ANOTHER COUNTABLE MARKOV PROCESS WITH ONLY INSTANTANEOUS STATES

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Let P be the transition function for a Markov process with a countable state space A and stationary transition probabilities; i.e., P is a nonnegative function defined for all triples (a, b, t) with $a \varepsilon A$, $b \varepsilon A$, and t a nonnegative real number, satisfying

$$(1) \qquad P(a, b, 0) = 1 \text{ if } a = b, \qquad 0 \text{ if } a \neq b,$$

$$(2) \qquad \sum_b P(a, b, t) = 1 \quad \text{for all} \quad a, t,$$

and

$$(3) \qquad P(a, b, s + t) = \sum_{c \varepsilon A} P(a, c, s)P(c, b, t) \quad \text{for all} \quad s \geqq 0, \qquad t \geqq 0, a, b.$$

We shall suppose, as usual, that P is continuous at $t = 0$; i.e.,

$$(4) \qquad P(a, a, t) \rightarrow 1 \text{ as } t \rightarrow 0 \text{ for all } a.$$

It is well known that, for any P satisfying (1), (2), (3), and (4), $P'(a, a, 0)$ exists for all a (it may be negatively infinite). Following P. Lévy [2], a state is called "instantaneous" if $P'(a, a, 0) = -\infty$. Examples of processes with all states instantaneous have been given by Feller and McKean [2] and by Dobrushin [1]. The purpose of this note is to describe a third example, somewhat simpler than those previously given.

We first describe the process informally, after which we define P and verify (1), (2), (3), and (4) and $P'(a, a, 0) = -\infty$ for all a directly. Let $X_1(t)$, $X_2(t)$, $\cdots$ be a sequence of Markov processes, independent of each other, each with two states 0 and 1. We suppose $X_n(0) = 0$ for all n. Let $X_n(t)$ be characterized by the parameters λ_n, μ_n:

$$\Pr\{X_n(t + h) = 1 \mid X_n(t) = 0\} = \lambda_n h + o(h),$$

$$\Pr\{X_n(t + h) = 0 \mid X_n(t) = 1\} = \mu_n h + o(h).$$

Our process $X(t)$ will be the joint process $X_1(t)$, $X_2(t)$, $\cdots$ which is clearly a Markov process. To insure that $X(t)$ has only a countable set of states, we

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determine λ_n, μ_n so that, at each time t, with probability 1, $X_n(t) = 0$ for all but a finite number of n. Since

$$\Pr(X_n(t) = 0 \mid X_n(0) = 0) = \frac{\mu_n}{\mu_n + \lambda_n} + \frac{\lambda_n}{\mu_n + \lambda_n} e^{-(\lambda_n+\mu_n)t}$$

$$\geqq \frac{\mu_n}{\mu_n + \lambda_n},$$

this will occur if

$$\prod_n \frac{\mu_n}{\mu_n + \lambda_n} > 0, \tag{5}$$

i.e.,

$$\sum_n \frac{\lambda_n}{\lambda_n + \mu_n} < \infty.$$

A state is instantaneous if and only if the probability of remaining in it throughout an interval is zero. Since the probability that $X_n(t) = 0$ throughout T, $T + h$ given that $X_n(T) = 0$ is $e^{-\lambda_n h}$, the chance that the state $X(T)$ with $X_n(T) = 0$ for $n \geqq N$ will persist throughout T, $T + h$ is at most

$$\prod_N^\infty e^{-\lambda_n h} \doteq e^{-h(\lambda_N + \lambda_{N+1} + \cdots)},$$

and will be zero if

$$\sum_n \lambda_n = \infty. \tag{6}$$

Thus any choice of $\{\lambda_n\}$, $\{\mu_n\}$ satisfying (5) and (6) yields an example of a process with only instantaneous states.

Formally, the set A of states is the set of all infinite sequences

$$a = (\epsilon_1, \epsilon_2, \cdots)$$

of 0's and 1's with only finitely many 1's. Let $\{\lambda_n\}$, $\{\mu_n\}$ be sequences of positive numbers satisfying (5) and (6), let

$$R_n(0, 0, t) = \frac{\mu_n}{\mu_n + \lambda_n} + \frac{\lambda_n}{\mu_n + \lambda_n} e^{-(\lambda_n+\mu_n)t},$$

$$R_n(1, 1, t) = \frac{\lambda_n}{\mu_n + \lambda_n} + \frac{\mu_n}{\mu_n + \lambda_n} e^{-(\lambda_n+\mu_n)t},$$

$$R_n(0, 1, t) = 1 - R_n(0, 0, t),$$

$$R_n(1, 0, t) = 1 - R_n(1, 1, t),$$

and define, for any two states $a = (\epsilon_1, \epsilon_2, \cdots)$ and $b = (\delta_1, \delta_2, \cdots)$ and any $t \geqq 0$,

$$P(a, b, t) = \prod_{n=1}^{\infty} R_n(\epsilon_n, \delta_n, t). \tag{7}$$

Denote by A_N the set of all states $a = (\epsilon_1, \epsilon_2, \cdots)$ with $\epsilon_n = 0$ for all $n > N$. For $a \varepsilon A_N$ and any $M \geqq N$, we have

$$\sum_{b \varepsilon A_M} P(a, b, t) = h_M(t) \sum_{\delta_1, \cdots, \delta_M} \prod_{1}^{M} R_n(\epsilon_n, \delta_n, t)$$

$$= h_M(t) \prod_{1}^{M} (R_n(\epsilon_n, 0, t) + R_n(\epsilon_n, 1, t)) = h_M(t),$$

where

$$h_M(t) = \prod_{M+1}^{\infty} R_n(0, 0, t) \geqq \prod_{M+1}^{\infty} \frac{\mu_n}{\mu_n + \lambda_n} = V_M. \tag{8}$$

From (8), $h_M(t) \to 1$ as $M \to \infty$, so that (2) is verified. For (3), say $a \epsilon A_N$, $b \epsilon A_N$. For $M \geqq N$,

$$\sum_{c \epsilon A_M} P(a, c, s)P(c, b, t)$$

$$= h_M(s)h_M(t) \sum_{\alpha_1, \cdots, \alpha_M} \prod_{n=1}^{M} R_n(\epsilon_n, \alpha_n, s)R_n(\alpha_n, \delta_n, t)$$

$$= h_M(s)h_M(t) \prod_{n=1}^{M} \left(\sum_{\alpha=0}^{1} R_n(\epsilon_n, \alpha, s)R_n(\alpha, \delta_n, t) \right)$$

$$= h_M(s)h_M(t) \prod_{n=1}^{M} R_n(\epsilon_n, \delta_n, s + t) \to P(a, b, s + t) \quad \text{as} \quad M \to \infty.$$

For (4), if $a \varepsilon A_N$ and $M \geqq N$,

$$P(a, a, t) \geqq \left(\prod_{1}^{M} R_n(\epsilon_n, \epsilon_n, t) \right) V_M,$$

so that

$$\liminf_{t \to 0} p(a,a,t) \geqq V_M.$$

Since this holds for all M and $V_M \to 1$ as $M \to \infty$, (4) is verified. Finally, since, for $a \epsilon A_N$ and $M \geqq N$ we have, for all $k \geqq 1$

$$P(a, a, t) \leqq h_{M,k}(t) = \prod_{M+1}^{M+k} R_n(0, 0, t),$$

and since $P(a, a, 0) = h_{M,k}(0) = 1$,

$$P'(a, a, 0) \leqq h'_{M,k}(0) = -\sum_{M+1}^{M+k} \lambda_n,$$

so that (6) implies $P'(a, a, 0) = -\infty$.

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SPACINGS GENERATED BY MIXED SAMPLES

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1. Summary and introduction. Suppose $X(1, 1), X(1, 2), \cdots, X(1, n_1), X(2, 1), \cdots, X(2, n_2), \cdots, X(k, 1), \cdots, X(k, n_k)$ are independent chance variables, $X(i, j)$ having the probability density function $f_i(x)$, for $j = 1, \cdots, n_i$, $i = 1, \cdots, k$. We assume that for each i, $f_i(x)$ is bounded and has at most a finite number of discontinuities. We denote $n_1 + n_2 + \cdots + n_k$ by N, and we assume that n_i/N is equal to r_i, where r_i is a given positive number. Let $Y_1 \leqq Y_2 \leqq \cdots \leqq Y_N$ denote the ordered values of the N observations

$$X(1, 1), \cdots, X(k, n_k).$$

Define W_i as $Y_{i+1} - Y_i$ for $i = 1, \cdots, N - 1$. For any given nonnegative t, let $R_N(t)$ denote the proportion of the values $W_1, \cdots, W_{N-1}$ which are greater than t/N. Let $S(t)$ denote

$$\int_{-\infty}^{\infty} (r_1f_1(x) + r_2f_2(x) + \cdots + r_kf_k(x)) \exp\{-t[r_1f_1(x) + \cdots + r_kf_k(x)]\}\, dx$$

and $V(N)$ denote $\sup_{t\geqq 0} | R_N(t) - S(t) |$. Then it is shown that $V(N)$ converges stochastically to zero as N increases. This is a generalization of [1], where k was equal to unity. The result is applied to find the asymptotic behavior of ranks in a k-sample problem.

2. Proof of the stochastic convergence of $V(N)$. As in [1], if it can be shown that $R_N(t)$ converges stochastically to $S(t)$ for each positive t, the convergence of $V(N)$ follows. Therefore we fix a positive value for t.

We define the chance variable $Z(i, j, N)$ to be equal to unity if no observations fall in the half-open interval $[(X(i, j), X(i, j) + t/N]$, and equal to zero otherwise. We denote $1/N\sum_{i=1}^{k}\sum_{j=1}^{n_i} Z(i, j, N)$ by $K(N)$. Clearly,

$$K(N) = (1 - 1/N)R_N(t) + 1/N,$$

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