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A NOTE ON P.B.I.B. DESIGN MATRICES

BY W. A. THOMPSON, JR.

University of Delaware

Summary. The notation P.B.I.B. (m) will mean partially balanced incomplete block design with m associative classes.

It is found that the C matrix of a P.B.I.B (m) may be expressed as a linear function of $m + 1$ commutative and linearly independent matrices. The author feels that this decomposition may be of interest to those studying the properties of P.B.I.B. designs.

1. The C matrix of a P.B.I.B. design. The reader should review the definition of partially balanced designs, and the relations among the parameters. See, for example, Bose and Shimamoto [2], or Bose [1], or Connor and Clatworthy [3].

The matrix

$$C = (c_{ij}),$$

where

$$c_{ii} = r(1 - 1/k),$$

$$c_{ij} = -\lambda_{ij}/k, \qquad i \neq j$$

is of special interest in incomplete block design theory.

In the case of a P.B.I.B. (m), the C matrix may be written in a particular form. We may write

$$kC = r(k - 1)I - \sum_{i=1}^{m} \lambda_i B_i, \tag{1.1}$$

where $B_s = [b_{tj}^{(s)}]$ for $s = 1, \cdots, m$, where $b_{tt}^{(s)} = 0$ and $b_{tj}^{(s)} = 1$ or 0 according as the treatments t and j are or are not sth associates. Note that $I, B_1, B_2, \cdots, B_m$ form a linearly independent set of matrices since a one in the (i, j)th position of any of them implies a zero in the (i, j)th position of all the others. $b_{hj}^{(s)} b_{ht}^{(s)}$ equals 1 if treatment j and treatment t are both sth associates of treatment h, but equals 0 otherwise. If $j \neq t$ then $\sum_i b_{ij}^{(s)} b_{it}^{(s)}$ is the number of treatments which are sth associates of both treatments j and t. But if j and t

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are rth associates, then this is the definition of p_{ss}^{r}. Note further that if $j = t$ then $\sum_i [b_{ij}^{(s)}]^2 = n_s$. Thus

$$
B_s B_s = [\sum_i b_{ji}^{(s)} b_{it}^{(s)}] = [\sum_i b_{ij}^{(s)} b_{it}^{(s)}]
= n_s I + \sum_i p_{ss}^{i} B_i . \tag{1.2}
$$

Similarly

$$
B_s B_t = [\sum_i b_{ij}^{(s)} b_{iq}^{(s)}]
= \sum_i p_{st}^{i} B_i . \tag{1.3}
$$

Consider the equations

$$
C = r(1 - 1/k)I - 1/k \sum_{i=1}^{m} \lambda_i B_i,
$$

$$
\begin{aligned}
CB_j &= r(1 - 1/k)B_j - 1/k \sum_{i=1}^{m} \lambda_i B_i B_j , \qquad j = 1 \cdots m, \\
&= r(1 - 1/k)B_j - 1/k \sum_{i \neq j} \lambda_i \left(\sum_{s=1}^{m} p_{ij}^{s} B_s \right) \\
&\quad - \lambda_j/k \left(n_j I + \sum_{i=1}^{m} p_{jj}^{i} B_i \right) \\
&= - \frac{n_j \lambda_j}{k} I + \left[[r(1 - 1/k) - 1/k \sum_i \lambda_i p_{ij}^{j} \right] B_j \\
&\quad - \sum_{s \neq j} 1/k \sum_i \lambda_i p_{ij}^{s} B_s \qquad j = 1 \cdots m.
\end{aligned}
$$

We may rewrite these equations as

$$
\begin{aligned}
C &= d_{00}I + d_{01}B_1 + \cdots + d_{0m}B_m , \\
CB_1 &= d_{10}I + d_{11}B_1 + \cdots + d_{1m}B_m , \\
&\vdots \\
CB_m &= d_{m0}I + d_{m1}B_1 + \cdots + d_{mm}B_m ,
\end{aligned} \tag{1.4}
$$

where

$$
\begin{aligned}
d_{00} &= r(1 - 1/k), \\
d_{0i} &= - \lambda_i/k, && i = 1 \cdots m, \\
d_{j0} &= - \frac{n_j \lambda_j}{k} , && j = 1 \cdots m, \\
d_{jj} &= r(1 - 1/k) - 1/k \sum_i \lambda_i p_{ij}^{j} , && j = 1 \cdots m, \\
d_{js} &= - 1/k \sum_i \lambda_i p_{ij}^{s} , && s = 1 \cdots m; j \neq s.
\end{aligned} \tag{1.5}
$$

If e is arbitrary, and I is a $v \times v$ matrix, then by subtracting eI from C in (1.4) we get the single matrix equation:

$$\begin{bmatrix} C - eI & & & 0 \\ & C - eI & & \\ & & \ddots & \\ 0 & & & C - eI \end{bmatrix} \begin{bmatrix} I \\ B_1 \\ \vdots \\ B_m \end{bmatrix}$$

$$= \begin{bmatrix} (d_{00} - e)I & d_{01} I & \cdots & d_{0m} I \\ d_{10} I & (d_{11} - e)I & \cdots & d_{1m} I \\ \cdots & \cdots & \cdots & \cdots \\ d_{m0} I & d_{m1} I & \cdots & (d_{mm} - e)I \end{bmatrix} \begin{bmatrix} I \\ B_1 \\ \vdots \\ B_m \end{bmatrix}.$$

Let D be the $(m + 1) \times (m + 1)$ square matrix:

$$D = \begin{bmatrix} d_{00} & d_{01} & \cdots & d_{0m} \\ d_{10} & d_{11} & \cdots & d_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ d_{m0} & d_{m1} & \cdots & d_{mm} \end{bmatrix}. \tag{1.7}$$

We could at this point use the B matrices to verify the following result:

THEOREM 1. *If e is a characteristic root of C then it is a characteristic root of D, and conversely if e is a characteristic root of D then it is a characteristic root of C.*

However, this theorem also follows from Lemma 3.1 of Connor and Clatworthy [3].

Using the matrices M and A of Lemma 3.1, with $z = kx - r(k - 1)$ we have

$$|M/k| = |xI - C|,$$

and

$$x|A/k| = |xI - D|.$$

This second relation follows by first adding all other rows of $|xI - D|$ to the first row and then subtracting the first column from all others. Theorem 1 then follows from Connor and Clatworthy's lemma.

2. The principal idempotent matrices of C. (If the reader is unfamiliar with the properties of principal idempotent matrices, then he may consult [4].) Let e be a characteristic root of C, and let $E(e)$ be the principal idempotent matrix of C corresponding to e. Theorem 1 then states that e is a root of D. B_0 will denote the identity matrix.

THEOREM 2. *$E(e) = \sum_{i=0}^{m} c_i B_i$, where $(c_0, c_1, \cdots, c_m)$ is a characteristic vector of D corresponding to e.*

PROOF. $E(e)$ must be a polynomial in C. Therefore, $E(e) = \sum_{i=0}^{m} c_i B_i$ according to (1.1), (1.2), and (1.3). At this point in the proof $c_0, c_1, \cdots, c_m$ are arbitrary constants. Now, $E(e)(C - eI) = 0$ since this is a property of principal idempotent matrices for C real and symmetric.

We rewrite this relation

$$(c_0 I, c_1 I, \cdots, c_m I) \begin{bmatrix} C - eI & & & 0 \\ & C - eI & & \\ & & \ddots & \\ 0 & & & C - eI \end{bmatrix} \begin{bmatrix} I \\ B_1 \\ \vdots \\ B_m \end{bmatrix} = 0. \tag{2.1}$$

Using 1.6 and the linear independence of the B's, 2.1 yields

$$(2.2)\quad (c_0 I, c_1 I, \cdots, c_m I)\begin{bmatrix} (d_{00}-e)I & d_{01} I & \cdots & d_{0m} I \\ d_{10} I & (d_{11}-e)I & & d_{1m} I \\ \vdots & \vdots & & \vdots \\ d_{m0} I & d_{m1} I & \cdots & (d_{mm}-e)I \end{bmatrix} = 0.$$

Therefore

$$(2.3)\qquad (c_0, c_1, \cdots, c_m)(D - eI) = 0.$$

If C has m^* distinct non-zero characteristic roots, $e_1, e_2, \cdots, e_{m^*}$, then we may write

$$C = e_1E(e_1) + e_2E(e_2) + \cdots + e_{m^*}E(e_{m^*}).$$

Now using Theorem 2 we have

THEOREM 3. *The C matrix of* a P.B.I.B. (m) *may be expressed as a linear function of the $m + 1$ commutative and linearly independent matrices $B_0, B_1, \cdots, B_m$.*

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ON A FACTORIZATION THEOREM IN THE THEORY OF ANALYTIC CHARACTERISTIC FUNCTIONS[1]

Dedicated to Paul Lévy on the occasion of his seventieth birthday

BY R. G. LAHA

The Catholic University of America

1. Introduction. Let $F(x)$ be a distribution function, that is, a non-decreasing right-continuous function such that $F(-\infty) = 0$ and $F(+\infty) = 1$. The characteristic function

$$(1.1)\qquad \phi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x)$$

of the distribution function $F(x)$ is defined for all real t. A characteristic function is said to be an *analytic characteristic function* if it coincides with a regular analytic function $\phi(z)$ in some neighborhood of the origin in the complex z-plane.

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