SUMS OF RANDOM PARTITIONS OF RANKS[1]

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1. Summary. Suppose that the integers 1, 2, $\cdots$, N are randomly distributed among k distinguishable classes with equal probability and without restrictions. It is natural to denote the class sums by $s_1, s_2, \cdots, s_k$ and largest of these by S. A generating function is obtained for the upper half of the range of S, namely $\frac{1}{4}N(N-1) \leqq S \leqq \frac{1}{2}N(N-1)$. For $k \leqq 6$, this is shown to provide the usual percentage points for N up to and beyond 10. Tables of 5% and 1% points are provided for $k = 2, 3, 6$ and $N = 1(1)10$.

For $k = 2$, the distribution is that of Wilcoxon's paired sample test [3]. This suggests the application of $k = 6$ to the six possible orders of three responses. This is a possible procedure but the peculiarities of its power are such that its use is not recommended.

However, when three treatments with a natural order are examined in randomized blocks, a significance procedure can be based on the same distribution which is specifically sensitive to average responses in either exactly the same or exactly the opposite order as the treatments. 5% and 1% levels are given for $N = 1(1)10$. The procedure may be promising.

The basic distribution used here is inappropriate for situations, such as analysis of variance of ranks, where the number of ranks in each class is restricted, as by being the same in all classes.

2. Discussion. The announced significance levels are given in Table 1, both in terms of the rank sum for all but the weightiest class, and in terms of the largest rank sum.

A set of observations on 3 treatments falls into one of 6 orders. Thus each block of a 3-treatment randomized block design falls into one of 6 classes. If we have assigned ranks to the blocks in some way that is independent of which treatment is which, we may regard the ranks as assigned to these 6 classes. If the complete null hypothesis is correct—if all three treatments are equivalent—then the ranks are assigned at random and the distribution applies.

A significant result thus corresponds to either (i) an unlikely event, or (ii) a situation where at least one treatment differs from the other two. Such a test is a portmanteau test, and, compared to other three-sample tests, may be expected to have somewhat better power when all three treatments differ notably, this increase being obtained at the expense of much decreased power when two out of the three treatments are equivalent.

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TABLE 1

5% and 1% significance levels for the sum of all other classes (out of k) but the weightiest (first entries, small values significant) and the largest sum of any one of k classes (last entries, large values significant) together with actual probability levels (in central parentheses)

N	k = 2		k = 3		k = 6	
	5%	1%	5%	1%	5%	1%
1	.( ).	.( ).	.( ).	.( ).	.( ).	.( ).
2	.( ).	.( ).	.( ).	.( ).	.( ).	.( )..
3	.( ).	.( ).	.( ).	.( ).	0(.027)6	.( ).
4	.( ).	.( ).	0(.037)10	.( ).	2(.051)8	0(.005)10
5	0(.062)15	.( ).	1(.037)14	0(.012)15	4(.055)11	2(.008)13
6	1(.062)20	.( ).	3(.045)18	1(.012)20	6(.039)15	4(.009)17
7	2(.047)26	.( ).	5(.037)23	2(.007)26	10(.053)18	7(.011)21
8	4(.039)32	0(.008)36	8(.044)28	4(.008)32	14(.045)22	10(.009)26
9	6(.055)39	2(.012)43	11(.042)34	7(.010)38	20(.049)25	15(.008)30
10	8(.049)47	3(.010)52	15(.045)40	10(.010)45	24(.043)31	19(.009)36

All this will be true, whatever basis we choose for ranking the blocks. We will ameliorate the situation if we favor blocks in which all three treatments appear quite distinct—favor them by assigning them high ranks. A plausible choice is to rank blocks according to the least difference among the three responses.

As an example, consider data on mean head breadths of termites due originally to Warren [2] and utilized by Tippett [1]. A portion of the data, where months are taken as blocks, is shown, and analyzed, in Table 2. The lowest rank sum for all other classes is 5, which does not reach the 5% level of 4.

This technique is not, for the present, recommended for use.

3. Ordered treatments. In the example just discussed, the order of nest numbers (presumably) was not expected to be related in any particular way to the order of the nest number averages. There are situations, however, where there is a natural order for the treatments, such that the treatment averages, if different, may be reasonably expected to fall either in the same order or in the opposite order. In such a situation, provided we agree to look only at (i) rank sums for classes not exactly in treatment order, or (ii) rank sums for classes just exactly opposite to treatment order, we can gain a factor of 3 in our significance calculations and may use the significance levels of Table 3. There is no need for us to retain the same system of ranking blocks. It now seems better to rank according to the *least differences between responses to adjacent treatments.*

For an example of the use of this table, we may return to the same data, taking nests as blocks, and the months of March, May and November as the ordered treatments. The data and analysis are given in Table 4. The rank sum outside of the exactly opposite order is 0, and Table 3 shows that this is far beyond the 1% level. (Reference to Table 6 shows it to be at the 0.03% level.)

TABLE 2

Analysis of part of Warren's [2] data on mean head breadths of termites in mm. Nests as unordered treatments. Months as blocks.

	Nest Number (less 670)			Min. Diff.	Rank	Order
	2	4	5			
Nov.	2.404	2.447	2.456	.009	1	245
Jan.	2.457	2.388	2.626	.079	5	425
Mar.	2.452	2.515	2.633	.063	4	245
May	2.396	2.445	2.487	.042	3	245
Aug.	2.279	2.312	2.410	.033	2	245

Order	Rank Sum	Sum for Other Orders
245	10	5
425	5	10
Other 4	0	15

TABLE 3

5% and 1% levels for the smallest rank sum for "A and other" or "B and other" (first entries, small values significant) when ranks are randomly allotted to A, to B, and to each of 4 other classes with equal probability (Actual probability levels in parentheses.)

N	5%	1%
1	—	—
2	0(.056)	—
3	1(.056)	0(.009)
4	3(.063)	1(.009)
5	5(.033)	3(.011)
6	9(.053)	6(.013)
7	13(.057)	9(.009)
8	17(.047)	13(.010)
9	23(.056)	18(.009)
10	?	23(.011)

It may well be that this sort of procedure for ordered treatments may prove useful. Further development, both of tables, and of methods of calculating tables, is likely to be required.

4. Derivation. Obviously, no two of the class sums s_i can both be greater than half the total. Thus, if $m > \frac{1}{4}N(N+1)$, the probability that S is greater than or equal to m is k times the probability that any one s_i, say s_1, is greater than m.

Write

$$s_1 = \tfrac{1}{2}N(N+1) - i,$$

where i is at present unrestricted. Let $a_{i,N}$ be the number of arrangements of

TABLE 4

Analysis of part of Warren's [2] data on mean head breadths of termites in mm. Months as ordered treatments. Nests as blocks. (Scoring: + for Mar. < May < Aug.; − for Mar. > May > Aug.; 0 for any other order)

Nest Number	Mar.	May	Aug.	Min. Diff.	Rank	Class
668	2.375	2.373	2.318	.002	1	—
670	2.613	2.557	2.377	.056	2.5	—
672	2.452	2.396	2.279	.056	2.5	—
674	2.515	2.445	2.312	.070	4	—
675	2.633	2.487	2.410	.077	5	—

Class	Rank Sum	Remainder
+	0	15
−	15	0
0	0	15

TABLE 5

Values of $b_{i,N}$ (see Sect. 4) for $k = 3$

i \ N	1		3	4	5	6	7	8	9	10
0	1	(Constant within rows)								
1	3									
2	3	5								
3	3	9	11							
4	3	9	15	17						
5	3	9	19	25	27					
6			27	37	43	45				
7				49	59	65	67			
8				57	79	89	95	97		
9				65	99	121	131	137	139	
10				81	131	165	187	197	203	205
11		(Constant within			155	209	243	265	275	281
12		columns)			179	265	319	353	375	385
13					195	313	403	457	491	513
14					211	369	499	589	643	677
15					243	441	619	753	843	897
k^{N-1}	1	3	9	27	81	243	729	2,187	6,561	19,683

1, 2, $\cdots$, N into k distinguishable classes for which i has a given value. Then the generating function of

$$i = s_2 + s_3 + \cdots + s_k$$

is

$$g_N(x) = \sum_i a_{i,N} x^i = \prod_{j=1}^{N} \{1 + (k-1)x^j\},$$

TABLE 6

Values of $b_{i,N}$ (see Sect. 4) for $k = 6$

i \ N	1	2	3	4	5	6	7	8	9	10
0	1	(Constant within a row)								
1	6									
2	6	11								
3	6	36	41							
4	6	36	66	71						
5	6	36	91	121	126					
6			216	271	301	306				
7				421	476	506	511			
8				546	751	806	836	841		
9				671	1,026	1,231	1,286	1,316	1,321	
10				1,296	1,901	2,256	2,461	2,516	2,546	2,551
11					2,651	3,281	3,636	3,841	3,896	3,926
12					3,401	4,906	5,536	5,891	6,096	6,151
13					4,026	6,406	7,936	8,566	8,921	9,126
14					4,651	8,406	10,936	12,466	13,096	13,451
15					7,776	12,906	16,936	19,491		21,651
16		(Constant within a column)				17,281	23,436	27,616	21,021	31,701
17						21,031	32,311	38,741	30,171	45,501
18						24,781	41,186	53,491	42,946	64,276
19						27,906	52,409	70,589	60,071	89,774
20						31,031	63,061	90,741	83,169	122,676
21						46,656	88,686	128,366	157,821	176,301
22							111,186	165,866	208,696	231,176
23							133,061	217,741	280,071	314,676
24							151,811	268,991	366,446	431,926
25							170,561	332,116	470,196	575,301
k^{N-1}	1	6	36	216	1,296	7,776	46,656	279,936	1,679,616	10,077,696
$3(6^{N-1})$	3	18	108	648	3,888	23,328	139,968	838,558	5,038,848	30,233,088

where the second form is obtained by the following argument: The integer j may be placed in the first class in one way, and in one of the others in $k - 1$ ways. Its contribution to i is zero in the first case and j in the others. This is represented by the factor $1 + (k - 1)x^j$.

The cumulative distribution of i has the generating function

$$h_N(x) = \frac{g_N(x)}{1 - x} = \sum b_{i,N} x^i,$$

where, of course,

$$b_{i,N} = \sum_{k \leq i} a_{k,N}.$$

Now,

$$g_N(x) = \{1 + (k - 1)x^N\} g_{N-1}(x),$$

so that

$$h_N(x) = \{1 + (k - 1)x^N\}h_{N-1}(x).$$

Thus,

$$b_{i,N} = b_{i,N-1} + (k - 1)b_{i-N,N-1}\,.$$

The probability that S is greater than or equal to $\frac{1}{2}N(N + 1) - i$ is, by our earlier argument,

$$\frac{kb_{i,N}}{k^N} = b_{i,N}\,k^{-(N-1)}$$

so long as $i < \frac{1}{4}N(N + 1)$.

In case only two of 6 classes are to be considered, we need only multiply by 2 instead of 6, obtaining

$$\frac{2b_{i,N}}{6^N} = \frac{b_{i,N}}{3(6^{N-1})}.$$

5. Values of $b_{i,N}$. We tabulate some values of $b_{i,N}$ for reference in Table 5($k = 3$) and Table 6($k = 6$).

Note that the values for $i \geqq N(N + 1)/4$ must be calculated for convenient recursion, although they are not related to the actual problem. The recursive process used can be illustrated from Table 5, where 139 = 137 + 2(1), 203 = 197 + 2(3), 275 = 265 + 2(5), 375 = 353 + 2(11), 491 = 457 + 2(17) in the next to last column, while 37 = 27 + 2(5), 49 = 27 + 2(11), 57 = 27 + 2(15), 65 = 27 + 2(19), 81 = 27 + 2(27) in the fourth column.

REFERENCES

[1] L. H. C. Tippett, *The Methods of Statistics*, 4th ed., John Wiley, New York, 1952, especially pp. 182–183.

[2] E. Warren, "Some statistical observations on termites," *Biometrika*, Vol. 6 (1909), pp. 329–347.

[3] Frank Wilcoxon, "Probability tables for individual comparisons by ranking methods," *Biometrics*, Vol. 3 (1947), pp. 119–122.