SUMS OF POWERS OF INDEPENDENT RANDOM VARIABLES[1]

By J. M. Shapiro

Ohio State University

1. Summary and introduction. Let (x_{nk}), $k = 1, \cdots, k_n$; $n = 1, 2, \cdots$ be a double sequence of infinitesimal random variables which are rowwise independent (i.e., $\lim_{n\to\infty} \max_{1\le k\le k_n} P(|x_{nk}| > \epsilon) = 0$ for evey $\epsilon > 0$, and for each n $x_{n1}, \cdots, x_{nk_n}$ are independent). Let $S_n = x_{n1} + \cdots + x_{nk_n} - A_n$ where the A_n are constants and let $F_n(x)$ be the distribution function of S_n. Necessary and sufficient conditions for $F_n(x)$ to converge to a distribution function $F(x)$ are known, and in particular we know that $F(x)$ is infinitely divisible.

In this paper we shall investigate the system of infinitesimal, rowwise independent random variables $(|x_{nk}|^r)$, $r \geqq 1$. In particular we shall be interested in large values of r. Specifically, let $S_n^r = |x_{n1}|^r + \cdots + |x_{nk_n}|^r - B_n(r)$, where $B_n(r)$ are suitably chosen constants. Let $F_n^r(x)$ be the distribution function of S_n^r. Necessary and sufficient conditions for $F_n^r(x)$ to converge $(n \to \infty)$ to a distribution function $F^r(x)$ are given, and also necessary and sufficient conditions for $F^r(x)$ to converge $(r \to \infty)$ to a distribution function $H(x)$ are given. The form that $H(x)$ must take is obtained and under rather general conditions it is shown that $H(x)$ is a Poisson distribution. In any case it is shown that $H(x)$ is the sum of two independent random variables, one Gaussian and the other Poisson (including their degenerate cases).

2. Notation. Let $F(x)$ be any infinitely divisible distribution function with characteristic function $\varphi(t)$. According to the formulas of Lévy and Khintchine (cf. [1]) we know that $\varphi(t)$ has the following representation:

$$\begin{aligned}\varphi(t) = \exp\Big\{ & i\gamma(\tau)t - \tfrac{1}{2}\sigma^2 t^2 + \int_{-\infty}^{-\tau} (e^{itu} - 1)\, dM(u) \\ & + \int_{\tau}^{\infty} (e^{itu} - 1)\, dN(u) + \int_{-\tau}^{0-} (e^{itu} - 1 - itu)\, dM(u) \\ & + \int_{0+}^{\tau} (e^{itu} - 1 - itu)\, dN(u), \end{aligned} \tag{2.1}$$

where $M(u)$ and $N(u)$ are respectively nondecreasing functions in the intervals $(-\infty, 0)$, $(0, +\infty)$ which satisfy $M(-\infty) = N(+\infty) = 0$ and $\int_{-\epsilon}^{0} u^2\, dM(u) + \int_0^{\epsilon} u^2\, dN(u) < \infty$ for every $\epsilon > 0$; σ is a nonnegative constant; τ and $-\tau$ are continuity points of $N(u)$ and $M(u)$; and $\gamma(\tau)$ is a constant depending only on τ.

It is well known that the distribution functions $F^r(x)$ and $H(x)$ referred to in Section 1 are infinitely divisible, and throughout this paper we let $M^r(u)$ and

Received August 28, 1957; revised November 12, 1957.

[1] Presented to the American Mathematical Society August 29, 1957.

$N^r(u)$ be associated with $F^r(x)$ and $M^*(u)$ and $N^*(u)$ be associated with $H(x)$, through the formulas given for their characteristic functions analogous to (2.1).

We let $F_{nk}(x)$ and $F^r_{nk}(x)$ be the distribution functions of x_{nk} and $| x_{nk} |^r$ respectively.

When speaking of a random variable (or its distribution function) being Poisson we shall mean it is either Poisson or its degenerate case (i.e., a random variable taking the value 0 with probability 1). The same applies to a Gaussian random variable) in this case the degenerate case is a random variable taking the value m with probability 1).

If $K(x)$ is a nondecreasing function when we write $\lim_{n\to\infty} K_n(x) = K(x)$ it is understood that this need only hold at continuity points of $K(x)$.

3. General results and proofs.

THEOREM 1. *Let* $\lim_{n\to\infty} F^r_n(x) = F^r(x)$ *for* $r \geqq r_0 \geqq 1$ *and* $\lim_{r\to\infty} F^r(x) = H(x)$, *where* $F^r(x)$ *and* $H(x)$ *are distribution functions. Then* $H(x)$ *is the distribution function of the sum of two independent random variables one of which is Gaussian and the other Poisson.*

We remark that Theorem 1 remains valid if we assume $\lim_{n\to\infty} F^r_n(x) = F^r(x)$ for some sequence of values of r becoming infinite in place of this condition holding for $r \geqq r_0$.

The proof of Theorem 1 requires the following lemma.

LEMMA 1. *If we add to the hypothesis of Theorem* 1 *the condition that* $\lim_{n\to\infty} F_n(x) = F(x)$, *the conclusion of Theorem* 1 *holds.*

Proof. Since $\lim_{n\to\infty} F_n(x) = F(x)$ by Theorem 1 on page 116 of [1], we see

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} F_{nk}(x) = M(x), \qquad x < 0,$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} (F_{nk}(x) - 1) = N(x), \qquad x > 0,$$

where $M(x)$ and $N(x)$ are given by (2.1). Now for $\alpha \geqq 0$,

$$F^r_{nk}(\alpha) \equiv P(| x_{nk} |^r \leqq \alpha) = F_{nk}(\alpha^{1/r}) - F_{nk}(- \alpha^{1/r} -)$$

and for $\alpha < 0$, $F^r_{nk}(\alpha) = 0$. Thus for $x < 0$, $\lim_{n\to\infty} \sum_{k=1}^{k_n} F^r_{nk}(x) = 0$, and for $x > 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} [F^r_{nk}(x) - 1] = \lim_{n\to\infty} \sum_{k=1}^{k_n} [F_{nk}(x^{1/r}) - 1] + \lim_{n\to\infty} \sum_{k=1}^{k_n} [-F_{nk}(-x^{1/r}-)].$$

Now assume that $x^{1/r}$ and $-x^{1/r}$ are continuity points of $N(x)$ and $M(x)$ respectively. Note that the set of points $x > 0$ for which this is true is dense on the positive axis. For such x we have $\lim_{n\to\infty} \sum_{k=1}^{k_n} [F^r_{nk}(x) - 1] = N(x^{1/r}) -$

$M(-x^{1/r})$. We note that the function $N(x^{1/r}) - M(-x^{1/r})$ and the function

$$\sum_{k=1}^{k_n} [F^r_{nk}(x) - 1]$$

are both nondecreasing for $x > 0$ and hence $\lim_{n\to\infty} \sum_{k=1}^{k_n} [F^r_{nk}(x) - 1] = N(x^{1/r}) - M(-x^{1/r})$ at all continuity points, $x > 0$, of $N(x^{1/r}) - M(-x^{1/r})$. Now since $\lim_{n\to\infty} F^r_n(x) = F^r(x)$ we see by Theorem 1 on page 116 of [1] that $M^r(x) \equiv 0$ and $N^r(x) = N(x^{1/r}) - M(-x^{1/r})$. (Note that since $\int_{-\epsilon}^{0} x^2\, dM(x) + \int_0^\epsilon x^2\, dN(x) < \infty$ it follows that for $r \geqq 1$, $\int_{-\epsilon}^{0} x^2\, dM^r(x) + \int_0^\epsilon x^2\, dN^r(x) < \infty$.) Now since $\lim_{r\to\infty} F^r(x) = H(x)$, it follows by Theorem 2 on page 88 of [1] that $\lim_{r\to\infty} M^r(x) = M^*(x)$ and $\lim_{r\to\infty} N^r(x) = N^*(x)$ at continuity points of $M^*(x)$ and $N^*(x)$. This shows that $M^*(x) \equiv 0$ and

$$N^*(x) = \lim_{r\to\infty} [N(x^{1/r}) - M(-x^{1/r})] = \begin{cases} N(1+) - M(-1-), & x > 1, \\ N(1-) - M(-1+), & 0 < x < 1. \end{cases}$$

This shows that $N^*(x)$ is constant for $0 < x < 1$ and for $x > 1$ and hence (since $M^*(-\infty) = N^*(+\infty) = 0$) we see that $N(1+) = 0$ and $M(-1-) = 0$. Thus we see $N^*(x)$ is either identically 0 or takes one jump at $x = 1$. (In fact if both $M(x)$ is continuous at -1, $N(x)$ is continuous at $+1$ then $N^*(x) = 0$; otherwise $N^*(x)$ takes one jump). Now let σ^* be the nonnegative constant associated with $H(x)$ by the formula (2.1). Then if $\sigma^* = 0$ and $N^*(x)$ takes one jump it is clear that $H(x)$ is Poisson or $H(x - m)$ is Poisson (m a constant). If $\sigma^* = 0$ and $N^*(x) \equiv 0$, $H(x)$ is a unitary distribution. If $\sigma^* = 0$ and $N^*(x) \equiv 0$, $H(x)$ is Gaussian; and if $\sigma^* = 0$ and $N^*(x)$ takes one jump, then (cf. [1]) it follows that $H(x)$ is the sum of two independent random variables one Gaussian and the other Poisson. This proves the lemma.

Proof of Theorem 1. Let $s \geqq r_0$ and let $y_{nk} = |x_{nk}|^s$. Then $|x_{nk}|^r = |y_{nk}|^{r/s}$. Then for $r/s \geqq 1$, under the conditions of Theorem 1 the conditions of Lemma 1 are satisfied with the system (x_{nk}) replaced by (y_{nk}). This proves Theorem 1.

LEMMA 2. *If* $\lim_{n\to\infty} F_n(x) = F(x)$, *then for suitably chosen constants* $B_n(r)$, $F^r_n(x)$ *converges to a distribution function* $F^r(x)$ *if and only if*[2]

$$\lim_{\epsilon\to 0} \operatorname*{lim\,{}^{sup}_{inf}}_{n\to\infty} \sum_{k=1}^{k_n} \left\{ \int_0^\epsilon x^{2r}\, d[F_{nk}(x) - F_{nk}(-x-)] - \left(\int_0^\epsilon x^r\, d[F_{nk}(x) - F_{nk}(-x-)] \right)^2 \right\} = \sigma_r^2 < \infty. \tag{3.1}$$

Proof. Suppose $\lim_{n\to\infty} F_n(x) = F(x)$ and that (3.1) holds. Then as in the proof Lemma 1, $\lim_{n\to\infty} \sum_{k=1}^{k_n} F^r_{nk}(x) = 0 \equiv M^r(x)$ for $x < 0$, and

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} (F^r_{nk}(x) - 1) = N(x^{1/r}) - M(-x^{1/r}) \equiv N^r(x) \qquad \text{for } x > 0.$$

[2] We use the notation $\operatorname{lim\,{}^{sup}_{inf}}$ to mean that the indicated condition is to hold for both lim inf and lim sup.

(Here we consider $N^r(x)$ and $M^r(x)$ only as functions just defined and not, at this point, as being associated with any distribution function.) We see that $M^r(-\infty) = N^r(+\infty) = 0$ and that $\int_{-\epsilon}^{0} x^2\, dM^r(x) + \int_0^{\epsilon} x^2\, dN^r(x) < \infty$. Consider

$$\lim_{\epsilon \to 0} \limsup_{n\to\infty}\inf \sum_{k=1}^{k_n} \left\{ \int_{|x|<\epsilon} x^2\, dF^r_{nk}(x) - \left(\int_{|x|<\epsilon} x\, dF^r_{nk}(x) \right)^2 \right\}$$

$$= \lim_{\epsilon \to 0} \limsup_{n\to\infty}\inf \sum_{k=1}^{k_n} \left\{ \int_0^{\epsilon} x^2\, d[F_{nk}(x^{1/r}) - F_{nk}(-x^{1/r}-)] \right.$$

$$(3.2) \qquad \left. - \left(\int_0^{\epsilon} x\, d[F_{nk}(x^{1/r}) - F_{nk}(-x^{1/r}-)] \right)^2 \right\}$$

$$= \lim_{\epsilon \to 0} \limsup_{n\to\infty}\inf \sum_{k=1}^{k_n} \left\{ \int_0^{\epsilon^{1/r}} x^{2r}\, d[F_{nk}(x) - F_{nk}(-x-)] \right.$$

$$\left. - \left(\int_0^{\epsilon^{1/r}} x^r\, d[F_{nk}(x) - F_{nk}(-x-)] \right)^2 \right\} = \sigma_r^2,$$

using condition (3.1). (Note r is fixed here.) Now by choosing

$$B_n(r) = \sum_{k=1}^{k_n} \int_{|x|<\tau} x\, dF^r_{nk}(x) - C_r + o\,(1),$$

where C_r is a constant and $o(1) \to 0$ as $n \to \infty$, we see by Theorem 1 on page 116 of [1] that $F^r_n(x)$ converges to a distribution $F^r(x)$. (We note that $M^r(x)$, $N^r(x)$ and σ_r^2 are associated with $F^r(x)$ through the formulas (2.1).)

Now suppose that $F^r_n(x) \to F^r(x)$. Then again using the theorem of [1] referred to above we see that (3.2) holds and hence that (3.1) holds.

THEOREM 2. *Under the conditions of Lemma 2 a necessary and sufficient condition for the distribution functions $F^r(x)$ to converge ($r \to \infty$) to a distribution function $H(x)$ for suitably chosen constants $B_n(r)$, is that*[3]

$$(3.3) \qquad M(x) = 0 \quad \text{for} \quad x < -1, \qquad N(x) = 0 \quad \text{for} \quad x > 1, \qquad \lim_{r\to\infty} \sigma_r^2 = (\sigma^*)^2.$$

Furthermore $H(x)$ is Gaussian if $M(x)$ is continuous at -1 and $N(x)$ is continuous at $+1$, $H(x - m)$ is Poisson if $\sigma^* = 0$ and either $M(x)$ is discontinuous at -1 or $N(x)$ is discontinuous at $+1$ where m is a constant, and $H(x)$ is the sum of two independent random variables, one Gaussian and the other Poisson otherwise.

Proof. Suppose $\lim_{r\to\infty} F^r(x) = H(x)$. Then as in the proof of Lemma 1 we see that $M(-1-) = 0$ and $N(1+) = 0$ and hence $M(x) = 0$ for $x < -1$ and $N(x) = 0$ for $x > 1$. Now by Theorem 2 on page 88 of [1] we have

$$\lim_{\epsilon \to 0} \limsup_{r\to\infty}\inf \left\{ \int_{-\epsilon}^{0} u^2\, dM^r(u) + \sigma_r^2 + \int_0^{\epsilon} u^2\, dN^r(u) \right\} = (\sigma^*)^2.$$

[3] Same notation as in the proofs of Lemmas 1 and 2.

Now

$$\left\{\int_{-\epsilon}^{0} u^2\, dM^r(u) + \int_0^{\epsilon} u^2\, dN^r(u)\right\} = \int_0^{\epsilon} u^2\, d[N(u^{1/r}) - M(-u^{1/r})]$$

$$= \int_0^{\epsilon^{1/r}} y^{2r}\, d[N(y) - M(-y)]$$

$$\leqq \epsilon\left\{\int_0^{1} y^2\, dN(y) + \int_{-1}^{0} y^2\, dM(y)\right\} \qquad \text{for } r > 1 \text{ and } 0 < \epsilon < 1.$$

Thus we see that $\lim\inf_{r\to\infty} \sigma_r^2 = (\sigma^*)^2 = \lim\sup_{r\to\infty} \sigma_r^2$.

Now suppose (3.3) holds. Then as in the proof of Lemma 1 we see

$$\lim_{r\to\infty} N^r(x) = N^*(x) = \begin{cases} N(1+) - M(-1-) & \text{for } x > 1 \\ N(1-) - M(-1+) & \text{for } 0 < x < 1 \end{cases}$$

and $\lim_{r\to\infty} M^r(x) = 0 \equiv M^*(x)$. (Here we consider M^* and N^* as functions just defined and not at this point as being associated with $H(x)$.) Now from (3.3) it follows that $N^*(+\infty) = M^*(-\infty) = 0$ and $\int_{-\epsilon}^{0} x^2\, dM^*(x) + \int_0^{\epsilon} x^2\, dN^*(x) < \infty$. Also since

$$\lim_{\epsilon\to 0} \lim_{r\to\infty}{}^{\sup}_{\inf} \left\{\int_{-\epsilon}^{0} u^2\, dM^r(u) + \int_0^{\epsilon} u^2\, dN^r(u)\right\} = 0$$

(from the first part of this proof), it follows from (3.3) that

$$\lim_{\epsilon\to 0} \lim_{r\to\infty}{}^{\sup}_{\inf} \left\{\int_{-\epsilon}^{0} u^2\, dM^r(u) + \sigma_r^2 + \int_0^{\epsilon} u^2\, dN^r(u)\right\} = (\sigma^*)^2.$$

Now by Theorem 1 on page 116 of [1] we see that $\gamma_r(\tau) = \sum_{k=1}^{k_n} \int_{|x|<t} x\, dF_{nk}^r(x) - B_n(r) + o(1)$, where $\gamma_r(\tau)$ is associated with $F^r(x)$ through the formulas (2.1). Thus by the proper choice of $B_n(r)$, $\gamma_r(\tau)$ converges $(r \to \infty)$ to some constant $\gamma_*(\tau)$, (τ fixed). But using Theorem 2 on page 88 of [1], we see that $\lim_{r\to\infty} F^r(x) = H(x)$, where $H(x)$ is the infinitely divisible distribution determined by M^*, N^*, $\gamma_*(\tau)$ and $(\sigma^*)^2$ given above. It remains to show the form for $H(x)$, but this follows as in the proof of Lemma 1.

4. Characterization of the Poisson distribution. In this section we give conditions which will insure that the distribution functions $F^r(x)$ will converge to the Poisson distribution. We use the same notation as in the previous sections. In particular $M(x)$ and $N(x)$ are associated with the distribution function $F(x)$, the limiting distribution of $F_n(x)$.

THEOREM 3. *If $F_n(x)$ converges to $F(x)$, $M(x) = 0$ for $x < -1$, $N(x) = 0$ for $x > 1$, and*

$$\text{(4.1)} \qquad \sum_{k=1}^{k_n} \int_{|x|<\epsilon} |x|^s\, dF_{nk}(x) \quad \textit{is bounded in } n \textit{ for some } s < 2r,$$

then for suitably chosen constants $B_n(r)$, $F_n^r(x)$ converges ($n \to \infty$) to a distribution function $F^r(x)$ and $F^r(x)$ converges ($r \to \infty$) to the Poisson distribution. (No matter what the choice of $B_n(r)$, if $F_n^r(x) \to F^r(x)$ and $F^r(x) \to H(x)$, then there exists a constant m such that $H(x - m)$ is Poisson.)

We postpone the proof of Theorem 3 as well as that of the next three theorems. In the rest of the paper it will be convenient to assume $r > 1$.

THEOREM 4. *Condition* (4.1) *of Theorem* 3 *may be replaced by*

(4.2) *The random variables* (x_{nk}) *are symmetric about the origin.*

THEOREM 5. *Let* $S_n = x_{n1} + \cdots + x_{nk_n}$ (i.e., *let* $A_n = 0$) *and suppose* $F_n(x) \to F(x)$. *Let* $N(x) = 0$ *for* $x > 1$. *Then if the* (x_{nk}) *are positive random variables the conclusion of Theorem* 3 *holds.*

THEOREM 6. *Let* $A_n = 0$, $F_n(x) \to F(x)$, $M(x) = 0$ *for* a < -1, *and* $N(x) = 0$ *for* $x > 1$. *Then if the* (x_{nk}) *are identically distributed within each row the conclusion of Theorem* 3 *holds.*

Proof of Theorem 3. We first show that condition (4.1) implies condition (3.1) of Lemma 2 with $\sigma_r^2 = 0$. We have

$$\sum_{k=1}^{k_n}\left\{\int_0^{\epsilon} x^{2r}\, d[F_{nk}(x) - F_{nk}(-x-)] - \left(\int_0^{\epsilon} x^r\, d[F_{nk}(x) - F_{nk}(-x-)]\right)^2\right\}$$

$$\leqq \sum_{k=1}^{k_n}\left\{\int_0^{\epsilon} x^{2r}\, d[F_{nk}(x) - F_{nk}(-x-)]\right\}$$

$$\leqq \epsilon^{2r-s} \sum_{k=1}^{k_n}\left\{\int_0^{\epsilon} |x|^s\, d[F_{nk}(x) - F_{nk}(-x-)]\right.$$

$$= \epsilon^{2r-s} \sum_{k=1}^{k_n}\left\{\int_0^{\epsilon} |x|^s\, dF_{nk}(x) + \int_{-\epsilon}^{0} |x|^s\, dF_{nk}(x-)\right\},$$

and since $2r - s > 0$ we see by (4.1) that (3.1) holds with $\sigma_r^2 = 0$. Thus from Lemma 2, $F_n^r(x) \to F^r(x)$. Also since $\lim_{r\to\infty} \sigma_r^2 = 0 = (\sigma^*)^2$, it follows from Theorem 2 that $F^r(x) \to H(x)$ and that $H(x - m)$ is a Poisson distribution. (This includes the possibility that $H(x)$ may be a degenerate Gaussian distribution.) We note that $B_n(r)$ could be chosen so as to make $m = 0$. This proves the theorem.

Proof of Theorem 4. We only need to show that (4.2) implies (4.1). Let $\alpha_{nk} = \int_{|x|<\tau} x\, dF_{nk}(x)$ for some $\tau > 0$. By Theorem 2 on page 111 of [1] we have

$$\sum_{k=1}^{k_n} \int_{-\infty}^{\infty} \frac{x^2}{1 + x^2}\, dF_{nk}(x + \alpha_{nk})$$

is bounded. But since the random variables are symmetric it follows that $\alpha_{nk} = 0$ and hence

$$\sum_{k=1}^{k_n} \int_{|x|<\epsilon} x^2\, dF_{nk}(x) \leqq (1+\epsilon^2) \sum_{k=1}^{k_n} \int_{|x|<\epsilon} \frac{x^2}{1+x^2}\, dF_{nk}(x)$$

$$\leqq (1+\epsilon^2) \sum_{k=1}^{k_n} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2}\, dF_{nk}(x)$$

is bounded. Thus (4.1) holds with $s = 2$, i.e., for $r > 1$.

Proof of Theorem 5. Since the x_{nk} are positive it follows from Theorem 1 on page 116 of [1] that $M(x) \equiv 0$ for $x < 0$, and that $\sum_{k=1}^{k_n} \int_{|x|<\tau} x\, dF_{nk}(x) = \sum_{k=1}^{k_n} \int_0^\tau x\, dF_{nk}(x)$ converges to a constant $\gamma(\tau)$ (note $A_n = 0$). Thus

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \left[\int_{|x|<\epsilon} x\, dF_{nk}(x)\right]^2 = \lim_{n\to\infty} \sum_{k=1}^{k_n} \left[\int_0^\epsilon x\, dF_{nk}(x)\right]^2$$

$$\leqq \lim_{n\to\infty} \left[\max_{1\leqq k\leqq k_n} \left(\int_0^\epsilon x\, dF_{nk}(x)\right)\right]\left[\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_0^\epsilon x\, dF_{nk}(x)\right] = 0,$$

since $\lim_{n\to\infty} \max_{1\leqq k\leqq k_n} \int_0^\epsilon x\, dF_{nk}(x) = 0$ (infinitesimalness). Now again from Theorem 1 on page 116 of [1] we have

$$\lim_{\epsilon\to 0} \lim{}^{\sup}_{\inf}{}_{n\to\infty} \sum_{k=1}^{k_n} \left\{\int_{|x|<\epsilon} x^2\, dF_{nk}(x) - \left(\int_{|x|<\epsilon} x\, dF_{nk}(x)\right)^2\right\} = \sigma^2$$

so that

$$\lim_{\epsilon\to 0} \lim{}^{\sup}_{\inf}{}_{n\to\infty} \sum_{k=1}^{k_n} \int_{|x|<\epsilon} x^2\, dF_{nk}(x) = \sigma^2 < \infty.$$

Thus $\sum_{k=1}^{k_n} \int_{|x|<\epsilon} x^2\, dF_{nk}(x)$ is bounded in n so that (4.1) holds with $s = 2$. This proves Theorem 5.

Proof of Theorem 6. Since $A_n = 0$ we again have

$$\sum_{k=1}^{k_n} \int_{|x|<\tau} x\, dF_{nk}(x) = k_n \int_{|x|<\tau} x\, dF_{n1}(x) \to \gamma(\tau).$$

Also

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \left(\int_{|x|<\epsilon} x\, dF_{nk}(x)\right)^2 = \lim_{n\to\infty} k_n \left(\int_{|x|<\epsilon} x\, dF_{n1}(x)\right)^2$$

$$= \lim_{n\to\infty} k_n \int_{|x|<\epsilon} x\, dF_{n1}(x) \cdot \lim_{n\to\infty} \int_{|x|<\epsilon} x\, dF_{n1}(x)$$

$$= \gamma(\tau)\cdot 0 = 0,$$

since the random variables (x_{nk}) are infinitesimal. From this point the proof is identical to that of Theorem 5.

The next theorem shows the existence of a double sequence of random variables $(|x_{nk}|^{r_n})$ such that the distribution functions of the row sums (minus a constant) converge to the Poisson distribution.

THEOREM 7. *Under the conditions of any one of the Theorems* 3 *through* 6 *there exists a sequence of numbers* $r_n \to \infty$ *such that the distribution functions of the sums* $|x_{n1}|^{r_n} + \cdots + |x_{nk_n}|^{r_n} - B_n(r_n)$, *($B_n(r_n)$ suitably chosen constants) converge to the Poisson distribution.*[4]

Proof. We have $\lim_{n\to\infty} F_n^r(x) = F^r(x)$ and $\lim_{r\to\infty} F^r(x) = H(x)$, where $H(x)$ is a Poisson distribution. (In particular the first limit relation holds for $r = 2, 3, \cdots$.) Let $\{\xi_k\}$, $k = 1, 2, \cdots$, be a countable dense set on the real line such that $F_n^r(\xi_k) \to F^r(\xi_k)$ for $r = 2, 3, \cdots$ and $F^r(\xi_k) \to H(\xi_k)$ for all k. Let $\{\epsilon_n\}$ be a positive decreasing sequence of real numbers such that $\epsilon_n \to 0$ as $n \to \infty$. Let $\{n_r\}$ be an increasing subsequence of the positive integers such that $n \geqq n_r$ implies that $|F_n^r(\xi_k) - F^r(\xi_k)| < \epsilon_r$ for $k = 1, 2, \cdots, r$ (r fixed). Consider the sequence of distribution functions S: $F_1^2(x), F_2^2(x), \cdots, F_{n_3-1}^2(x), F_{n_3}^3(x), \cdots, F_{n_4-1}^3(x), F_{n_4}^4(x), \cdots F_{n_5-1}^4(x), \cdots$. We claim this sequence converges to $H(x)$ for $x = \xi_k$ for $k = 1, 2, \cdots$. Consider ξ_k. Let $\epsilon > 0$ be given. Let r_0, ($r_0 > k$) be such that $r \geqq r_0$ implies $\epsilon_r < \epsilon/2$. Let $r_1 \geqq r_0$ be such that $r \geqq r_1$ implies $|F^r(\xi_k) - H(\xi_k)| < \epsilon/2$. Then for $n > N(\xi_k) = n_{r_1}$ consider

$$|F_u^r(\xi_k) - H(\xi_k)| \leqq |F_n^r(\xi_k) - F^r(\xi_k)| + |F^r(\xi_k) - H(\xi_k)|.$$

Since we are considering only elements of the sequence S we have $n > n_{r_1}$ implies $r \geqq r_1 \geqq r_0 > k$. Therefore $|F_n^r(\xi_k) - F^r(\xi_k)| < \epsilon_r < \epsilon/2$ and $|F^r(\xi_k) - H(\xi_k)| < \epsilon/2$. Thus $|F_n^r(\xi_k) - H(\xi_k)| < \epsilon$ and we see that the sequence S converges to $H(x)$ for $x = \xi_k$, $k = 1, 2, \cdots$. But since $\{\xi_k\}$ is dense, the sequence S converges to $H(x)$ at every continuity point of $H(x)$. Now if we let $r_n = 2$ for $n = 1, \cdots, n_3 - 1$ and $r_n = m$ for $n = n_m, \cdots, n_{m+1} - 1$ ($m > 2$), we see that the distribution function of $|x_{n1}|^{r_n} + \cdots + |x_{nk_n}|^{r_n} - B_n(r_n)$ is $F_n^{r_n}(x)$, which is the nth element of the sequence S. This proves the theorem.

REFERENCE

[1] B. V. GNEDENKO AND A. N. KOLMOGOROV, *Limit Distributions for Sums of Independent Random Variables*, translation by K. L. Chung Addison-Wesley, 1954.

[4] An analogous theorem holds for the conditions of theorem 2.