A SMOOTH INVERTIBILITY THEOREM[1]

By John W. Tukey

Princeton University

1. Introduction. In connection with discussions of fiducial inference (e.g., see 3), it is often desirable to consider the invertibility of certain mappings. We shall say that a mapping is smoothly invertible (of class α) if (condition (1) of [3] is irrelevant here):

(2) the mapping is 1-1 and hence has a single-valued inverse,

(3) this inverse is a continuous function (and has continuous derivatives of all orders up to α).

All too often, as has been emphasized to the author by L. J. Savage, the question of invertibility has been "answered" by showing that a Jacobian is of constant sign. It is, of course, well known that this does not suffice to give uniqueness in the large.

Explicit conditions sufficient for uniqueness in the large do not seem to be given frequently in the literature. The present note records an explicit theorem in a form which seems likely to be of service in such conditions.

2. A smooth invertibility theorem. We now state the smooth invertibility theorem as follows:

Any α times continuously differentiable mapping from an arcwise connected open domain (in n dimensions) *to a simply connected range, whose Jacobian determinant is continuous and of one sign throughout the domain, and whose inverse carries compact sets into compact sets is smoothly invertible of order α.*

In our application it is convenient to use the

Observation. *If the open domain and the simply connected range are both the whole plane (or the whole of any Euclidian space) then the inverse will carry compact sets into compact sets provided that it carries bounded sets into bounded sets.*

The proof of this observation follows immediately from the remarks that (i) the inverse image of closed sets by a continuous mapping are always closed, (ii) in the whole plane the compact sets are just those which are closed and bounded.

3. Proof. The proof of the theorem rests on a classical result about local inversion (which makes no use of arcwise connectedness, simple connectedness, or the hypothesis about the inverse taking compact sets into compact sets) and a purely topological result relating local uniqueness of inverses to global uniqueness (which makes no explicit use of the differentiability conditions).

We say that N is a local inverse neighborhood of x if

(1) $f(N)$ is a neighborhood of $y = f(x)$,

Received March 4, 1957.

[1] Prepared in connection with research sponsored by the Office of Naval Research.

(2) there is a choice $g_N(y')$ defined for y' in $f(N)$ such that $f(g_n(y')) = y'$ for any y' in $f(N)$.

If there is but one inverse image x' in N for each y' in $f(N)$, N is a unique local inverse neighborhood. If every x has a unique local inverse neighborhood, we say that $y = f(x)$ has unique local inverses.

The classical result (given for example in [1] on pp. 257–258 of Vol. 2) asserts that under our differentiability and Jacobian hypothesis, $y = f(x)$ has unique local inverses, and that, if we restrict the neighborhoods sufficiently, these inverses are α times continuously differentiable.

All that is lacking is the knowledge that these local inverses are unique, not only when we must go back to a neighborhood N of a particular solution, x of $y = f(x)$, but when we are free to go back to any x. This will follow from a special case of the covering homotopy theorem, which will be derived from a more usual form in the next section, namely: *If $y = f(x)$ is continuous and has unique local inverses, if X is a compact metric space, if x_t is a continuous image of the unit interval $0 \leqq t \leqq 1$ in X, if $y_{t,s}$ is a continuous image of the unit square $0 < t,\ s < 1$ in $f(X)$, and if $y_{t,1} = f(x_t)$, then it is possible to define a continuous image $x_{t,s}$ of the unit square in X so that*

$$x_{t,1} = x_t, \tag{1}$$

$$f(x_{t,s}) = y_{t,s} \tag{2}$$

(and indeed this can be done in at most one way.)

Using this result we can complete the proof of the smooth invertibility theorem as follows: Let x_0 and x_1 be any two solutions (possibly coincident) of $y_0 = f(x)$. Since X is arcwise connected, we may join x_0 to x_1 by an arc x_t for $0 \leqq t \leqq 1$. Let $y_{t,1} = f(x_t)$. Since $y_{0,1} = y_{1,1} = y_0$ the image of this arc is a closed curve (in $f(X)$). Since $f(X)$ is simply connected, this curve can be shrunk to the point y_0, keeping its ends at y_0; that is to say, we can define $y_{t,s}$ for $0 \leqq t,\ s < 1$ as a continuous extension of $y_{t,1}$ with $y_{t,0} \equiv y_{0,s} \equiv y_{1,s} \equiv y_0$. Let H be the set of all y of the form $y_{t,s}$ for $0 \leqq t,\ s < 1$. As the continuous image of a compact space this will be compact and will hence be closed in $f(X)$. Let G be the set of x for which $y = f(x)$ lies in H. Because of our hypothesis, G will also be compact. Surely G contains x_t for $0 \leqq t \leqq 1$.

Now apply the topological result to G. We have then a continuous image $x_{t,s}$ of $0 \leqq t,\ s \leqq 1$ which satisfies $x_{0,1} = x_0$, $x_{1,1} = x_1$, and $f(x_{t,0}) \equiv f(x_{0,s}) \equiv f(x_{1,s}) = y_0$.

The images of the three sides $s = 0$, $t = 0$, and $t = 1$ of the unit square thus provide an arc leading from x_0 to x_1 every point of which maps into y_0. The local uniqueness of inverses now ensures that this arc is a constant mapping, and hence that, in particular $x_0 = x_1$. Thus the solution of $y_0 = f(x)$ is shown to be unique, and the proof of the smooth invertibility theorem is concluded.

There is some interest in the need for all the topological hypotheses of this theorem, so examples are given in Section 5 to show that no one of these three hypotheses can be removed without the conclusion failing.

4. Proof of topological result. The textbook of Seifert and Threlfall [2] gives, as Satz I and Satz II (pp. 186–188), theorems from which we can immediately derive a close analog of the result we used. The differences will be as follows: (i) the result wil be restricted to a topological complex (which would suffice, since we wish to apply it only to n-dimensional open domains), (ii) the condition that the inverse image of compact sets is compact would be absent, (iii) the condition of unique local inverses would be strengthened to local homeomorphism. We need then only show that our extra condition implies local homeomorphism.

Let y be a point of $f(X)$. Let $x_1, x_2, \cdots, x_k$ be its inverse images. Let N_1, $N_2, \cdots, N_k$ be corresponding unique local inverse neighborhoods. Let K be the part of X not in any of these N_1. Consider L, the complement of the closure of $f(K)$. If y is in L, then the intersection of L with all $f(N_1)$ is a neighborhood of y each of whose points has exactly k inverse images, one each in N_1, N_2, $\cdots, N_k$, and since these local inverses are continuous, we have the desired local homeomorphism.

If y is not in L, then it is in f(K), the closure of $f(K)$, and there is a sequence y_j in $f(K)$ which converges to y. Let $Y = y, y_1, y_2, \cdots$. Then Y is compact.

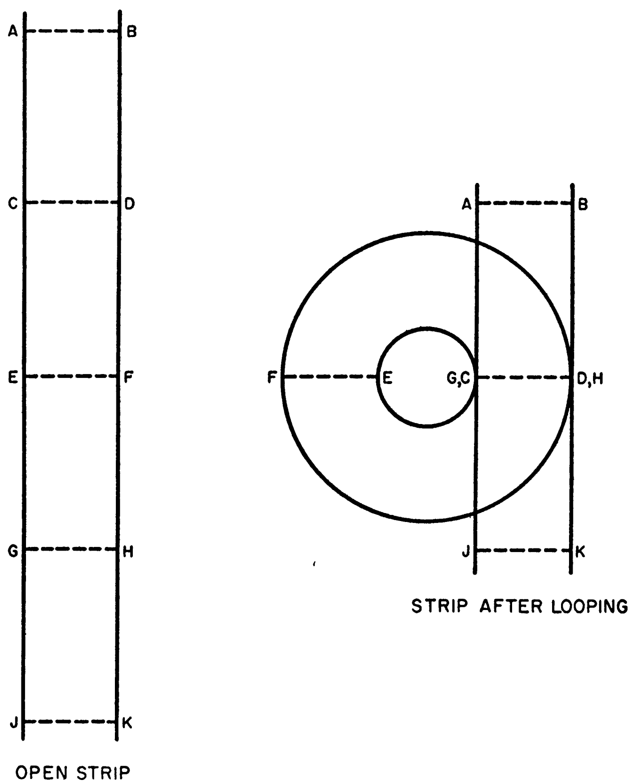

FIG. 1. Second step in the mapping

Let x_j in K satisfy $f(x_j) = y_j$. Then x_j is in the inverse image of Y which is compact. Hence we can extract a subsequence x_{j_n} which converges to some x_0. Clearly $f(x_0) = \lim f(x_{j_n}) = \lim y_{j_n} = y$. Hence x_0 is in some N_j, and hence not in K, which is a contradiction. The desired result is thus proved.

5. Remarks and counter examples. In this section we show that no one of the three topological hypotheses can be omitted. In each case we show that uniqueness of the inverse is immediately lost. For the first two hypotheses the example can be very simple and one-dimensional. For the third a two-dimensional, not-too-simple example is provided.

If we drop the connectedness requirement for X, then we may take X as two non-intersecting straight lines and $f(x)$ as a rigid application of each on a third. Uniqueness is lost. (Consideration shows, indeed, that ϵ-connectedness for all $\epsilon > 0$ would suffice.)

If we drop the simple connectedness of $f(X)$, then we may take X as a circle, $f(X)$ as a circle of half that diameter, and the mapping $f(X)$, as the winding of an inextensible loop of string around the smaller circle. See Fig. 1. (We cannot use the whole infinite line without having the inverse image of a compact set become non-compact.)

In the third example we drop the requirement on compactness of inverse images. Here the example is a mapping of the plane into part of itself which is most easily described qualitatively and geometrically. We begin by deforming the plane into a long open strip, which can clearly be done with a positive, non-zero Jacobian. We now consider a transformation of the strip into a simply looped strip in which uniqueness of inverse has been lost but simple connectedness of range has not been achieved. Graphically, corresponding points appear as in Fig. 1. The failure of simple connectivity is due to the small circular disk bounded by the image of the arc CEG. If we loop the strip again in the same way, we can use the new loop to cover this hole, meanwhile placing the new hole on the old loop. The resulting transformation will have a simply connected range, but the inverses will not be unique. This is the desired third example.

REFERENCES

[1] A. Ostrowski, *Vorlesungen uber Diffential und Integralrechnung*, Basel, Birkhauser, 1951.

[2] H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Leipzig, Taubner, 1938, and New York, Chelsea, 1947.

[3] John W. Tukey, "Some examples with fiducial relevance," *Ann. Math. Stat.*, Vol. 28, (1957), pp. 687–695.