ON THE EFFICIENCY OF ESTIMATES OF TREND IN THE ORNSTEIN UHLENBECK PROCESS

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1. Summary. The problem is that of estimating the trend of a normal process when the trend function is known up to a finite number of coefficients. That is,

$$y_t = x_t + f(t), \qquad 0 \leqq t \leqq T,$$

where x_t is a normal process with mean zero and covariance function

$$E[X_u, X_v] = C(u, v)$$

and

$$f(t) = k_1\phi_1(t) + \cdots + k_s\phi_s(t).$$

The $\phi_i(t)$ are known functions and the k_i are to be estimated.

The standard procedure in such a case is to derive the estimates by the maximum likelihood method. However, if the covariance function $C(u, v)$ is not completely known, this is usually impossible, and it is essential to find an alternative procedure. The method of least squares has been proposed by Mann [1]. The estimates obtained by this method are independent of $C(u, v)$ and have the additional advantage of being easily computed. Mann and Moranda [2] showed that for the Ornstein Uhlenbeck process the asymptotic efficiency of the least square estimate relative to the maximum likelihood estimate is one, in the special case that the $\phi_i(t)$ are polynomials or trigonometric polynomials. Mann defines the efficiency $\bar{e}(T)$ of an estimate $\bar{f}(t)$

$$\bar{e}(T) = \frac{E\left[\int_0^T [\hat{f}(t) - f(t)]^2\, dt\right]}{E\left[\int_0^T [\bar{f}(t) - f(t)]^2\, dt\right]},$$

where $\hat{f}(t)$ is the maximum likelihood estimate. For the cases that shall be of particular interest—the Ornstein Uhlenbeck process with $\bar{f}(t)$ a linear unbiased estimate—Mann and Moranda [2] have shown that $\bar{e}(T) \leqq 1$.

In the present paper the asymptotic efficiency of the least square estimates will be computed for a wider class of functions $\phi_i(t)$. It will be shown that except for a special case just slightly broader than the one treated by Mann and Moranda, the asymptotic efficiency is actually less than one. Thus except for this special case, the least square estimates could be improved upon. An alternative estimate $\bar{k}_i(\alpha)$ is proposed. It will be shown that for $a \geqq \beta$, where β is the true correlation parameter in the Ornstein Uhlenbeck process, the estimates $\bar{k}_i(\alpha)$ are

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asymptotically more efficient than the least square estimates, and in fact as $\alpha \to \beta$ from above the efficiency increases (strictly) to one.

2. Introduction. The least square estimate is obtained by minimizing the expression

$$\int_0^T (y_t - f(t))^2\, dt$$

and is given by

$$\bar{k}_i = \sum_{j=1}^{s} G^{ij}(T) \int_0^T \phi_i(t) y_t\, dt,$$

where

$$G_{ij}(T) = \int_0^T \phi_i(t)\phi_j(t)\, dt. \tag{1}$$

The maximum likelihood estimates $\hat{k}_i$ minimize

$$\int_0^T \int_0^T [y_u - f(u)][y_v - f(v)] C^{-1}(u, v)\, du\, dv$$

and are given by

$$\hat{k}_i = \sum_{j=1}^{s} \Phi^{ij}(T) \int_0^T \int_0^T \phi_j(u) y_v C^{-1}(u, v)\, du\, dv,$$

where

$$\Phi_{ij}(T) = \int_0^T \int_0^T \phi_i(u)\phi_j(v) C^{-1}(u, v)\, du\, dv. \tag{2}$$

It will be assumed that the $\phi_i(t)$ and $C(u, v)$ are such that these integrals exist. The efficiency of the least square estimates can now be computed.

$$\bar{e}(T) = \frac{t[G(T)\Phi^{-1}(T)]}{t[\Psi(T)G^{-1}(T)]},$$

where

$$\Psi(T) = \int_0^T \int_0^T \phi_i(u)\phi_j(v) C(u, v)\, du\, dv. \tag{3}$$

The trace of the matrix is t.

It will further be assumed that there are functions $H_i(T)$ such that the limits

$$\begin{aligned} \lim_{T\to\infty} \frac{G_{ij}(T)}{H_i(T)H_j(T)} &= G_{ij}, \\ \lim_{T\to\infty} \frac{\Phi_{ij}(T)}{H_i(T)H_j(T)} &= \Phi_{ij}, \\ \lim_{T\to\infty} \frac{\Psi_{ij}(T)}{H_i(T)H_j(T)} &= \Psi_{ij} \end{aligned} \tag{4}$$

exist and are positive definite matrices. The asymptotic efficiency then is

$$\bar{e} = \lim_{T\to\infty} \bar{e}(T) = \frac{t(G\Phi^{-1})}{t(\Psi G^{-1})}. \tag{5}$$

Necessary and sufficient conditions that $\bar{e} = 1$ will be found for two classes of G, Φ, Ψ. The first, which includes the cases treated by Mann and Moranda, requires that G, Φ, Ψ be of the form

$$\begin{aligned} G &= \sum_{n=1}^{N} G_n, \\ \Phi &= \sum_{n=1}^{N} c_n G_n, \\ \Psi &= \sum_{n=1}^{N} \frac{1}{c_n} G_n, \end{aligned} \tag{6}$$

where the G_n are positive semi-definite matrices and the c_n are distinct positive real numbers. The second requires that

$$\begin{aligned} \Phi &= BGB^T \\ \Psi &= B^{-1}GB^{-1T} + C, \end{aligned} \tag{7}$$

where B is positive definite, and C is positive semi-definite.

Results will be applied to the case that x_t is an Ornstein Uhlenbeck process and the $\phi_i(t)$ are of the form first

$$\phi_i(t) = \sum_{n=1}^{N} \sum_{r=1}^{\gamma_i} t^r(a_{inr} \sin \omega_n t + b_{inr} \cos \omega_n t)$$

and second

$$\phi_i(t) = e^{a_i t}, \qquad a_i > 0.$$

When the covariance $C(u, v)$ involves some unknown parameters an attempt can be made to estimate them along with the k_i by the maximum likelihood method. However, this frequently leads to equations which cannot be solved. In this case, a natural procedure is to make an estimate $C^*(u, v)$ of $C(u, v)$ by any convenient method and then use the maximum likelihood estimates of the k_i based on the covariance $C^*(u, v)$.

For the Ornstein Uhlenbeck process

$$C(u, v) = \sigma^2 e^{-\beta|u-v|}.$$

Let

$$C^*(u, v) = \begin{cases} \sigma^2 & \text{if } u = v, \\ 0 & \text{otherwise.} \end{cases}$$

This covariance function yields the least square estimates. If the true value β is replaced by α, a family of estimates is obtained by this method.

$$\bar{k}_i(\alpha) \tag{8}$$
$$= \sum_{j=1}^{s} \Phi_\alpha^{ij}(T) \frac{1}{2}\left[\phi_j(T)y_T + \phi_j(0)y_0 + \frac{1}{\alpha}\int_0^T \phi_j'(t)\,dy_t + \alpha\int_0^T \phi_j(t)y_t\,dt\right],$$

where

$$\Phi_{\alpha ij}(T) \tag{9}$$
$$= \frac{1}{2}\left[\phi_i(T)\phi_j(T) + \phi_i(0)\phi_j(0) + \frac{1}{\alpha}\int_0^T \phi_i'(t)\phi_j'(t)\,dt + \alpha\int_0^T \phi_i(t)\phi_j(t)\,dt\right].$$

Clearly

$$\lim_{\alpha\to\infty} \bar{k}_i(\alpha) = \bar{k}_i,$$
$$\bar{k}_i(\beta) = \hat{k}_i.$$

3. Efficiency of estimates for G, Φ, Ψ, of form (6). Assume that G, Φ, Ψ, defined by (1), (2), (3), and (4) are the special form (6). Then from (5)

$$1 - \bar{e} = \frac{t(\Psi G^{-1} - G\Phi^{-1})}{t(\Psi G^{-1})} = \frac{t[G^{-1}\Phi^{-1}(\Phi\Psi - GG)]}{t(\Psi G^{-1})},$$

$$(\Phi\Psi - GG) + (\Phi\Psi - GG)^T$$
$$= \sum_{n=1}^{N}\sum_{m=1}^{N}\left[\frac{c_n}{c_m} G_n G_m - G_n G_m + \frac{c_m}{c_n} G_n G_m - G_n G_m\right] \tag{10}$$
$$= \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{(c_m - c_n)^2}{c_n c_m} G_n G_m.$$

$G_m G_n$ are positive semi-definite and

$$\frac{(c_m - c_n)^2}{c_n c_m} > 0, \qquad \text{all} \quad m \neq n.$$

Thus, $\Phi\Psi - GG$ is positive semi-definite. In order that $1 - \bar{e} = 0$, it is necessary and sufficient that

$$\Phi\Psi - GG = 0.$$

This is equivalent to requiring

$$t(G_m G_n) = 0, \qquad \text{all } m \neq n.$$

This result will be stated as a theorem.

THEOREM 1. *If G, Φ, Ψ are nonsingular and of the form*

$$G = \sum_{n=1}^{N} G_n ,$$

$$\Phi = \sum_{n=1}^{N} c_n G_n ,$$

$$\Psi = \sum_{n=1}^{N} \frac{1}{c_n} G_n,$$

where the G_n are positive semi-definite matrices and the c_n are distinct positive real numbers, then

$$\bar{e} = \frac{t(G\Phi^{-1})}{t(\Psi G^{-1})} = 1$$

if and only if

$$t(G_m G_n) = 0, \qquad \text{all } m \neq n.$$

For the Ornstein Uhlenbeck process the theorem can be applied to obtain the special result.

THEOREM 2. *Let*

$$y_t = x_t + f(t),$$

where x_t is an Ornstein Uhlenbeck process with mean zero, and

$$f(t) = k_1\phi_1(t) + \cdots + k_s\phi_s(t).$$

Suppose

$$\phi_i(t) = \sum_{n=1}^{N} \sum_{r=1}^{\gamma_i} t^r(a_{inr} \sin \omega_n t + b_{inr} \cos \omega_n t)$$

are such that

$$\phi_i^*(t) = t^{\gamma_i} \sum_{n=1}^{N} (a_{in\gamma_i} \sin \omega_n t + b_{in\gamma_i} \cos \omega_n t)$$

are linearly independent. Then the asymptotic efficiency of the least square estimates of the k_j is one, if and only if

$$\begin{aligned} \sum a_{in\gamma}a_{im\gamma} &= 0, \\ \sum a_{in\gamma}b_{im\gamma} &= 0, \\ \sum b_{in\gamma}b_{im\gamma} &= 0, \end{aligned} \tag{11}$$

for all γ and $m \neq n$. The sums extend over all i for which $\gamma_i = \gamma$.

PROOF. Let $H_i(T) = T^{\gamma_i+1/2}$. The only terms which appear in the limits (4) will be those of maximum order, that is, those of the $\phi^*(t)$. Denote $a_{in\gamma_i}$ by a_{in} and $b_{in\gamma_i}$ by b_{in}. Then G, Φ, Ψ can be computed and are of the form (6) with

$$G_{nij} = \frac{a_{in}\,a_{jn} + b_{in}\,b_{jn}}{(\gamma_i + \gamma_j + 1)\gamma_i\,!\gamma_j\,!},$$

$$c_n = \frac{\beta^2 + \omega_n^2}{2\beta}.$$

The G_n can easily be shown to be positive semi-definite. Thus by theorem 1, $\bar{e} = 1$ if and only if

$$t(G_mG_n) = 0, \qquad \text{all } m \neq n.$$

$$t(G_mG_n) = \sum_{i=1}^{s}\sum_{j=1}^{s} \frac{a_{in}\,a_{im}\,a_{jn}\,a_{jm} + a_{in}\,b_{im}\,a_{jn}\,b_{jm} + a_{im}\,b_{in}\,a_{jm}\,b_{jn} + b_{in}\,b_{im}\,b_{jn}\,b_{jm}}{(\gamma_i + \gamma_j + 1)^2(\gamma_i\,!)^2(\gamma_j\,!)^2}$$

$$= \sum_{\gamma}\sum_{\delta} \frac{A_{mn\gamma}\,A_{mn\delta} + C_{mn\gamma}\,C_{mn\delta} + C_{nm\gamma}\,C_{nm\delta} + B_{mn\gamma}\,B_{mn\delta}}{(\gamma + \delta + 1)^2(\gamma\,!)^2(\delta\,!)^2}.$$

γ and δ are summed over all distinct values of γ_i and

$$A_{mn\gamma} = \sum a_{in}a_{im},$$
$$B_{mn\gamma} = \sum b_{in}b_{im},$$
$$C_{mn\gamma} = \sum a_{in}b_{im}.$$

The summations extend over all values of i for which $\gamma_i = \gamma$. Since

$$\frac{1}{(\gamma + \delta + 1)^2}$$

is a positive definite matrix for γ, δ ranging over distinct integers,

$$t(G_mG_n) = 0, \qquad \text{all } m \neq n$$

if and only if

$$A_{mn\gamma} = B_{mn\gamma} = C_{mn\gamma} = 0$$

for all γ and $m \neq n$.

Thus, unless the special conditions (11) are satisfied, $\bar{e}$ will be strictly less than one. For example,

$$f(t) = k_1 + k_2 \sin t + k_3 \sin 2t$$

can be estimated efficiently by least squares, but

$$f(t) = k_1 + k_2 (\sin t + \sin 2t)$$

cannot.

Grenander and Rosenblatt [3] in Section 7.6 obtain results very similar to those of Theorem 2.

THEOREM 3. *If y_t and $\phi_i(t)$ are as in the hypothesis of theorem 2, then the asymptotic efficiency $\bar{e}(\alpha)$ of the estimate $\bar{k}_i(\alpha)$ (8) is monotone decreasing from 1 at $\alpha = \beta$ to $\bar{e}$ as $\alpha \to \infty$. If $\bar{e} \neq 1$, then it is strictly decreasing.*

PROOF. First the efficiency of the $\bar{k}_i(\alpha)$ estimates must be computed.

$$\begin{aligned} E(T,\alpha) &= E\left[\int_0^T (\bar{f}_\alpha(t) - f(t))^2\, dt\right] \\ &= t[\Sigma_{\bar{k}_i(\alpha)\bar{k}_j(\alpha)}(T) G_{ij}(T)]. \\ \Sigma_{\bar{k}_i(\alpha)\bar{k}_j(\alpha)} &= E[(\bar{k}_i(\alpha) - k_i)(\bar{k}_j(\alpha) - k_j)] \\ &= \Phi_\alpha^{-1}(T)\left[\frac{(\alpha^2-\beta^2)}{4}\Psi_\alpha(T) + \frac{\beta}{\alpha}\Phi_\alpha(T)\right]\Phi_\alpha^{-1}(T). \\ \Psi_{\alpha ij}(T) &= \int_0^T\int_0^T (\phi_i(u) + \frac{1}{\alpha}\phi_i'(u))(\phi_j(v) + \frac{1}{\alpha}\phi_j'(v)) e^{-\beta|u-v|}\, du\, dv, \end{aligned}$$

and $\Phi_\alpha(T)$ is defined by (9).

For $\phi_i(t)$ as in theorem 2 and

$$H_i(T) = T^{\gamma_i + 1/2},$$

$$\Phi_{\alpha ij} = \lim_{T\to\infty} \frac{\Phi_{\alpha ij}(T)}{H_i(T)H_j(T)} = \frac{1}{2\alpha}[\alpha^2 G_{ij} + \zeta_{ij}],$$

and

$$\Phi = \frac{1}{2\beta}(\beta^2 G + \zeta),$$

where

$$\zeta_{ij} = \lim_{T\to\infty} \frac{\int_0^T \phi_i'(t)\phi_j'(t)\, dt}{H_i(T)H_j(T)}.$$

$$\Psi_{\alpha ij} = \lim_{T\to\infty} \frac{\Psi_{\alpha ij}(T)}{H_i(T)H_j(T)} = \frac{(\alpha^2-\beta^2)}{\alpha^2}\Psi_{ij} + \frac{2\beta}{\alpha^2}G_{ij}$$

Thus,

$$\Phi_\alpha = \frac{1}{2\alpha}[(\alpha^2 - \beta^2)G + 2\beta\Phi].$$

Let

$$A = (\alpha^2 - \beta^2)G + 2\beta\Phi.$$

Then

$$\begin{aligned} E(\alpha) &= \lim_{T\to\infty} E(T,\alpha) \\ &= t\{A^{-1}A^{-1}[(\alpha^2-\beta^2)^2\Psi + 4\beta(\alpha^2-\beta^2)G + 4\beta^2\Phi]\}, \end{aligned}$$

and

$$\frac{\partial E(\alpha)}{\partial \alpha} = 8\alpha\beta(\alpha^2 - \beta^2)t[A^{-1}A^{-1}GA^{-1}\Phi(G^{-1}\Psi - \Phi^{-1}G)].$$

It follows from (10) that

$$G^{-1}\Psi - \Phi^{-1}G$$

is positive semi-definite. Thus, for $\alpha > \beta$, the derivative of $E(\alpha)$ is nonnegative and $E(\alpha)$ is monotone increasing. If $\bar{e} \neq 1$, then

$$t(G^{-1}\Psi - \Phi^{-1}G) > 0,$$

and at least one characteristic root must be nonzero. Since $A^{-1}A^{-1}GA^{-1}\Phi$ is positive definite $\partial E(\alpha)/\partial\alpha$ will then be positive, and $E(\alpha)$ is strictly increasing.

$$\bar{e}(\alpha) = \lim_{T\to\infty} \frac{E\left[\int_0^T (\hat{f}(t) - f(t))^2\,dt\right]}{E\left[\int_0^T (\hat{f}_\alpha(t) - f(t))^2\,dt\right]} = \frac{t(\Phi^{-1}G)}{E(\alpha)}.$$

4. Efficiency of estimates for exponential $\phi_i(t)$. Assume G, Φ, Ψ are of the form (7). Then

$$1 - \bar{e} = \frac{t(\Psi G^{-1} - \Phi^{-1}G)}{t(\Psi G^{-1})} = \frac{t(CG^{-1})}{t(\Psi G^{-1})},$$

and $\bar{e} = 1$ if and only if $C = 0$.

Let

$$\phi_i(t) = e^{a_i t}$$

and

$$H_i(\alpha) = e^{a_i T},$$

where the a_i are positive and distinct. Then

$$G_{ij} = \frac{1}{a_i + a_j},$$

$$\Phi_{ij} = \frac{(\beta + a_i)(\beta + a_j)}{a_i + a_j},$$

$$\Psi_{ij} = \frac{2\beta}{(\beta + a_i)(\beta + a_j)(a_i + a_j)} + \frac{1}{(\beta + a_i)(\beta + a_j)}.$$

Thus,

$$B_{ij} = \frac{(\beta + a_i)\delta_{ij}}{\sqrt{2\beta}},$$

$$C = \frac{1}{2\beta} B^{-1} B \neq 0,$$

and hence

$$\bar{e} < 1.$$

Since the least square estimates are not asymptotically efficient, it is of interest to compute the efficiency of the $\bar{k}_i(\alpha)$ estimates. In this case

$$\Phi_{\alpha ij} = \frac{(\alpha + a_i)(\alpha + a_j)}{2\alpha(a_i + a_j)} = \frac{1}{2\alpha} AGA,$$

where

$$A_{ij} = (\alpha + a_i)\delta_{ij}$$

$$\Psi_\alpha = \frac{1}{\alpha^2} A\Psi A$$

$$\dot{E}(\alpha) = t\{G^{-1}A^{-1}GA^{-1}G^{-1}[(\alpha^2 - \beta^2)\Psi + 2\beta G]\}$$

$$\frac{\partial E(\alpha)}{\partial \alpha} = 2(\alpha - \beta)t[A^{-1}G^{-1}D],$$

where

$$D_{ij} = \frac{(\alpha - \beta)a_i + (\alpha + \beta)a_j + 2a_i a_j}{(a_i + a_j)(\alpha + a_i)^2(\beta + a_i)(\alpha + a_j)}.$$

For $\alpha > \beta$ this matrix is positive definite. Thus, $\partial E(\alpha)/\partial\alpha$ is positive, and $E(\alpha)$ is strictly increasing. Thus, for $\alpha > \beta$, the $\bar{k}_i(\alpha)$ estimates are more efficient than the least square estimates.

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