for all N sufficiently large. By Minkowski's inequality

$$m_{2n}(\phi)^{1/2n} \geqq |\sin(\phi - \theta)| \left(\frac{1}{N}\sum \xi_j^{2n}\right)^{1/2n} - \left(\frac{1}{N}\sum (u_j \sin\phi - v_j \cos\phi)^{2n}\right)^{1/2n}$$

Therefore with probability greater than $1 - \epsilon$,

$$\frac{\max_n (m_{2n}(\phi))^{1/2n}}{n} > \frac{\max_n (m_{2n}(\theta))^{1/2n}}{n}$$

for all N sufficiently large for all ϕ not in the interval $(\theta - \delta, \theta + \delta)$ (mod π).

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ON THE DECOMPOSITION OF CERTAIN χ^2 VARIABLES

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It is well known that if the sum, say $Q = Q_1 + Q_2$, of two stochastically independent variables is χ^2 with r d.f., and if Q_1 is also χ^2 with r_1 d.f., then Q_2 is likewise χ^2 with $r_2 = r - r_1$ d.f. If the hypothesis of stochastic independence is removed, little can be said about Q_2. It seems to us quite interesting that if the variables under consideration are real symmetric quadratic forms in either central or non-central, stochastically independent or dependent normal variables, and if the hypothesis of stochastic independence of Q_1 and Q_2 is replaced by the weaker hypothesis $Q_2 \geqq 0$, then Q_1 and Q_2 are stochastically independent so that Q_2 is itself a χ^2 variable with $r_2 = r - r_1$ d.f.

Before we state our theorem, we recall [1] that the real symmetric quadratic form $Y'BY$ in n mutually stochastically independent normal variables $Y' = (y_1, y_2, \cdots, y_n)$ with unit variances and means $U' = (u_1, u_2, \cdots, u_n)$ has a non-central χ^2 distribution whose characteristic function is

$$\varphi(t) = \exp\left[\frac{it\theta}{1 - 2it}\right] \Big/ (1 - 2it)^{r/2}$$

if and only if $B^2 = B$. Here, $\theta = U'BU$ and r is the rank of B.

THEOREM. *Let $Q = Q_1 + \cdots + Q_{k-1} + Q_k$, where $Q = X'AX$ and $Q_j = X'A_jX$, $j = 1, 2, \cdots, k$, are real symmetric quadratic forms in n normally distributed variables $X' = (x_1, x_2, \cdots, x_n)$ with means $M' = (m_1, m_2, \cdots, m_n)$ and real symmetric definite positive variance-covariance matrix V. Let $Q, Q_1, \cdots,$*

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Q_{k-1} *have non-central* χ^2 *distributions with parameters* r, θ *and* r_j, θ_j, $(j = 1, \cdots, k - 1)$, *respectively and let* Q_k *be non-negative. Then* $Q_1, Q_2, \cdots, Q_k$ *are mutually stochastically independent and* Q_k *has a non-central* χ^2 *distribution with parameters* $r_k = r - \sum_1^{k-1} r_j$, $\theta_k = \theta - \sum_1^{k-1} \theta_j$.

PROOF. We first prove the theorem for $k = 2$. There exists a real symmetric positive definite matrix C such that $C'C = V$. If we let $X = CY$, $Y' = (y_1, y_2, \cdots, y_n)$, and at the same time let $M = CU$, $U' = (u_1, u_2, \cdots, u_n)$, then $y_1, y_2, \cdots, y_n$ are mutually stochastically independent normal variables with unit variances and means $U' = (u_1, u_2, \cdots, u_n)$. Also

$$X'AX = X'A_1X + X'A_2X$$

becomes $Y'BY = Y'B_1Y + Y'B_2Y$, where $B = C'AC$, $B_1 = C'A_1C$, $B_2 = C'A_2C$, and $B = B_1 + B_2$. By hypothesis, $Y'BY$ and $Y'B_1Y$ have non-central χ^2 distributions and $Y'B_2Y \geqq 0$. Thus $B^2 = B$ and $B_1^2 = B_1$. With a suitably chosen orthogonal matrix L, $L'BL$ is a diagonal matrix having r ones and $n - r$ zeros on the principal diagonal. Since B_1 and B_2 are semi-definite positive, each element on the principal diagonal of $L'B_1L$ and $L'B_2L$ is non-negative and hence each of these matrices has a zero on the principal diagonal corresponding to each zero on that of $L'BL$. Moreover all elements in the rows and columns of $L'B_1L$ and $L'B_2L$ in which these zeros appear are likewise zero. If we properly choose our notation we may the write $L'BL = L'B_1L + L'B_2L$, using sub-matrices, as

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} G_r & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} H_r & 0 \\ 0 & 0 \end{pmatrix}.$$

If we multiply on the left by $L'B_1L$ and make use of $B_1^2 = B_1$, we have

$$L'B_1B_2L = 0.$$

That is, $B_1B_2 = 0$, so, by a result of Carpenter [1], $Y'B_1Y$ and $Y'B_2Y$ (that is, Q_1 and Q_2) are stochastically independent. Since Q and Q_1 have non-central χ^2 distributions it follows that Q_2 has a non-central χ^2 distribution with parameters $r_2 = r - r_1$, $\theta_2 = \theta - \theta_1$. For $k > 2$, the proof of the theorem is easily completed by induction.

As an example, let $(x_1, y_1), \cdots, (x_n, y_n)$ denote a random sample from a bivariate normal distribution having unit variances, means m_x and m_y, and correlation coefficient ρ. It is fairly obvious that the left member and the first term of the right member of

$$\sum_1^n (x_i^2 - 2\rho x_i y_i + y_i^2)/(1 - \rho^2) = (n\bar{x}^2 - 2\rho n\bar{x}\bar{y} + n\bar{y}^2)/(1 - \rho^2)$$
$$+ \sum_1^n [(x_i - \bar{x})^2 - 2\rho(x_i - \bar{x})(y_i - \bar{y}) + (y_i - \bar{y})^2]/(1 - \rho^2)$$

have non-central χ^2 distributions with parameters $r = 2n$,

$$\theta = n(m_x^2 - 2\rho m_x m_y + m_y^2) \,/\, (1 - \rho^2)$$

and $r_1 = 2$, $\theta_1 = n(m_x^2 - 2\rho m_x m_y + m_y^2)/(1 - \rho^2)$ respectively. Accordingly, the non-negative form

$$\sum_1^n [(x_i - \bar{x})^2 - 2\rho(x_i - \bar{x})(y_i - \bar{y}) + (y_i - \bar{y})^2]/(1 - \rho^2)$$

has a central χ^2 distribution with $2n - 2$ degrees of freedom.

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A NOTE ON THE GENERATION OF RANDOM NORMAL DEVIATES[1]

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1. Introduction. Sampling experiments often require the generation of large numbers of random normal deviates. When an electronic computer is used it is desirable to arrange for the generation of such normal deviates within the machine itself rather than to rely on tables. Pseudo random numbers can be generated by a variety of methods within the machine and the purpose of this note is to give what is believed to be a new method for generating normal deviates from independent random numbers. This approach can be used on small as well as large scale computers. A detailed comparison of the utility of this approach with other known methods (such as: (1) the inverse Gaussian function of the uniform deviates, (2) Teichroew's approach, (3) a rational approximation such as that developed by Hastings, (4) the sum of a fixed number of uniform deviates and (5) rejection-type approach), has been made elsewhere [1] by one of the authors (M.M.). It is shown that the present approach not only gives higher accuracy than previous methods but also compares in speed very favourably with other methods.

2. Method. The following approach may be used to generate a pair of random deviates from the same normal distribution starting from a pair of random numbers.

Method: Let U_1, U_2 be independent random variables from the same rectangular density function on the interval (0, 1). Consider the random variables:

$$X_1 = (-2 \log_e U_1)^{1/2} \cos 2\pi U_2$$
$$X_2 = (-2 \log_e U_1)^{1/2} \sin 2\pi U_2 \quad (1)$$

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