ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Ames, Iowa Meeting of the Institute, April 3–5, 1958.)

11. **Bias and Confidence in Not-quite Large Samples.** (Preliminary Report) JOHN W. TUKEY, Princeton University, (By Title).

The linear combination of estimates based on all the data with estimates based on parts thereof seems to have been first treated in print as a means of reducing bias by Jones (*J. Amer. Stat. Assn.*, Vol. 51 (1956), pp. 54–83). Let $y_{(.)}$ be the estimate based on all the data, $y_{(i)}$ that based on all but the ith piece, $\bar{y}_{(i)}$ the average of the $y_{(i)}$. Quenouille (*Biometrika*, Vol. 43 (1956), pp. 353–560) has pointed out some of the advantages of $ny_{(.)} - (n-1)\bar{y}_{(i)}$ as such an estimate of much reduced bias. Actually, the individual expressions $ny_{(.)} - (n-1)y_{(i)}$ may, to a good approximation, be treated as though they were n independent estimates. Not only is each nearly unbiased, but their average sum of squares of deviations is nearly $n(n-1)$ times the variance of their mean, etc. In a wide class of situations they behave rather like projections from a non-linear situation on to a tangent linear situation. They may thus be used in connection with standard confidence procedures to set closely approximate confidence limits on the estimand. (Received December 26, 1957.)

12. **Limiting Distributions of k-sample Test Criteria of Kolmogorov-Smirnov-v. Mises Type.** J. KIEFER, Cornell University, (By Title).

Let S_j be the sample d.f. of n_j independent, indentically distributed random variables with common unknown continuous d.f. F_j $(1 \leqq j \leqq k)$, the S_j being independent. For testing the hypothesis $H:F_1 = \cdots = F_k$, several criteria were suggested by the author in *Ann. Math. Stat.*, Vol. 26 (1955), p. 775. Among these are $T = \sup_x \sum_j n_j[S_j(x) - \bar{S}(x)]^2$ and

$$W = \int \sum_j n_j[S_j(x) - \bar{S}(x)]^2 dS^*(x),$$

where $\bar{S} = \sum_j n_j S_j / \sum n_j$ and $S^* = \sum_j a_j S_j$ with $\sum_j a_j = 1$. It is proved by the method indicated in the above reference that, under H, the limit of $P\{T < a^2\}$ as all $n_j \to \infty$, is

$$\frac{2^{(5-2k)/2} a^{1-k}}{\Gamma((k-1)/2)} \sum_{n=1}^{\infty} \frac{\alpha_n^{k-3} \exp[-\alpha_n^2/2a^2]}{[J_{(k-1)/2}(\alpha_n)]^2},$$

where $a > 0$ and α_n is the nth positive zero of the Bessel function $J_{(k-3)/2}$; alternative expressions are also given. When $k = 4$ or $k = 2$ the summand above reduces to an elementary function; the latter case gives the Kolmogorov-Smirnov distribution, since $T^{1/2}$ is the Smirnov statistic when $k = 2$. The limiting d.f. of W is expressible in a series involving Hermite polynomials when k is odd and Bessel functions when k is even. For $k = 2$, W is the test suggested by Lehmann and Rosenblatt, and the above d.f. is the limiting ω^2 d.f. in the form given by Anderson and Darling. (Received January 6, 1958.)

13. **A Rule for Action Based on Percentage Changes in the Sample Mean.** D. B. OWEN, Sandia Corporation, (By Title).

A random selection is made of n items from a normal population X, each item is measured once, and the sample mean $\bar{x}$ is computed. The sample items are identified by some means and the sample and the remaining population are mixed at random. They are then subjected to some condition, such as storage, after which the same items that were first sampled are measured again, giving a new mean $\bar{y}$. Some action is taken only if the new mean $\bar{y}$ differs from the old mean $\bar{x}$ by more than $p\bar{x}$, where $p > 0$. The probability of taking action using the above rule is shown to be expressible in terms of the bivariate normal cumulative probability function. If there is no change in the mean (from X to Y), the probability for action increases monotonically with an increase in the standard deviation. If there is no increase in the standard deviation (from X to Y), the probability for action increases monotonically with any increase in the mean. However, for a fixed (relatively large) increase in the mean, the probability for action drops with increasing standard deviation and then increases. (Received February 5, 1958.)

14. **On a Multivariate Gamma Distribution.** P. R. KRISHNAIAH and M. M. RAO, University of Minnesota.

From the relation between the univariate Gamma and Gaussian distributions one naturally considers the corresponding p-variate cases. Some writers in the past implied this in their approach to this problem. But the properties of the latter distribution are not well utilized. The derivation of a p-variate Gamma distribution by Krishnamoorthy and Parthasarathy (these *Annals*, 1951) and later used by Gurland (these *Annals*, 1955) was not too direct. In this paper a simple derivation using the too familiar properties of the Normal and Wishart distributions is given. Also some interesting connections between the Gamma and Gaussian distributions are discussed. A special case for $p = 2$ (for correlations) has been given in Cramér's book (p. 317). But that property is shown to be true for $p > 2$. Next the "Arithmetical Character" of this distribution in the sense of P. Lévy (*Proc. Cambridge Philos. Soc.* (1948), p. 295) is considered. Some small and large sample properties are also discussed. (Received February 6, 1958.)

15. **An Expression for the Cumulative Distribution Function of the Noncentral t-Distribution.** D. B. OWEN, Sandia Corporation, (By Title).

The cumulative distribution function of the noncentral t-distribution may be expressed in terms of the univariate normal integral and elementary functions for an even number of degrees of freedom and for an odd number of degrees of freedom in terms of the univariate normal integral, elementary functions, and the $T(h, a)$ function. The $T(h, a)$ function was tabulated by the present author in *Ann. Math. Stat.*, Vol. 27 (1956), pp. 1075–1090. The above results were obtained by repeated integration by parts. For example, with one degree of freedom $\Pr(T \leqq t) = G(-\delta/\sqrt{1+t^2}) + 2T(\delta/\sqrt{1+t^2}, t)$, where $G(x)$ is the univariate normal integral from minus infinity to x, and δ is the noncentrality parameter. This expression is especially useful since the noncentral t-distribution has not been tabulated for one degree of freedom. (Received February 6, 1958.)

16. **The Fourth Product Moment of a Binary Random Process.** J. A. MCFADDEN, Purdue University, (introduced by Judah Rosenblatt) (By Title).

Let $x(t)$ describe a stationary random process, and let $y(t) = 1$ when $x(t) \geqq 0$ and $y(t) = -1$ when $x(t) < 0$. Let $s(\tau_1, \tau_2, \tau_3)$ denote the fourth product moment,

$$E[y(t)y(t+\tau_1)y(t+\tau_2)y(t+\tau_3)],$$

where $0 \leqq \tau_1 \leqq \tau_2 \leqq \tau_3$. If $x(t)$ is a Gaussian process, then s is related to the quadrivariate normal integral, which apparently cannot be expressed in closed form. For practical applications it seems advisable to make different assumptions about $x(t)$ (or about $y(t)$. Let $E[y(t)] = 0$ and let all product moments of odd degree in $y(t)$ be zero. Consider furthermore the zeros of the function $x(t)$. If the zeros obey the Poisson distribution, then a particularly simple result follows for s and for all higher moments. Another assumption is the following: Let unspecified events occur at times $t_1, t_2, \cdots$, according to the Poisson distribution, the average number of events per unit time being denoted by α. If alternate events (those at $t_1, t_3, \cdots$) are designated as zeros of $x(t)$, then the autocorrelation function of $y(t)$ is $E[y(t)y(t+\tau)] = e^{-\alpha\tau} \cos \alpha\tau$, and the desired fourth moment is

$$s = e^{-(u+w)} \cos u \cos w - e^{-(u+2v+w)} \sin u \sin w,$$

where $u = \alpha\tau_1$, $v = \alpha(\tau_2 - \tau_1)$, and $w = \alpha(\tau_3 - \tau_2)$. (Received February 10, 1958.)

17. Approximate Solutions for the Probability Density of Zero-Crossing Intervals in a Gaussian Process. J. A. McFadden, Purdue University, (introduced by Judah Rosenblatt).

Let $x(t)$ be a stationary Gaussian process, and let $P_0(\tau)$ be the probability density of the lengths of intervals between successive zeros in this process. Under the assumption that the length of a given zero-crossing interval is independent of the sum of the previous $(2m + 2)$ interval lengths, where m takes on *all* values, $m = 0, 1, 2, \cdots$, then the following integral equation can be derived for $P_0(\tau)$: $P_0(\tau) = Q_1(\tau) - \int_0^\tau Q_2(l)P_0(\tau - l)\, dl$, where $Q_1(\tau)\, d\tau$ is the conditional probability of a zero with *negative* slope in the interval $(t + \tau, t + \tau + d\tau)$, given a zero with positive slope at time t, and $Q_2(\tau)\, d\tau$ is the conditional probability of a zero with *positive* slope in $(t + \tau, t + \tau + d\tau)$, given a zero with positive slope at t. Using expressions for $Q_1(\tau)$ and $Q_2(\tau)$ given by S. O. Rice, the integral equation has been solved numerically for several choices of spectral density. The results compare favorably with experiment, and the agreement is much better than can be obtained by the usual renewal methods, i.e., assuming that all interval lengths are independent. (Received February 10, 1958.)

18. Minimal Complete Classes of Tests. D. L. Burkholder, University of Illinois.

Minimal complete classes of tests are found for a number of common testing problems including, for example, those listed by Lehmann and Scheffé in *Sankhyā*, Vol. 15 (1955), p. 224, with respect to the exponential family of distributions. The proofs are based partly on the theory of complete and sufficient statistics and partly on other ideas needed and developed for those cases in which the hypothesis set ω and the alternative set $\Omega - \omega$ are separated by an indifference zone. The kinds of results obtained are illustrated in the following special case: Let X_1 and X_2 be independent random variables where X_i is binomial (n_i, p_i), $0 < p_i < 1$, $i = 1, 2$. Let ω be a subset of $\{(p_1, p_2) \mid p_1 \geqq p_2\}$, $\Omega - \omega$ be a subset of $\{(p_1, p_2) \mid p_1 < p_2\}$, and suppose there are positive numbers m and M such that if

$$S = \{(p_1, p_2) \mid m < p_1/p_2 < M\}$$

then each of $S \cap \omega$ and $S \cap (\Omega - \omega)$ has the origin as a limit point. Let C be the class of all tests φ of the form: $\varphi(x_1, x_2) = 1$ if $x_1 < c(x_1 + x_2)$, $= a(x_1 + x_2)$ if equality holds, $= 0$ otherwise. Then C is the minimal complete class of tests for the problem of testing $(p_1, p_2) \in \omega$ against $(p_1, p_2) \in \Omega - \omega$. Thus, both Fisher's exact test and the classical test are admissible since both are in C. (Received February 11, 1958.)

19. On the Fitting of Some Contagious Distributions, S. K. Katti and John Gurland, Iowa State College.

A number of compound and generalized distributions are compared by using such characteristics as skewness, kurtosis, and the ratio of the first two frequencies. A study has also been made of the limiting forms of the distributions. Some of these distributions have been fitted to sampled data by estimating the parameters by various methods in order to gain some empirical knowledge of the usefulness of these distributions and the relative merits or demerits of the methods of estimation. (Received February 12, 1958.)

20. Notes on the Spearman-Karber Procedures in Bioassay. (Preliminary Report) BYRON WM. BROWN, JR., University of Minnesota.

The maximum bias of the Spearman-Karber estimator of the L.D. 50 over possible choices of dose levels is examined under various conditions on the distribution function, such as unimodality and symmetry. The maximum mean square error of the estimator is examined also. The results are compared with actual values for several distributions. The results are also used to make some comparisons of the Spearman-Karber estimator with some commonly used parametric methods of estimating the L.D. 50. (Received February 12, 1958.)

21. Biases in Prediction by Regression for Certain Incompletely Specified Models. HAROLD LARSEN, Iowa State College, (transmitted by H. T. David).

An experimenter doesn't know whether to assume a "full" population regression model $E(y_j) = \sum_{i=1}^{k} \beta_i x_{ij}$ or a "partial" population regression model $E(y_j) = \sum_{i=1}^{m} \beta_i x_{ij}$, $k \geqq m$. He decides the matter by the natural preliminary F-test of the hypothesis that the last $(k - m)\beta_i$'s are zero. He uses the full model for subsequent predictions if the hypothesis is rejected, and uses the partial model for subsequent predictions if the hypothesis is not rejected. Call this predictor y^*.

The full model is assumed to be true, the error terms being normally distributed with zero mean. Under these assumptions the expected value of the estimator y^* is derived. The expected value of the estimated variance of y^* is also derived if a certain sometimes-pool procedure is used. (Received February 12, 1958.)

22. Independence of Statistics and Characterization of the Multivariate Normal Distribution. S. G. GHURYE, University of Chicago and Ingram Olkin, Michigan State University, (By Title).

Some of the results proved are: If x, y are independent p-dimensional random vectors and A is a non-singular matrix such that $x + y$ and $x + Ay$ are independent, then x, y are normal. If $x_1, \cdots, x_n$ are independent random vectors, $A_1, \cdots, A_n, B_1, \cdots, B_n$ are non-singular commutative symmetric matrices such that $\sum A_i x_i$ and $\sum B_i x_i$ are independent, then the x_i are normal. If $f_1(t), \cdots, f_n(t)$ are c.f.'s and there exist positive numbers $\alpha_1, \cdots, \alpha_n$ such that in some neighborhood of the origin $\prod f_i^{\alpha_i}(t)$ agrees with an entire function of finite order p, where p is larger than the exponent of convergence of the zeros of the function, then p cannot exceed 2. This is applied to characterize the normal distribution by the independence of a sum of independent r.v.'s (not all of which need be identically distributed) and a polynomial of special, specified type.

Let $x_1, \cdots, x_n$ be independent p-variate normal random vectors. Let $z_j = (x_{j1}, \cdots, x_{jn})$. NASC are given for the independence of (1) $q_{ij}^{(1)} = z_i A_{ij} z_j' + z_i a_i' + z_j a_j'$ and $q_{ij}^{(2)} = z_i B_{ij} z_j' + z_i b_i' + z_j b_j'$, (2) (q_{ij}) and $\sum A_i x_i'$, and (3) $\sum A_i x_i'$ and $\sum B_i x_i'$. (Received February 4, 1958.)

23. Contributions to the Theory of Rank Order Statistics—The One-Sample Case. I. RICHARD SAVAGE, University of Minnesota.

The testing that a distribution has median zero against slippage is considered using the techniques developed earlier (these *Annals*, Vol. 27 (1956) pp. 590–615, and Vol. 28 (1957) pp. 967–977). Let $Z = (Z_1, \cdots, Z_N)$ be a random vector with $Z_i = 1(0)$ if the ith largest in absolute value in a sample of N from the density $f(x)$ is positive (negative). Then

$$P(Z = z) = N! \int_{0 \leq y_1 \leq \cdots \leq y_N \leq \infty} \cdots \int \prod_{i=1}^{N} [f^{1-z_i}(-y_i) f^{z_i}(y_i)\, dy_i]$$

Conditions are found implying $P(Z = z) > P(Z = z')$ where z is derived from z' by replacing a 0 by a 1, or interchanging a 0 and 1 in z' by moving the 1 to the left. These conditions are met by the normal and other symmetric exponential distributions. (Received February 17, 1958.)

24. **An Identity of Use in Non-Linear Least Squares.** M. B. Wilk, Bell Telephone Laboratories.

Under rather general conditions the identity $f(x) = f(x_0) + (x - x_0)f'[(x + x_0)/2]$ is a necessary and sufficient condition that $f(x)$ be a quadratic function. The identity generalizes immediately and in the same form to p variables. A procedure due to Gauss for iterative non-linear least squares fitting of observations y_i to a function $f(x_i\,;\,\theta)$, involves essentially the repeated linear regression of $[y_i - f(x_i\,;\,\theta_0)]$ on $[\partial f(x_i\,;\,\theta)/\partial\theta]_{\theta=\theta_0}$ with the regression coefficient $\hat{\delta}$ giving "improved" estimates of θ by $(\theta_0 + \hat{\delta})$. The generalization to p parameters is immediate.

This process can oscillate wildly (for example, out of computer range) and does not necessarily converge. A modification of this "Linear Gauss" procedure, based on the identity above, will approximate a "Quadratic Gauss" procedure while always solving only sets of linear equations. Advantages are a damping of the oscillations of the Linear Gauss, possible decrease in the extent of computing, and possible improvements in convergence characteristics. (Received February 17, 1958.)

25. **Unbiased Regression Estimators.** W. H. Williams, Iowa State College.

In sample surveys one desires unbiased estimators of population characteristics such as the mean $\bar{Y}$ of a variate y, and that these estimates be made with good precision. There are many ways of improving precision, one of which is the use of auxiliary information. In particular, this information is sometimes used in a regression estimator obtained by evaluating the line of best fit at the point $\bar{X}$. The properties of this estimator are derived from the stochastic model $y_i = A + Bx_i + e_i$ where the e_i are random errors which have expectation zero, common variance and are uncorrelated with each other. The estimator $\hat{y}_b$ of the population mean $\bar{Y}$ is then of the form $\hat{y}_b = \bar{y} + b(\bar{X} - \bar{x})$ where $\bar{y}$ and $\bar{x}$ are sample means and b is the least squares estimator of the regression slope. If the paired observations $y_i x_i$ satisfy the above linear model then $\hat{y}_b$ has expectation $\bar{Y}$. However, it is often unrealistic to assume that such a model is satisfied by the data and in such an event $\bar{y}_b$ will usually be biased. For large populations the expectation of $\hat{y}_b$ is given by $\bar{Y} - \text{Cov}(xb)$ so that $\hat{y}_b$ has a bias of $-\text{Cov}(\bar{x}b)$. $\text{Cov}(\bar{x}b)$ refers to the joint distribution of $\bar{x}$ and b in random samples of size n. An unbiased estimator of $\bar{Y}$ is obtained which has favorable efficiency. This estimator is easily generalized to the multivariate situation. (Received February 20, 1958.)

26. **Maximum Likelihood Estimation from Incomplete Data for Continuous Distributions.** Scott A. Krane, Iowa State College.

A method is given for obtaining the maximum likelihood estimates of parameters of continuous distributions from sample data which is "incomplete" due to truncation, censoring or grouping. The method may be applied to any distribution for which the likelihood equations are soluble in the complete data case. No special functions are required.

The likelihood equations for incomplete samples contain two types of terms: (a) the differentials of the likelihood evaluated at observed variate values x_i, and (b) integrals of the above differentials over intervals of missing variate values. The method presented replaces the integrals in (b) by weighted sums of terms similar to (a) evaluated at variate values, z_h, "representative" of the intervals of missing values. The likelihood equations for the incomplete sample are then identical with those for a complete sample of values x_i and z_h. The z_h values and weights required are functions of the parameters, so that an iterative procedure is used to obtain the estimates. (Received February 26, 1958.)

27. Unbiased Ratio Estimators in Stratified Sampling. JOSE NIETO DE PASCUAL, (transmitted by W. H. Williams).

The paper presents some theory of unbiased ratio estimators of the population mean $\overline{Y}$, in stratified sampling, computed from samples of k drawn from each of L strata ($k \ll L$).

Two unbiased ratio estimators and their exact variances, as well as unbiased estimates of the latter, are given. The derivations follow the lines of an unbiased ratio estimator for simple random sampling, y', introduced by Hartley and Ross (*Nature* (174), August 7, 1954, p. 270). The two estimators are (a) An unbiased "separate" ratio estimator formed by obtaining the y' estimator for each stratum, and (b) An unbiased "combined" ratio estimator computed by the y' formulae from k pairs of $\bar{y}_{st}$, $\bar{x}_{st}$, where $\bar{y}_{st}$, $\bar{x}_{st}$ are the familiar unbiased estimators of $\overline{Y}$, $\overline{X}$, computed from stratified samples of one unit drawn from each of the L strata.

These two unbiased estimators are then compared with the "combined" ratio estimator (Hansen, Hurwitz, and Gurney, *J. Amer. Stat. Assn.*, Vol. 41 (1946), pp. 173–189), and conditions on the population characteristics are described when the unbiased estimators are more efficient. Generalizations and the special case $k = 2$ are discussed in detail. (Received February 27, 1958.)

28. On the Laws of Cauchy and Gauss. R. G. LAHA, The Catholic University of America.

The following theorems are proved: THEOREM 1. Let x and y be two independently and identically distributed random variables having a common distribution function $F(x)$. Let the quotient $\omega = x/y$ follow the Cauchy law distributed symmetrically about the origin. Then $F(x)$ has the following general propeties: (1) it is symmetric about the origin, absolutely continuous, and has a continuous probability density function $f(x) = F'(x)$; (2) the random variable x has an unbounded range; (3) the probability density function $f(x)$ satisfies the equation $\int_0^\infty f(x)f(\omega x)x\,dx = c_0/(1 + \omega^2)$ holding for all ω, where c_0 is a constant. THEOREM 2. In addition to the conditions of Theorem 1, let $F(x)$ have finite moments of all orders. Then $F(x)$ is normal. (Received February 28, 1958.)

(Abstracts of papers presented at the Gatlinburg, Tennessee Meeting of the Institute, April 10–12, 1958.)

29. On the Simple von Neumann Model of Dynamic Economic Equilibrium as a Markov Chain. (Preliminary Report) DAVID ROSENBLATT, American University, (By Title).

The simple von Neumann model of dynamic economic equilibrium (the special case in which (i) there is the same number of "goods" as of basic "productive processes" and (ii) there is a single "output" for each "productive process") is simply transformed and structurally related to two stationary Markov chains. Results are obtained for aggregation and consolidation in the simple von Neumann model and these are compared and contrasted with analogous results for macro-statistical input-output formulations. (Received February 5, 1958.)

30. Tests on a Variance-Covariance Matrix. Nathan Mantel, National Cancer Institute.

A class of tests on the elements of the variance-covariance matrix is proposed. The class includes as a special case Pitman's Test for equality of two correlated variances. Depending on the assumptions made the test may be one for uniformity, for equality of variances or for equality of covariances. The test may be adapted so as to provide more specific contrasts. Tests on the corresponding correlation matrix through the use of either empirical or population standardizing factors are also considered.

An interesting adaptation of the procedure is one which permits testing the interaction in a two-day classification in the absence of replication.

The testing procedures depend on the fact that when the row sums of the variance-covariance matrix are equal the mean of a set of observations is uncorrelated with any of the deviations from the mean. The test is primarily one on the significance of the multiple correlation of the mean on the set of deviations. The power efficiency of the test for specific alternatives may be increased by testing the correlation of the mean with a subtest of deviations or linear combinations of deviations. Efficiency may also be increased by shifting attention from the original variables to linear transforms. In some instances a single change in sign of some of the variables can increase efficiency. (Received February 10, 1958.)

31. An Upper Bound for the Variance of Certain Statistics. Wassily Hoeffding, University of North Carolina.

It is shown that if X_1, X_2, X_3 are independent and identically distributed random variables, if $0 \leqq f(X_1, X_2) = f(X_2, X_1) \leqq 1$, and $Ef(X_1, X_2) = p$, then $Ef(X_1, X_2)f(X_1, X_3) - p^2 \leqq H(p)$, where $H(p) = p^{3/2} - p^2$, $\frac{1}{2} \leqq p \leqq 1$, and $H(p) = (1-p)^{3/2} - (1-p)^2$, $0 \leqq p \leqq \frac{1}{2}$. The sign of equality holds if, with probability one, $f(X_1, X_2) = g(X_1)g(X_2)$ (for $p \geqq \frac{1}{2}$) or $f(X_1, X_2) = 1 - g(X_1)g(X_2)$ (for $p \leqq \frac{1}{2}$), where $g(X)$ takes the values 0 and 1 only. This inequality implies an upper bound for the variance of the statistic

$$U = \sum_{1 \leqq i \neq j \leqq n} f(X_i, X_j)[n(n-1)]^{-1}$$

in terms of its mean. This class of statistics includes M. G. Kendall's rank correlation coefficient t and (except for a minor difference) the Cramér–von Mises statistic ω^2. In the former case the inequality has been conjectured by Daniels and Kendall. (Received February 12, 1958.)

32. On a Test for the Equality of Several Means. K. V. Ramachandran, Demographic Training and Research Centre, India, (By Title).

Let $x_{ij} (i = 1, 2, \cdots, k; j = 1, 2, \cdots, n)$ be random samples of sizes n from k univariate normal populations with means μ_i and variance $\sigma^2 (i = 1, 2, \cdots, k)$. The hypothesis $H_0: \mu_1 = \mu_2 = \cdots = \mu_k$ against H: Not H_0 is equivalent to the union of $H_{0i}: \mu_i = \mu$ (say) against $H_i: \mu_i \neq \mu$ (where μ is unknown) for every $i = 1, 2, \cdots, k$. To test $H_{0i}: \mu_i = \mu$ against $H_i: \mu_i \neq \mu$ for any given i we have the test based on $t_i = [(\bar{x}_i - \bar{x})/S][nk/(k-1)]^{1/2}$ where $n\bar{x}_i = \sum_{j=1}^{n} x_{ij}$, $k\bar{x} = \sum_{i=1}^{k} \bar{x}_i$ and $k(n-1)S^2 = \sum_{i=1}^{k} \sum_{j=1}^{n} (x_{ij} - \bar{x}_i)^2$. We accept H_{0i} against

H_i for any given i if $| t_i | \leqq t_{\alpha/2}$ where $t_{\alpha/2}$ is the upper $\alpha/2$ per cent point of Student's t distribution with $k(n - 1)$ d.f. Hence we accept H_0 against $H \neq H_0$ iff (if and only if) $\max_i | t_i | \leqq t_{\alpha/2}$, i.e., iff $\max_i | [(\bar{x}_i - \bar{x})/S][nk/(k - 1)]^{1/2} | \leqq t_{\alpha/2}$. This two-sided version of the Nair Statistic provides an alternative test in the analysis of variance situation and gives simultaneous confidence bounds on all $\mu_i - \mu (i = 1, 2, \cdots, k)$ with a confidence coefficient $1 - \alpha$. Power properties and multivariate and other generalizations of these tests are being investigated. (Received February 14, 1958.)

33. **On a Test for the Equality of Several Variances.** K. V. RAMACHANDRAN, Demographic Training and Research Centre, India, (By Title).

Let $x_{ij} (i = 1, 2, \cdots, k; j = 1, 2, \cdots, n)$ be random samples of sizes n from k univariate normal populations with means μ_i and variances σ_i^2 $(i = 1, 2, \cdots, k)$. The hypothesis $H_0: \sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2$ against H: Not H_0 is equivalent to the union of $H_{0i}: \sigma_i^2 = \sigma^2$ (say) against $H_i: \sigma_i^2 \neq \sigma^2$ (where σ^2 is unknown) for every $i = 1, 2, \cdots, k$. To test $H_{0i}: \sigma_i^2 = \sigma^2$ against $H_i: \sigma_i^2 \neq \sigma^2$ for any given i we have the test based on $F_i = S_i^2 / \sum_{l=1}^k S_l^2$ where $(n - 1) S_i^2 = \sum_{j=1}^n (x_{ij} - \bar{x}_i)^2 l \neq i$. We accept H_{0i} against H_i for any given i if $F_1' \leqq F_i \leqq F_2'$ where $\Pr\{F_i \leqq F_1' | H_{0i}\} + \Pr\{F_i \geqq F_2' | H_{0i}\} = \alpha$ and F_i has an F distribution with $(n - 1), (k - 1)(n - 1)$ d.f. Hence we accept H_0 against $H \neq H_0$ iff $F_1' \leqq F_{\min} \leqq F_{\max} \leqq F_2'$, where $F_{\min} = S_{\min}^2 / \sum_{l=1, l \neq (\min)}^k S_l^2$ and $F_{\max} = S_{\max}^2 / \sum_{l=1, l \neq (\max)}^k S_l^2$, i.e., iff $g_1' \leqq S_{\min}^2 / \sum_{l=1}^k S_l^2 \leqq S_{\max}^2 / \sum_{l=1}^k S_l^2 \leqq g_2'$. This two-sided version of Cochran's statistic provides an alternative test in the homogeneity of variances situation. The distribution problem, power properties and multivariate and other generalizations of these tests are being investigated. (Received February 14, 1958.)

34. **An Optimum Property of Some Bechhofer-Type Non-Sequential Multiple-Decision Rules.** WM. JACKSON HALL, University of North Carolina.

R. E. Bechhofer has proposed a single-sample multiple-decision procedure for ranking means of normal populations with known variances, and, with M. Sobel, a procedure for ranking variances of normal populations (*Ann. Math. Stat.*, Vol. 25 (1954), pp. 16–39 and 273–289). We assume that the sample sizes for the populations are equal, and, in the first case, that the variances are equal. Their rules guarantee a correct ranking with prescribed probability when the population parameters are sufficiently distinct (in a prescribed way). This paper proves that no other rules can accomplish this with a smaller sample size; that is, their rules are "most economical". This is not true if the sample sizes are unequal, but it is true for any analogous procedure for ranking populations according to a parameter when, for each sample, there is a numerical sufficient statistic with a monotone likelihood ratio and the parameter is a location or scale (but not range) parameter in the distribution of the statistic. These results are obtained from application of "most economical decision theory" (*Ann. Math. Stat.*, Vol. 25 (1954), p. 814). (Received February 17, 1958.)

35. **Second Order Rotatable Designs in Three or More Factors.** R. C. BOSE and NORMAN R. DRAPER, University of North Carolina.

Previous attempts to obtain second-order rotatable designs for three factors made use of the regular figures in three dimensions. A new method that has been successfully developed employs various sets of points which satisfy the conditions: $\sum x^2 = \sum y^2 = \sum z^2$, $\sum x^4 = \sum y^4 = \sum z^4$, $\sum x^2y^2 = \sum y^2z^2 = \sum z^2x^2$, all odd moments up to and including order four being zero. The basic sets may be combined in various ways to give a number of infinite classes of rotatable designs, each class dependent on a parameter. This parameter may take any value in a specified range, which depends only on the number of points in the design. By giving specific values to the parameters in the various classes, all of the second-order designs suggested in the Institute of Statistics Mimeo Series No. 149 by D. A. Gardiner, A. H. E. Grandage, and R. J. Hader were derived as special cases. An example of such a class is as follows. The $N = 20 + n_0$ points $(\pm a, \pm a, \pm a)$; $(\pm c_1, 0, 0)$, $(0, \pm c_1, 0)$, $(0, 0, \pm c_1)$; $(\pm c_2, 0, 0)$, $(0, \pm c_2, 0)$, $(0, 0, \pm c_2)$; $(0, 0, 0)$ (n_0 times) where $c_i^2 = \{N - 8a^2 \pm [N(16a^2 - N)]^{1/2}\}/12$ $(i = 1, 2)$ form a second-order rotatable design if $.0738N \geqq a^2 \geqq .0625N$. When $c_2 = 0$, the well-known cube and octahedron design, with center points, is obtained. The method has been used additionally to construct designs for both higher-order rotatability and higher dimension (number of factors). (Received February 17, 1958.)

36. A Markov Chain Resulting From a Certain Sorting Problem. A. BRUCE CLARKE, University of Michigan.

Consider the following sorting problem: Objects are chosen consecutively from an infinite population consisting of r different categories in proportions $p_1, p_2, \cdots, p_r \sum p_i = 1$. The objects chosen are sorted by category and placed in r piles. Periodically one of the categories $1, 2, \cdots, r$ is selected at random with probabilities $q_1, q_2, \cdots, q_r$, $\sum q_i = 1$, and the pile of elements of the selected category is removed from the system. Denoting the number of elements in the ith pile immediately preceding the tth pile removal by x_{ti}, the distribution of the random vector $x_t = (x_{ti}, \cdots, x_{tr})$ is studied as $t \to \infty$. This forms a stationary Markov chain. The limiting distributions of the individual components x_{ti}, $t \to \infty$, are obtained explicitly, and a recursion formula is established which leads to the limiting distribution of x_t. One result is that the mean total number of individuals in the system at any time, $E[\sum_{i=1}^r x_{ti}]$, is minimized if the probabilities q_i are chosen proportional to $\sqrt{p_i}$. (Received February 17, 1958.)

37. Fitting the Logistic by Maximum Likelihood. J. L. HODGES, JR., University of California, (By Title).

A method is presented by means of which the maximum likelihood estimates of the logistic response function may be quickly obtained to graphical accuracy, without the use of a computing machine or special charts. The basic idea is to replace the observed response numbers by equivalent ones for which the estimates are obvious. (Received February 19, 1958.)

38. Useful Bayes Solutions for Multiple Comparisons Problems. I. (Preliminary Report) DAVID B. DUNCAN, University of North Carolina.

A Bayes solution is developed for the common t-test problem of testing the hypothesis $\theta \leqq 0$ against the alternative $\theta > 0$ given observed values of x and s where x is normally distributed with θ as mean and variance σ^2 and s^2 is an independent estimate of σ^2 distributed as $\chi_\nu^2\sigma^2/\nu$. The ultimate objective is to solve many forms of multiple comparisons problems generated by the restricted products (Lehmann, *Ann. Math. Stat.*, 1957, pp. 1–25) of problems of the given form, the Bayes solutions to be obtained as corresponding products of solutions of the form developed. The loss function assumes losses proportional to $|\theta|$, the factor for type I errors being k times that for type II errors, $k \geqq 1$. The Bayes function is a normal density with mean 0 and variance $\gamma^2\sigma^2$. These functions fit, at least to a satisfactory degree of approximation, a wide variety of problems met in practice. The solution (restricted to invariant procedures) has the critical region $x/s > t$ where t is a function of the degrees of freedom ν, loss ratio k and dispersion ratio γ^2. A brief table of t with these three arguments is presented. (Research jointly supported by the U.S. Public Health Serv-

ice and by the U.S. Air Force through the Office of Scientific Research of the Air Research and Development Command.) (Received February 20, 1958.)

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39. Determining Bounds on Integrals with Applications to Cataloging Problems. BERNARD HARRIS.

Assume that a random sample of size N has been drawn from a multinomial population with an unknown and perhaps countably infinite number of classes. The experimenter wishes to predict $d(\alpha)$, the number of classes that will be observed in a second sample of size αN, $\alpha > 1$, (or when the sample size is increased by $(\alpha - 1)N$ additional observations); and $C(\alpha)$, the coverage of a second sample (or augmented sample), where $C(\alpha) = \sum p_i$, the sum is to be taken over those classes for which at least one representative has been observed in the sample. It is shown that $Ed(\alpha) \sim d + n_1 E\{[1 - e^{-(\alpha-1)x}]/x\}$, and $EC(\alpha) \sim 1 - (n_1/N) + (n_1/N)E\{1 - e^{-(\alpha-1)x}\}$ where d is the number of classes observed, n_1 is the number of classes occurring once in the sample, and the expectation is taken with respect to a distribution function unknown to the experimenter, but estimates of the moments are available. Hence a reasonable procedure is to compute upper and lower predictors of $d(\alpha)$ and $C(\alpha)$ by determining the suprema and infima of the above expected values subject to moment constraints.

Several results are given concerning bounds on integrals subject to moment constraints, and a method of determining the sharpest bounds is shown. The explicit solutions are computed for 0, 1, 2, 3 moment constraints and applied to several examples. (Received January 23, 1958.)

40. Single Server Queuing Processes with a Finite Number of Sources. GERALD HARRISON, The Teleregister Corporation.

A service system is considered which consists of a single server and a finite number of sources. The sources are assumed to be non-interacting and to have the same negative exponential idle time distribution. The service time is assumed to have an arbitrary distribution with a finite mean. There are no defections from the waiting line, and the service time is independent of the length of the waiting line. The stationary behavior of this service system is studied. The relations between load factor, mean delay, mean service time, mean source idle time, and proportion of calls delayed are obtained. The length of the waiting line at instants of termination of service is a Markov chain and its stationary distribution is thus reduced to solving a system of linear equations which, because of the form of the transition matrix, reduces to a simple iterative procedure. Under the assumption of the queuing discipline of service in the order of arrival the waiting time distribution is obtained. These results are specialized to the cases of constant and negative exponential service time distributions. (Received February 13, 1958.)