fourier transform

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x) \tag{1}$$

is called the characteristic function of the distribution $F(x)$. Characteristic functions are very important in probability theory, and in the following discussion we shall use some of their well-known properties which may be found in books [1], [5] on the subject. We first derive a theorem which shows how given characteristic functions may be transformed into new characteristic functions.

Theorem 1. *Let $\{\phi_v(t)\}$ be an arbitrary sequence of characteristic functions and $\{a_v\}$ be a sequence of real numbers. The necessary and sufficient condition that*

$$f(t) = \sum_{v=0}^{\infty} a_v\, \phi_v(t) \tag{2}$$

should be a characteristic function for every sequence $\{\phi_v(t)\}$ of characteristic functions is that

$$a_v \geqq 0, \qquad \sum_{v=0}^{\infty} a_v = 1. \tag{3}$$

We first show that the condition is sufficient. Let m ($m \geqq 0$) be a subscript such that a_m is the first non-vanishing element of the sequence $\{a_v\}$. We denote by

$$g_n(t) = \left[\sum_{v=0}^{m+n} a_v\, \phi_v(t)\right] \Big/ \left[\sum_{v=0}^{m+n} a_v\right] \qquad \text{for } n = 0, 1, 2, \cdots.$$

If (3) is satisfied, then $g_n(t)$ is a linear combination of a finite number of characteristic functions. The coefficients in this linear combination are non-negative and their sum is one; therefore $g_n(t)$ is also a characteristic function. We see

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then from P. Lévy's continuity theorem that $f(t) = \lim_{n\to\infty} g_n(t)$ is also a characteristic function.

To prove the necessity of the condition, we assume that $f(t)$ as given by (2) is a characteristic function for any sequence $\phi_v(t)$ of characteristic functions. Let $\phi_v(t) = e^{itv}$; then $f(t) = \sum_{v=0}^{\infty} a_v e^{itv}$. This is the Fourier transform of a step function with jumps a_v at the points $v = 0, 1, 2, \cdots$. Since $f(t)$ is by assumption a characteristic function, this step function must be a discrete probability distribution; therefore $a_v \geqq 0$, $\sum a_v = 1$, so that Theorem 1 is established.

An application of some interest is obtained by putting $\phi_n(t) = n^{-it} = \exp[-it(\ln n)]$ and $a_n = n^{-\sigma}/\sum_1^{\infty} n^{-\sigma}$, where $\sigma > 1$. It follows then from Theorem 1 that the function $f(t) = \zeta(\sigma + it)/\zeta(\sigma)$ is a characteristic function for $\sigma > 1$. Here $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is Riemann's zeta function and $s = \sigma + it$, with σ, t real and $\sigma > 1$.

This result was already obtained in a different manner by B. V. Gnedenko and A. N. Kolmogorov ([7], p. 75) who showed that $\zeta(\sigma + it)/\zeta(\sigma)$ is the characteristic function of an infinitely divisible distribution.

Next, we let $\phi(t)$ be an arbitrary characteristic function and put $\phi_v(t) = [\phi(t)]^v$, $v = 0, 1, 2, \cdots$. We obtain then

Corollary to Theorem 1. *Let $\phi(t)$ be a characteristic function and let $G(z)$ be a function of the complex variable z, which is regular in $|z| < R$ where $R > 1$. The function $G[\phi(t)]$ is also a characteristic function if, and only if, $G(z)$ has a power-series expansion about the origin with non-negative coefficients and if $G(1) = 1$.*

It is worth while to remark that the class of functions $G(z)$ which have the property that $G[\phi(t)]$ is a characteristic function whenever $\phi(t)$ is a characteristic function includes also functions which are not analytic. An example is the function $G(z) = |z|^2$. The restriction in the corollary that $G(z)$ should be regular is therefore somewhat artificial.

Definition. A distribution is said to be infinitely divisible if for every positive integer n its characteristic function is the nth power of some characteristic function.

By means of the corollary to Theorem 1 we obtain the following result.

Theorem 2. *Let $\phi(t)$ be an arbitrary characteristic function and p a real number such that $p > 1$; then*

$$\psi(t) = \frac{p-1}{p - \phi(t)} \tag{4}$$

is the characteristic function of an infinitely divisible law.

To prove Theorem 2 we let n be a positive integer and consider the function

$$G(z) = \left[\frac{p-1}{p-z}\right]^{1/n}.$$

Here it is understood that $G(z)$ is the principal value of the power on the right-hand side. Clearly

$$G(z) = \left[\frac{p-1}{p}\right]^{1/n} \left[\frac{1}{1-z/p}\right]^{1/n} = \left[\frac{p-1}{p}\right]^{1/n} \left[1 - \frac{z}{p}\right]^{-1/n}.$$

We expand $G(z)$ according to the binomial theorem and see that

$$G(z) = \left[\frac{p-1}{p}\right]^{1/n} \left\{1 + \sum_{k=1}^{\infty} \frac{(1+n)(1+2n)\cdots(1+\overline{k-1}n)}{(np)^k k!} z^k\right\}.$$

This shows that for any positive integer n the conditions of the corollary are satisfied. The function $G[\phi(t)] = \{[p-1]/[p-\phi(t)]\}^{1/n}$ is therefore a characteristic function for any positive integer n; in other words, $\psi(t)$, as given by (4), is the characteristic function of an infinitely divisible law.

In a similar manner we derive from the corollary to Theorem 1 a theorem which is due to Bruno de Finetti [4].

THEOREM OF DE FINETTI. *If $\phi(t)$ is an arbitrary characteristic function, and if p is a positive real number, then $\psi(t) = \exp\{p[\phi(t) - 1]\}$ is the characteristic function of an infinitely divisible law.*

The function $G(z) = e^{p(z-1)}$ satisfies the assumptions of the corollary, so that we see immediately that $\psi(t)$ is a characteristic function for any $p > 0$. It follows then from its functional form that it must be the characteristic function of an infinitely divisible law.

3. A remark concerning analytic characteristic functions. A characteristic function is said to be an analytic characteristic function if it is an analytic function which coincides in some neighborhood of the origin with a characteristic function.

In an earlier paper [6] the following result was obtained:

THEOREM 4 of [6]. *Let $\phi(t)$ be the characteristic function of an infinitely divisible law and assume that $\phi(t)$ is an analytic characteristic function. Then $\phi(t)$ has no zeros inside its strip of convergence.*

In the following we show that this statement cannot be improved. This is done by constructing an analytic characteristic function of an infinitely divisible law which has zeros on the boundary of its strip of convergence.

Let $a > 0$, $b > 0$ be two real numbers and put $w = a + ib$; the function

$$\phi(t) = \frac{(1 - it/w)(1 - it/\bar{w})}{(1 - it/a)^2} \tag{5}$$

is then a characteristic function. This is seen immediately if we write $\phi(t) = p + (1-p)(1 - it/a)^{-2}$, where $p = a^2/(a^2 + b^2)$. We define

$$m = 2\int_0^\infty \frac{e^{-ax}(1 - \cos bx)}{1 + x^2}\, dx \tag{6a}$$

$$M(x) = \begin{cases} 0 & \text{for } x < 0 \\ -2\int_x^\infty e^{-at}(1 - \cos bt)t^{-1}\, dt & \text{for } x > 0. \end{cases} \tag{6b}$$

The function $M(x)$ is real and non-decreasing in $(-\infty, 0)$ and in $(0, \infty)$. Moreover, $M(-\infty) = M(+\infty) = 0$; and the integral $\int u^2\, dM(u)$ is finite over every finite interval. According to P. Lévy's representation theorem ([5] p. 180), the function

$$\psi(t) = mit + \int_{-\infty}^{\infty} \left(e^{itx} - 1 - \frac{itx}{1 + x^2}\right) dM(x) \tag{7}$$

is the logarithm of the characteristic function of an infinitely divisible law. We write

$$I(t) = \int_0^{\infty} \left(e^{itx} - 1 - \frac{itx}{1 + x^2}\right) e^{-ax}(1 - \cos bx)\, \frac{dx}{x}$$

and obtain $\psi(t) = mix + 2I(t)$.

It is easily seen that it is permissible to differentiate $I(t)$ under the integral sign. A simple computation gives

$$2I'(t) = 2\, \frac{i/a}{1 - it/a} - \frac{i/w}{1 - it/w} - \frac{i/\bar{w}}{1 - it/\bar{w}} - im.$$

Considering $I(0) = 0$, we see that $\psi(t) = \log \phi(t)$, where $\phi(t)$ is given by (5).

We finally remark that it is possible to use Theorem 2 to construct characteristic functions of infinitely divisible laws which have zeros arbitrarily close to the boundary of the strip of convergence. As an example we mention $\psi(t) = (p - 1)/[p - \phi(t)]$, where $\phi(t) = (1 - it/\alpha)^{-1}$. The function $\psi(t)$ then has the zero $t_0 = -i\alpha$, and the boundary of its region of convergence is the line $\mathrm{Im}\,(t) = -\alpha(p - 1)/p$. By selecting p large enough, the distance between the boundary and t_0 can be made arbitrarily small.

4. A question raised by D. Dugué. In this section we are concerned with certain factorizations of non-infinitely divisible laws. The uniform (rectangular) distribution has the characteristic function $(\sin t)/t$; it is not infinitely divisible, since it has real zeros. It is well known that it has the factor $\sin (t/n)/(t/n)$ for every positive integer n. The uniform distribution is therefore an example of a law which is not infinitely divisible but has an enumerable infinity of different factors. These factors, with characteristic function $\sin (t/n)/(t/n)$, depend on a discrete parameter n.

In a recent paper [3], D. Dugué raised the question of whether there exists a law which is not infinitely divisible but has a non-enumerable set of factors depending on a continuous parameter. As an example for such a distribution Dugué uses the Laplace distribution. This example is, however, invalid, since the Laplace distribution is infinitely divisible. The purpose of this section is to answer Dugué's question in the affirmative by giving an example of a probability law with the desired properties. This example will be a rational characteristic function; for its construction we use the following lemma.

LEMMA 1. *Let*

$$(9)\qquad \phi(t) = \frac{\left(1+\dfrac{it}{w}\right)\left(1+\dfrac{it}{\bar{w}}\right)}{\left(1-\dfrac{it}{a}\right)\left(1-\dfrac{it}{v}\right)\left(1-\dfrac{it}{\bar{v}}\right)},$$

where $v = a + ib$, $w = \alpha + i\beta$, *and* $a > 0$, $b > 0$, $\alpha > 0$, $\beta > 0$. *The function* $\phi(t)$ *is a characteristic function if, and only if, one of the following two, mutually exclusive, conditions holds:*

(i) $\beta = \sqrt{b^2 - (a+\alpha)^2} \geqq \sqrt{3}(a+\alpha)$;
(ii) $\beta \neq \sqrt{b^2 - (a+\alpha)^2}$ *and simultaneously* $\beta^2 \geqq (a+\alpha)^2 + b^2/2$.

PROOF. We denote by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)\,dt$$

and obtain by a simple computation

$$(10)\qquad f(x) = \begin{cases} Ce^{-ax}\left\{1 - \left[\dfrac{c^2-b^2}{c^2}\right]\cos bx - \dfrac{2bd}{c^2}\sin bx\right\} & \text{if } x > 0; \\ 0 \quad \text{otherwise.} \end{cases}$$

Here

$$(10\text{a})\qquad \begin{cases} d = a + \alpha, \\ c^2 = (a+\alpha)^2 + \beta^2 = d^2 + \beta^2, \\ C = \dfrac{av\bar{v}c^2}{w\bar{w}b^2}. \end{cases}$$

The function $f(x)$ is real and $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Therefore we conclude that the function $\phi(t)$ is a characteristic function if, and only if, the trigonometric polynomial

$$(11)\qquad h(x) = 1 - \left[\frac{c^2-b^2}{c^2}\right]\cos bx - \frac{2bd}{c^2}\sin bx$$

is nonnegative.

We assume first that $c^2 = b^2$ or, considering (10a), that $\beta^2 = b^2 - (a+\alpha)^2$. Then $h(x) = 1 - 2d/b\sin bx$ and $h(x) \geqq 0$ if, and only if, $2d/b \leqq 1$. It is seen by simple algebra that this is equivalent to condition (i). We suppose next that $c^2 \neq b^2$ and determine by elementary considerations the smallest value of $h(x)$, which is

$$\min_{-\infty<x<+\infty} h(x) = 1 - \frac{|c^2-b^2|}{c^2}\cdot\frac{1}{\cos bx_0},$$

where $-\pi/2b < x_0 < +\pi/2b$ and $\tan bx_0 = 2bd/(c^2 - b^2)$. The function $h(x)$ is therefore non-negative if, and only if, $1 \geqq |c^2 - b^2|/(c^2 \cos bx_0)$. This condition leads easily to (ii), so that the lemma is established.

We use this lemma, together with the theorem quoted at the beginning of Section 3, to construct a characteristic function with the desired properties.

Let $a, \alpha_1, \alpha_2, \beta_1, \beta_2, b$ be arbitrary positive numbers such that $\alpha_2 > \alpha_1$ and also

$$\beta_j^2 > \max\left[b^2 - (a + \alpha_j)^2; \qquad (a + \alpha_j)^2 + \frac{b^2}{2}\right] \qquad (j = 1, 2). \tag{12}$$

We define $v = a + ib$, $w_1 = \alpha_1 + i\beta_1$, and $w_2 = \alpha_2 + i\beta_2$. The functions

$$\phi_j(t) = \frac{\left(1 + \frac{it}{w_j}\right)\left(1 + \frac{it}{\bar{w}_j}\right)}{\left(1 - \frac{it}{a}\right)\left(1 - \frac{it}{v}\right)\left(1 - \frac{it}{\bar{v}}\right)} \qquad (j = 1, 2)$$

are then characteristic functions, since they satisfy condition (ii) of the preceding lemma. Therefore

$$\phi_3(t) = \phi_1(t)\phi_2(t) = \frac{\left(1 + \frac{it}{w_1}\right)\left(1 + \frac{it}{\bar{w}_1}\right)\left(1 + \frac{it}{w_2}\right)\left(1 + \frac{it}{\bar{w}_2}\right)}{\left(1 - \frac{it^2}{a}\right)\left(1 - \frac{it}{v}\right)^2\left(1 - \frac{it}{v}\right)^2} \tag{13}$$

is also a characteristic function. It is also known that

$$\phi_4(t) = \left[\left(1 + \frac{it}{\alpha_2}\right)\left(1 + \frac{it}{w_2}\right)\left(1 + \frac{it}{\bar{w}_2}\right)\right]^{-1}$$

is a characteristic function; we conclude then that this is also true for $\phi(t) = \phi_4(t)\phi_3(t)$; i.e.,

$$\phi(t) = \frac{\left(1 + \frac{it}{w_1}\right)\left(1 + \frac{it}{\bar{w}_1}\right)}{\left(1 + \frac{it}{\alpha_2}\right)\left(1 - \frac{it}{a}\right)^2\left(1 - \frac{it}{v}\right)^2\left(1 - \frac{it}{\bar{v}}\right)^2}. \tag{14}$$

The function $\phi(t)$ is an analytic characteristic function which has the strip $\alpha_2 > I(t) > -a$ as its strip of convergence. Its zeros iw_1 and $i\bar{w}_1$ are located inside this strip, so that $\phi(t)$ cannot be the function of an infinitely divisible law. Similarly, $\phi_3(t)$ is not infinitely divisible, since its strip of convergence is the half plane $I(t) > -a$ in which it has four zeros. We have, therefore, an example of a law $\phi(t)$, which is not infinitely divisible and which has, nevertheless, a non-enumerable infinity of not infinitely divisible factors $\phi_3(t)$. These factors depend on a continuous parameter β_2, which is subject only to the restriction (12).

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