ON RENEWAL PROCESSES RELATED TO TYPE I AND TYPE II COUNTER MODELS[1]

By Ronald Pyke

Stanford University

Summary. Several renewal processes related to the Type I and Type II counter models are defined and studied. The distribution and characteristic functions for the secondary (or output) process of the Type I counter model are obtained explicitly. Both the non-stationary and stationary probabilities of the state of the counter, (locked or unlocked), are derived. Integral equations determining the distribution and characteristic functions for the secondary process of the Type II counter model are obtained. Also it is shown that a more general model proposed by Albert and Nelson [1] may be solved explicitly in terms of a corresponding Type II counter model. An example of this general model is given. Related with each model is a discrete renewal process which is also studied.

1. Introduction and Notation. Two important classes of counting devices are the Type I and Type II counters defined as follows. A counter for detecting radioactive impulses is placed within range of a radioactive material. By "an event has happened", we mean that an impulse has been emitted by the material and by "an event has been registered", we mean that an impulse emitted by the material has been detected and recorded by the counter. Due to the inertia of the counting device, all impulses will probably not be counted. The time during which the device is unable to record an impulse is referred to as deadtime.

Definition. A Type I counter is one in which deadtime is produced only after an event has been registered. A Type II counter is one in which dead time is produced after each event has happened. Examples of Type I and Type II counters are the Geiger-Müller counters and electron multipliers respectively.

In sections 4 to 7, attention will be given only to the Type I problem. It is stated theoretically as follows. Let X, Y and Z be random variables (r.v.) with distribution functions (d.f.) F, G and H respectively. Let $\{X_i\}_{i=1}^{\infty}$, $\{Y_j\}_{j=0}^{\infty}$ be independent X- and Y-renewal processes; that is $\{X_i, Y_j : i \geqq 1, j \geqq 0\}$ is a family of mutually independent r.v.'s and each X_i and Y_j has d.f. F and G respectively. Set $X_0 = 0$ (a.s.) and $S_k = \sum_{i=0}^{k} X_i$ for $k = 0, 1, 2, \cdots$. Assume throughout this discussion that $F(0) = G(0-) = 0$, F is a non-lattice distribution and that all d.f.'s are right continuous. Define $n_0 = 0$ and

$$n_j = \min\{k \, \varepsilon \, I^+ : S_k > Y_{j-1} + S_{n_{j-1}}\}$$

for $j = 1, 2, 3, \cdots$, where I^+ is the set of positive integers. The above definitions are valid with probability one.

Received October 29, 1957; revised March 28, 1958.

[1] This work was sponsored in part by the Office of Naval Research. Reproduction in whole or in part is permitted for any purpose of the United States Government.

The secondary renewal process, $\{Z_i\}_{i=1}^{\infty}$ (to be referred to as the Z-process) is defined by

$$Z_i = S_{n_i} - S_{n_{i-1}} \qquad (i \varepsilon I^+).$$

This is clearly a renewal process since the S_j's are sums of independent r.v.'s and since $\{n_j - n_{j-1}\}_{j=1}^{\infty}$, a sequence of identically and independently distributed r.v.'s, is itself a renewal process. $\{n_j - n_{j-1}\}_{j=1}^{\infty}$ shall be referred to as the N-process, and H shall denote the common c.d.f. of the Z-process. It will be shown that $E(n_1)$ denotes the asymptotic bias of the counter.

One may define a related stochastic process which is of interest in counting problems. Let $\{V_t : t \geqq 0\}$ be a stochastic process, having a two point range space, with joint distribution functions derived from its definition which is: $V_0 = 0$ (a.s.) and

$$V_t = \begin{cases} 1 \text{ if } Z_k + Y_k \leqq t < Z_{k+1} \text{ for some } k \varepsilon I^+ \\ 0 \text{ otherwise} \end{cases}$$

Set

$$P_1(t) = 1 - P_0(t) = \Pr[V_t = 1]$$

and

$$P_1 = 1 - P_0 = \lim_{t \to \infty} P_1(t)$$

if the limit exists.

A subscript, j say, affixed to any distribution function will denote its jth convolution with itself. The zero subscript will denote the c.d.f. degenerate at zero.

In sections 8 and 9 the Type II problem is studied. Its theoretical formulation differs from the Type I problem only in the definition of the N-process, which for the Type II problem is $n_0 = 0$ and

$$(1) \qquad n_j = \min\{k \varepsilon I^+ : k > n_{j-1}, S_k > S_r + Y_r, r = n_{j-1}, \cdots, k-1\}.$$

In all other instances, the definitions remain unchanged. For example, the secondary renewal process is still given by

$$Z_i = S_{n_i} - S_{n_{i-1}} \qquad (i \varepsilon I^+),$$

although, it is clearly a different process. The same notation is used for both models in order to emphasize to the reader the common interpretation of the various symbols.

In section 10 a more general model, suggested by Albert and Nelson [1], is studied. It is shown that the solution of this more general model is an immediate consequence of the solution of a corresponding Type II problem.

We shall begin in section 3 by proving a theorem from which the quantities $P_1(t)$ are immediately deducible.

To understand the connection between the above notation and the counter problem itself, let Y_j represent the deadtime caused respectively by the registration of an event at time S_{n_j} in the Type I model and the happening of an event at time S_j in the Type II model (time being measured from the registration of some event) and let X_k be the time between the kth and $(k + 1)$-st impulses. The secondary renewal process is determined by the r.v. Z, which denotes the time between successive counts or registrations. The event $[V_t = 1]$ corresponds to the counter being unlocked at time t. For a more detailed description of the physical problem, the reader is referred to the references. (See e.g., Feller [2].)

Throughout this paper, the integrals that appear are to be considered as Lebesque-Stieltjes integrals. This will avoid the special considerations that would otherwise be required in cases where the integrand has a set of discontinuities of positive measure with respect to the Stieltjes measure. Notice that the ordinary integration-by-parts formula holds for the Lesbesque-Stieltjes integrals that appear in this paper. A proof of this is possible by probabilistic methods.

2. The literature and known results. The Type I and Type II counter problems have been studied by several people. Most of these studies deal with the special case in which the input process is Poisson. Not only does the Poissonian input make the problems involved more tractable, but in this instance, it serves to make the statistical model very realistic, since the impulses from a radioactive material behave randomly over time, at least in time intervals which are short relative to the half-life of the material. For an extensive bibliography, the reader is referred to Takàcs [3].

It is important, however, to study the more general non-poissonian models for several reasons. First of all, it is necessary at times to make successive counts and it is known that the secondary process of the first counter, which would serve as the input process for the second counter, is not a Poisson process even though the original process was. Secondly, these same theoretical models have arisen in other contexts in which the Poisson process is not so easily justified (e.g., in inventory theory, Arrow, Karlin and Scarf [4]).

In his recent paper, [5], received by this author after completion of the first draft of this paper, Takacs also studies the general counter problem. Although there is some overlap, there are many differences in approach and coverage between the two treatments of the problem. Theorem 2 is equivalent to results obtained by Tackacs in [3] and again in [5], for the case of continuous F and G. Even for this case, however, our result (4) is a simplification in that a double integral has been replaced by a single one. Attention should also be given to a recent paper of Smith [6], in which the Type II counter model with Poissonian input (and related quasi-Poissonian inputs) as well as the model with constant deadtime, is studied.

3. A related renewal problem. In this section, we shall consider two alternating renewal processes, not necessarily independent, and obtain explicitly the probabilities, both finite and stationary, of one of the processes being in effect at any given instant of time. To be more precise, let $\{U_i\}_{i=1}^{\infty}$, $\{V_i\}_{i=1}^{\infty}$, be two renewal processes with common c.d.f. K and R respectively. By definition U_i and $U_j(i \neq j$ are independent and similarly for V_i and V_j. Concerning the relationship between the two processes assume only that $\{U_i + V_i\}_{i=1}^{\infty}$ forms a renewal process; that is, independence of U_i and V_i is not assumed. Let H denote the common c.d.f. of $U_i + V_i$ for all i. Define $T_0 = 0$ and, for $j \geqq 1$, set

$$T_{2j} = U_1 + V_1 + U_2 + V_2 + \cdots + U_j + V_j$$

$$T_{2j-1} = U_1 + V_1 + U_2 + V_2 + \cdots + U_j .$$

Define

$$A(t) = \begin{cases} 1 \text{ if } T_{2j-1} < t \leqq T_{2j} \text{ for some } j > 1 \\ 0 \quad \text{otherwise} \end{cases}$$

and

$$P_0(t) = 1 - P_1(t) = \Pr[A(t) = 0].$$

THEOREM 1. *For all* $t \geqq 0$

$$P_0(t) = \int_{0-}^{t} [1 - K(t - x)]\, dN(x)$$

where $N(x) = \sum_{j=0}^{\infty} H_j(x)$ *and* H_j *is the* c.d.f. *of* T_{2j} i.e., *the* jth *convolution of* H. *Moreover,*

$$P_0 = \lim_{t\to\infty} P_0(t) = \frac{E(U)}{E(U) + E(V)}$$

whenever at least one term of the denominator is finite. P_0 *is interpreted as being zero when* $E(V) = \infty$ *and one when* $E(U) = \infty$.

PROOF. By definition,

$$\begin{aligned} P_0(t) &= \sum_{j=0}^{\infty} \Pr[T_{2j} \leqq t < T_{2j+1}] \\ &= \sum_{j=0}^{\infty} \int_{0-}^{t} \Pr[T_{2j} \leqq t < T_{2j} + U_{j+1} \mid T_{2j} = x]\, dH_j(x) \\ &= \sum_{j=0}^{\infty} \int_{0-}^{t} [1 - K(t - x)]\, dH_j(x) \\ &= \int_{0-}^{t} [1 - K(t - x)]\, dN(x) \end{aligned}$$

as required. Since we are working with an at most countable family of r.v.'s, the conditional probability argument used above and in proofs which follow is valid. The second statement of the theorem is an immediate application of a theorem of Smith ([7], Theorem 1) which we quote in a particular form for further reference.

THEOREM S: *If $k(x)$ is any bounded function, zero for negative argument, integrable, non-increasing in $(0, \infty)$ for which $k(x) \to 0$ as $x \to \infty$; if H is a non-negative non-lattice distribution function and*

$$N(x) = \sum_{n=0}^{\infty}{}' H_n(x)$$

then

$$\lim_{t\to\infty} \int_{0-}^{t} k(t-x)\, dN(x) = \int_0^\infty k(x)\, dx \left\{ \int_{0-}^{\infty} y\, dH(t) \right\}^{-1}.$$

The right hand side is to be taken as zero whenever the denominator is infinite.

In connection with the last statement of Theorem 1, observe that $P_1(t)$ converges to the stated limit since the function $k(x) = 1 - K(x)$ satisfies the conditions of Theorem S. We mention also that the last statement of Theorem 1 is actually a special case of a result concerning semi-Markov processes, given by Smith ([8], cf. Theorem 5).

4. The N-Process of the Type I Model. Set $p_0 = 0 = r_0$,

$$p_k = \Pr[n_1 = k] = \Pr[n_j - n_{j-1} = k] \qquad (j, k \,\varepsilon\, I^+)$$

and

$$r_k = \Pr[n_j = k \text{ for some } j] \qquad (k \,\varepsilon\, I^+).$$

Moreover, define the corresponding generating functions, for $|s| < 1$,

$$P(s) = \sum_{k=1}^{\infty} p_k s^k, \qquad R(s) = \sum_{k=1}^{\infty} r_k s^k.$$

The N-Process may be considered as a sampling of the positive integers I^+; that is, $n_1 < n_2 < n_3 < \cdots$ and $\{n_j, j \geqq 1\} \subset I^+$. In this context, one may speak of the event E, "an integer is sampled." One may show that, in the terminology of Feller [9], this event is recurrent. Since, for all $k \,\varepsilon\, I^+$

$$r_k = p_k + \sum_{j=1}^{k-1} p_j r_{k-j}$$

one obtains directly the known relationships

$$P(s) = \frac{R(s)}{1 + R(s)}, \qquad R(s) = \frac{P(s)}{1 - P(s)}$$

Moreover, it is known that (cf. [9])

$$\lim_{k\to\infty} r_{mk} = \lim_{s\to 1-} \frac{(1 - s^m)P(s)}{1 - P(s)} = m/E(n_1) \tag{2}$$

where m is the g.c.d. of those indices n for which $p_n > 0$. The right hand side of (2) is to be interpreted as zero whenever $E(n_1)$, the 'mean recurrence time,' is infinite.

The probabilities p_k are readily computed from the relation

$$p_k = \Pr[S_{k-1} \leqq Y_0 < S_k] \qquad (k \varepsilon I^+).$$

They are given in

LEMMA 1. *For all* $k \varepsilon I^+$

$$p_k = \int_{0-}^{\infty} [F_{k-1}(y) - F_k(y)]\, dG(y).$$

Observe that the event E is a certain event. That is

$$\sum_{k=1}^{\infty} p_k = 1 - \lim_{n\to\infty} \int_{0-}^{\infty} F_n(y)\, dG(y) = 1$$

since $\lim_{n\to\infty} F_n(y) = 0$ for all $y \geqq 0$ if and only if $F(0) < 1$, a condition which has been assumed.

Define the r.v. N_y for $y > 0$ as the smallest index k for which $S_k > y$. Set $Q_y(s)$ as the generating function of the probabilities associated with N_y. One may then easily show that for $|s| < 1$

$$P(s) = \int_{0-}^{\infty} Q_y(s)\, dG(y) = (1 - s) \sum_{k=0}^{\infty} s^k \int_{0-}^{\infty} G(y-)\, dF_k(y)$$

Consequently, setting $M_k(y) = E(N_y^k)$, one obtains

$$E(n_1^k) = \int_{0-}^{\infty} M_k(y)\, dG(y) \qquad (k \varepsilon I^+).$$

In particular

$$E(n_1) = \int_{0-}^{\infty} M_1(y)\, dG(y) = \int_{0-}^{\infty} [1 - G(y-)]\, dM(y). \tag{3}$$

It is well known, and easily proven that

$$M_1(y) = \sum_{j=0}^{\infty} F_j(y)$$

$M_1(y)$ will be used very frequently throughout this paper. We shall therefore drop the subscript and write $M(y) = M_1(y)$.

Set $\mu = E(X)$ and $\nu = E(Y)$. It is well known, (cf. Smith [7]) that if $\mu < \infty$, $M(y) = y/\mu + o(y)$ as $y \to \infty$. Thus if $\mu < \infty$, by (3) $E(n_1) < \infty$ if and only if $\nu < \infty$. Similarly, if $\mu = \infty$, then $M(y) = o(y)$ and, hence $E(n_1) < \infty$ whenever $\nu < \infty$. The case of $\mu = \infty = \nu$ is special and will not be studied here.

5. The Z-renewal process. In this section the c.d.f. of Z as well as its Laplace-Stieltjes transform will be obtained. Consider the notation

$$\varphi(s) = \int_0^{\infty} e^{-sx}\, dF(x). \qquad \psi(s) = \int_{0-}^{\infty} e^{-sx}\, dG(x)$$

$$\Phi(s) = \int_0^{\infty} e^{-sx}\, dH(x), \qquad \psi^*(s) = \int_0^{\infty} e^{-sx} G(x-)\, dM(x)$$

for all $s \geqq 0$. One then obtains

THEOREM 2. *For all* $z \geqq 0$, $s \in R$

$$H(z) = \int_0^z G(u-)[1 - F(z-u)]\, dM(u) \tag{4}$$

$$\Phi(s) = [1 - \varphi(s)]\psi^*(s). \tag{5}$$

PROOF. Clearly

$$H(z) = \Pr[Z \leqq z] = \sum_{k=1}^{\infty} \Pr[S_{k-1} \leqq Y < S_k \leqq z].$$

For $k \geqq 2$

$$\begin{aligned}
\Pr[S_{k-1} \leqq Y < S_k \leqq z] &= \int_{0-}^{z}\int_0^y [F(z-u) - F(y-u)]\, dF_{k-1}(u)\, dG(y) \\
&= \int_{0-}^{z} [G(z) - G(u-)]F(z-u)\, dF_{k-1}(u) - \int_{0-}^{z} F_k(y)\, dG(y) \\
&= G(z)F_k(z) - \int_0^z G(u-)F(z-u)\, dF_{k-1}(u) - \int_{0-}^{z} F_k(y)\, dG(y) \\
&= \int_0^z G(u-)\, dF_k(u) - \int_0^z G(u-)F(z-u)\, dF_{k-1}(u).
\end{aligned}$$

For $k = 1$

$$\Pr[Y < S_1 \leqq Z] = \int_0^z G(u-)\, dF_1(u)$$

and (4) follows by summation over k. To obtain (5) for $s > 0$, write

$$\begin{aligned}
\frac{1}{s}\Phi(s) &= \int_0^{\infty} e^{-sz}H(z)\, dz \\
&= \int_0^{\infty}\int_{u-}^{\infty} e^{-sz}G(u-)\, dz\, dM(u) - \int_0^{\infty}\int_{u-}^{\infty} e^{-sz}F(z-u)G(u-)\, dz\, dM(u) \\
&= \frac{1}{s}\psi^*(s) - \frac{1}{s}\varphi(s)\psi^*(s)
\end{aligned}$$

as required. At $s = 0$, Φ may properly be defined by $\Phi(0) = 1$. This follows by an application of an Abelian theorem to (5). That is, consider

$$\begin{aligned}
\lim_{s\to 0+} [1 - \varphi(s)]\psi^*(s) &= \lim_{s\to 0+} \int_{0-}^{\infty} e^{-sx}G(x-)\, dM(x) \left\{\int_{0-}^{\infty} e^{-sx}\, dM(x)\right\}^{-1} \\
&= \lim_{x\to\infty} G(x) = 1
\end{aligned}$$

since $M(x) \to \infty$ as $x \to \infty$.

Of particular importance to the counter problem is the expectation of the secondary renewal process. One obtains

THEOREM 3. *$E(Z) < \infty$, if and only if $\nu < \infty$ and $\mu < \infty$. Moreover*

$$E(Z) = \mu E(n_1) = \mu \int_{0-}^{\infty} M(y)\, dG(y). \tag{6}$$

PROOF. The first statement follows from the relationship

$$\max\,(Y_0, X_1) \leqq Z_1 \leqq Y_0 + X_{n_1} \qquad \text{(a.s.)}.$$

The second statement is a consequence of a well known result in Sequential Analysis, for by it

$$E(Z \mid Y_0 = y) = \mu E(N_y)$$

and (6) follows by integration with respect to $dG(y)$. $E(Z)$ is to be interpreted as infinity whenever ν or $E(n_1)$ is infinite. Of course, (6) could also be proven directly from Theorem 2 using (5).

Let $N(x)$ denote the expected number of partial sums of the Z-process less than or equal to x; that is

$$N(x) = \sum_{j=0}^{\infty} H_j(x).$$

Define the bias of the counter at time x by $B(x) = M(x)/N(x)$. Then as a consequence of Theorem 3 and a known asymptotic renewal theorem, one obtains

LEMMA 2. *If $\mu < \infty$, then*

$$\lim_{x \to \infty} B(x) = 1/E(n_1)$$

where the right hand side is to be interpreted as zero when $\nu = +\infty$.

It may be easily shown that this result is also valid for the Type II *and Albert and Nelson models.*

6. The distribution of free-time. Let $W = Z_1 - Y_0$ represent the length of time the counter is free during successive registrations. Denote its c.d.f. and L-S transform by K and k respectively. Clearly $E(W) = \mu E(n_1) - \nu$. Moreover,

$$K(x) = \int_{0-}^{\infty} \Pr\,[Z_1 \leqq x + y \mid Y = y]\, dG(y).$$

Under the condition $[Y = y]$, Z_1 has a c.d.f. given by (4), but with G degenerate at y, i.e., $G(u) = 1$ if $u \geqq y$ and $G(u) = 0$ otherwise. Therefore,

$$K(x) = \int_{0-}^{\infty} \int_{y}^{x+y} [1 - F(x + y - u)]\, dM(u)\, dG(y)$$

$$= 1 - \int_{0-}^{\infty} \int_{0-}^{y} [1 - F(x + y - u)]\, dM(u)\, dG(y).$$

It follows similarly that $k(s)$ is the expectation w.r.t. $dG(y)$ of the L-S transform of $Z_1 - y$ obtained under the condition $[Y = y]$. By (5) this is seen to be

$$k(s) = [1 - \varphi(s)] \int_{0-}^{\infty} \int_{y}^{\infty} e^{-s(x-y)}\, dM(x)\, dG(y). \tag{7}$$

According to its definition in section 1, $P_1(t)$ is the probability that the counter is free at time t. Setting U and V of section 3 equal to Y and W, we have as a consequence of Theorem 1, the following result: for all $t \geqq 0$

$$P_0(t) = \int_{0-}^{t} [1 - G(t - x)]\, dN(x) \tag{8}$$

where

$$N(x) = \sum_{j=0}^{\infty} H_j(x).$$

This formula differs from equation (26) of Takacs [5]. Moreover, in the limit

$$P_0 = \lim_{t\to\infty} P_0(t) = \nu/\mu E(n_1).$$

Let the L-S transform of $P_0(t)$ be denoted by

$$\pi(s) = \int_{0-}^{\infty} e^{-st}\, dP_0(t).$$

Then, by direct computation one obtains from (8)

$$\pi(s) = \frac{1 - \psi(s)}{1 - \Phi(s)}.$$

7. Examples of the Type I counter problem. (a) $F(x) = 1 - e^{-\lambda x}$: This is the well known Poisson input counter problem which with various assumptions on G has been studied by several authors. For arbitrary G, the problem was treated by Takacs [3]. Because of special properties possessed by the exponential distribution, this particular example may be (and indeed has been) solved in several different ways. In [6], Smith has shown that much of the essential simplicity of this case carries over in asymptotic considerations to a wider class of F which generate so-called quasi-Poisson processes. For the present example, $\mu = 1/\lambda$ and $M(x) = \lambda x + 1$ for $x \geqq 0$ and $M(x) = 0$ for $x < 0$. The formulae of the previous sections become

$$Q_y(s) = se^{-y\lambda(1-s)}$$

$$P(s) = s\psi(\lambda - \lambda s)$$

$$E(n_1) = \lambda\nu + 1$$

$$H(z) = \int_{0-}^{z} [1 - e^{-\lambda(z-y)}]\, dG(y)$$

$$\Phi(s) = \varphi(s)\psi(s) = \frac{\lambda\psi(s)}{\lambda + s}.$$

These last two results may, of course, be obtained immediately from the known characterization of the exponential distribution that truncation on the left does not change the form of the distribution function. This implies that $Z_1 = Y_0 + X$

where X is exponentially distributed and is independent of Y_0. Finally, for this example, we have

$$\pi(s) = \frac{(\lambda + s)[1 - \psi(s)]}{\lambda + s - \lambda\psi(s)}$$

(b) $Y = d$ (a.s.): This important oft-studied case is applicable to counters for which the deadtime is independent of the intensity or amplitude of the incoming radioactive pulses. For this case $G(x) = 0$ or 1 according as $x <$ or $\geqq d$, and the formulae of the previous sections become

$$p_k = F_{k-1}(d) - F_k(d)$$

$$E(n_1) = M(d)$$

$$H(z) = \begin{cases} 0 & \text{if } z \leqq d \\ \int_d^z [1 - F(z - u)]\, dM(u) & \text{if } z > d \end{cases}$$

$$\Phi(s) = [1 - \varphi(s)] \int_d^\infty e^{-sx}\, dM(x) = 1 - [1 - \varphi(s)] \int_{0-}^d e^{-sx}\, dM(x)$$

and

$$\pi(s) = (1 - e^{-sd}) \left\{[1 - \phi(s)] \int_{0-}^d e^{-sx}\, dM(x)\right\}^{-1}$$

(c) $G(y) = 1 - e^{-y\beta}$: The above two cases have been studied previously, whereas, the present case has not, to this author's knowledge, as yet been considered. In a different context, (7) has been employed by Scarf [4] for G exponential. For this example, we have $\nu = 1/\beta$.

$$p_k = [\varphi(\beta)^{k-1}[1 - \varphi(\beta)]$$

$$P(s) = \frac{s[1 - \varphi(\beta)]}{1 - s\varphi(\beta)}$$

$$E(n_1) = [1 - \varphi(\beta)]^{-1}$$

$$H(z) = F(z) - \int_{0-}^z e^{-\beta x}[1 - F(z - x)]\, dM(x)$$

$$\Phi(s) = \frac{\varphi(s) - \varphi(\beta + s)}{1 - \varphi(\beta + s)}$$

$$\pi(s) = \frac{s[1 - \varphi(\beta + s)]}{(\beta + s)[1 - \varphi(s)]}$$

and

$$k(s) = \frac{\beta[\varphi(s) - \varphi(\beta - s)]}{(\beta - s)[1 - \varphi(s)][1 - \varphi(\beta - s)]}.$$

8. The general Type II counter. This problem is a very difficult one to solve in general. Discussions of the general problem have been given by Takacs [5] and Pollaczek [10]. Certain particular cases have been studied in the literature in greater detail. For example: Poisson input and constant deadtime by Feller [2], Poisson input and general deadtime by Takacs [3] and, with a different approach, by Chernoff and Daly [11], and exponentially distributed deadtime by Takacs [5]. The same notation as that used for the Type I problem will be employed in this section, but with the corresponding definition of n_j, namely (1).

In this model, it is simpler to evaluate r_k than p_k, contrary to what was observed in the Type I problem. For $k \geqq 1$

$$r_k = \Pr[S_r + Y_r < S_k\ r = 0, 1, \cdots, k-1] = \underbrace{\int_0^\infty \int_0^\infty \cdots \int_0^\infty}_{k} \tag{9}$$
$$\cdot G(x_1 + x_2 + \cdots + x_{k-1}-) \cdots G(x_1-)\, dF(x_1) \cdots dF(x_k).$$

Therefore, by the same argument leading to (2), one obtains the relationship

$$E(n_1) = m/\lim_{k\to\infty} r_{mk} \tag{10}$$

where m is the g.c.d. of those integers n for which $p_n > 0$. If $X \leqq Y$ (a.s.) set $m = 0$ and $n_1 = \infty$ (a.s.). In all other cases $\Pr[X > Y] > 0$. However, since $p_1 = \Pr[X > Y]$, one obtains $m = 1$. That is to say, whenever $\Pr[X > Y] > 0$

$$E(n_1) = 1/\lim_{k\to\infty} r_k$$

With a knowledge of r_k, one is able to compute $E(n_1)$ and hence the expectation of the secondary renewal process.

As before, set $Z_0 = 0$ (a.s.) and $Z_i = S_{n_i} - S_{n_{i-1}}$. Clearly the Z_i's form a renewal process. The problem of deriving an explicit expression for H, the common c.d.f. of the Z-renewal process, is extremely difficult. However, it is possible to display an integral equation which formally, but not always in practice, determines H. In section 10, an example will be given for which the solution is readily attained from this integral equation whereas it is not easily derived by other methods. Takacs [5] has, for the Type II problem, obtained an integral equation in $N(t)$, the expected number of counts (partial sums of the Z-process) in $[0, t]$ for all $t \geqq 0$. These two representations are equivalent in the sense that H and N are uniquely determined one by the other. More precisely, for $s \geqq 0$, the relationship between H and N is given by

$$\int_{0-}^\infty e^{-st}\, dN(t) = \sum_{j=0}^\infty \int_{0-}^\infty e^{-st}\, dH_j(t) = \frac{1}{1 - \Phi(s)}. \tag{11}$$

THEOREM 4. *For all* $z \geqq 0$

$$H(z) = \int_0^z \int_0^{z-x} [1 - H(z - x - t)] G(x + t-)\, dN(t)\, dF(x) \tag{12}$$

and for $s > 0$

$$\Phi(s) = \lambda(s)[1 + \lambda(s)]^{-1} \tag{13}$$

where

$$\lambda(s) = \int_{0-}^{\infty} \int_{0}^{\infty} e^{-s(x+t)} G(x + t-)\, dF(x)\, dN(t). \tag{14}$$

(Notice that because of (11), $\lambda(s) + 1$ is the L-S transform of N.)

PROOF. (12) is obtained as follows. $[Z_1 \leqq z]$ is the union of two disjoint events, A and B say, where

$$A = [Y_0 < X_1 \leqq z]$$

and

$$B = [0 \leqq Y_0 - X_1 < Z_j \leqq z - X_1 \text{ for some } j \geqq 1].$$

Clearly

$$\Pr(A) = \int_0^z G(x-)\, dF(x).$$

Under the condition, $[z > Y_0 = y \geqq x = X_1] = C$ say,

$$\begin{aligned} \Pr(B \mid C) &= 1 - \sum_{j=0}^{\infty} \Pr[Z_j \leqq y - x, Z_{j+1} > z - x] \\ &= 1 - \int_{0-}^{y-x} [1 - H(z - x - t)]\, dN(t) \\ &= \int_{y-x}^{z-x} [1 - H(z - x - t)]\, dN(t). \end{aligned}$$

Therefore,

$$H(z) = \int_{0-}^{z} G(x-)\, dF(x) + \int_{0-}^{z-} \int_{x-}^{z-} \int_{y-x}^{z-x} [1 - H(z - x - t)]\, dN(t)\, dG(y)\, dF(x)$$

and an interchange of integration gives

$$\begin{aligned} H(z) &= N(0)[F(z) - F(z-)]G(z-) \\ &\quad + \int_{0-}^{z-} \int_{0-}^{z-x} [1 - H(z - x - t)]G(x + t-)\, dN(t)\, dF(x) \\ &= \int_{0}^{z} \int_{0-}^{z-x} [1 - H(z - x - t)]G(x + t-)\, dN(t)\, dF(x) \end{aligned}$$

as required. For the proof of (13), consider changes of integration according to

$$\int_0^{\infty} dz \int_0^z dx \int_{0-}^{z-x} dt = \int_0^{\infty} dx \int_{x-}^{\infty} dz \int_{0-}^{z-x} dt = \int_0^{\infty} dx \int_{0-}^{\infty} dt \int_{(x+t)-}^{\infty} dz.$$

It then follows that, for $x > 0$,

$$\frac{1}{s}\Phi(s) = \int_0^\infty e^{-sz} H(z)\,dz = \int_0^\infty \int_{0-}^\infty e^{-s(x+t)} \frac{1}{s}[1 - \Phi(s)]G(x + t-)\,dN(t)\,dF(x).$$

Solving for $\Phi(s)$ gives the desired result. As was stated earlier, Takacs [5] derived an integral equation in $N(t)$, which may be shortened to read

$$N(t) - 1 = \int_{0-}^t G(x-)\,dW(x)$$

where

$$W(x) = \int_{0-}^x F(x - y)\,dN(y).$$

Upon taking Laplace transforms of both sides one may check that it satisfies the relationship (11).

Of particular interest is the expectation of the secondary renewal process, namely $E(Z)$. As in the Type I problem, it follows from known results of Sequential Analysis that

$$E(Z) = E\left(\sum_{j=1}^{n_1} X_j\right) = \mu E(n_1). \tag{15}$$

From the above theorem, one obtains

$$E(Z) = \lim_{s\to 0}\frac{1 - \Phi(s)}{s} = 1/\lim_{s\to 0} s\lambda(s).$$

Thus, by (10), one obtains a double relationship

$$1/E(n_1) = \nu \lim_{s\to 0} s\lambda(s) = \lim_{k\to\infty} r_k\,.$$

Although it may well be that in a particular example one of the above limits will be computable, in most cases they will be unwieldy. For example, even in the case of Poisson input, the quantities r_k are complicated expressions, although $E(n_1)$ is a simple expression best obtained in an entirely different way using the particular properties of the exponential distribution. The p_k's may be expressed in terms of the r_j's as follows;

$$p_n = -\sum{}^* \prod_{j=1}^n \left((-r_j)^{k_j}\frac{1}{k_j\,!}\right) k.\,! \tag{16}$$

where $k. = \sum_{j=1}^n k_j$ and $\sum^*$ denotes summation over all vectors of integers $(k_1, k_2, \cdots, k_n)$ for which $\sum_{j=1}^n jk_j = n$. However, (16) will be, in most cases, very unwieldy, especially when one recalls the complicated structure of the r_j's.

9. The case of constant deadtime. Partial results for this example have been given by Takacs [5] for the Albert and Nelson model to be studied in the next section. Also, this case has been studied from a different viewpoint by Smith [6]. We shall study this special case in full. Set $Y = d$ (a.s.). Then $G(x) = 0$ or 1 according as $x <$ or $\geqq d$. From (9) we obtain for $k \geqq 1$

$$r_k = 1 - F(d) \equiv q \text{ say.}$$

Consequently by (10)

$$E(n_1) = [1 - F(d)]^{-1} = q^{-1} \tag{17}$$

which is interpreted as being equal to ∞ if $1 = F(d)$. Using the notation introduced in section 4, one obtains

$$R(s) = \sum_{k=1}^{\infty} r_k s^k = qs(1 - s)^{-1}$$

and hence

$$P(s) = \frac{sq}{sq + 1 - s}.$$

From this relationship, or by direct computation, one obtains

$$p_k = q(1 - q)^{k-1}.$$

Therefore, n_1 has a Pascal (or geometric) distribution. The quickest way to obtain H and Φ for this example is as follows. Clearly $H(z) = 0$ for $z \leqq d$. For $z \geqq d$

$$\begin{aligned} H(z) &= \sum_{n=1}^{\infty} \Pr [S_n \leqq z \mid n_1 = n] \, p_n \\ &= q \sum_{n=1}^{\infty} \Pr [S_n \leqq z \mid n_1 = n](1 - q)^{n-1}. \end{aligned}$$

Now

$$\begin{aligned} \Pr [S_n \leqq z \mid n_1 = n] &= \Pr [S_n \leqq z \mid X_j < d, 1 \leqq j < n - 1, X_n > d] \\ &= \Pr [U_1 + U_2 + \cdots + U_{n-1} + V \leqq z] \end{aligned}$$

where the U_i's and V, are mutually independent with c.d.f.'s given by

$$\Pr [U_i \leqq u] \equiv K(u) = F(u)/F(d); \quad (u \leqq d, 1 \leqq i < n)$$

$$\Pr [V \leqq u] \equiv L(u) = \frac{F(u) - F(d)}{1 - F(d)} \qquad (u \geqq d).$$

Therefore, for $z \geqq$ d

$$H(z) = q \sum_{n=1}^{\infty} (1 - q)^{n-1} \int_{0-}^{z} K_n(z - u) \, dL(u) \tag{18}$$

where K_n denotes the nth convolution of K with itself. It is then immediate that

$$\Phi(s) = \frac{\int_d^\infty e^{-sx}\, dF(x)}{1 - \int_0^d e^{-sx}\, dF(x)}. \tag{19}$$

One may check that expressions (18) and (19) satisfy the equations of Theorem 4. In [6], Smith has obtained for this case $N(t) = 1$ for $0 \leqq t \leqq d$ and

$$N(t) - 1 = \int_{0-}^{t-d} [F(t - x) - F(d)]\, dM(x)$$

for $t > d$. By means of (11), one may show that this expression agrees with (19). (19) has also been obtained by Takacs [3]. From (15) and (17) it follows that $E(Z) = \mu q^{-1}$. One may also compute

$$\text{var } (Z) = q^{-1} \text{ var } (X) + 2\mu q^{-2} \int_0^d x\, dF(x)$$

which disagrees with the expression given in Theorem 7 of [6].

For this example, not only is it possible to compute $P_0(t)$, the probability that the counter is free, but one may also derive the quantities $P_k(t)$ defined by

$$P_k(t) = \Pr [S_j + Y_j \geqq t \text{ for exactly } k \text{ values of } j]$$

for $k = 0, 1, 2, \cdots$. That is, $P_k(t)$ denotes the probability that k impulses are in process at time t. Now then

$$\begin{aligned} P_k(t) &= \sum_{j=0}^{\infty} \Pr [S_j \leqq t - d < S_{j+1} \leqq S_{j+k} < t \leqq S_{j+k+1}] \\ &= \int_{0-}^{t-d} \int_{t-d-x}^{t-x} [F_{k-1}(t - x - y) - F_k(t - x - y)]\, dF(y)\, dM(x). \end{aligned}$$

In particular

$$P_0(t) = \int_{0-}^{t-d} [1 - F(t - x)]\, dM(x). \tag{20}$$

Define the real functions h_m ($m \geqq 0$) as follows: for $v \leqq d$ set $h_m(v) = 1$ and for $v \geqq d$ set

$$h_m(v) = 1 - \int_0^{v-d} F_m(v - y)\, dF(y).$$

With these definitions we may write for $k \geqq 1$

$$P_k(t) = \int_{0-}^{t-d} [F_k(t - x) - F_{k+1}(t - x) - h_k(t - x) + h_{k-1}(t - x)]\, dM(x).$$

The functions h_m and $1 - F_m$ $(m \geqq 0)$ clearly satisfy the conditions of Theorem S, by which

$$(21)\qquad P_k = \lim_{t\to\infty} P_k(t) = \mu^{-1}\int_d^\infty [F_k(v) - F_{k+1}(v) - h_k(v) + h_{k-1}(v)]\,dv.$$

Moreover, by definition

$$\begin{aligned}\int_0^\infty [h_k(v) - h_{k-1}(v)]\,dv &= \int_d^\infty \int_0^{v-d} [F_{k-1}(v-y) - F_k(v-y)]\,dF(y)\,dv \\ &= \int_0^\infty \int_{y+d-}^\infty [F_{k-1}(v-y) - F_k(v-y)]\,dv\,dF(y) \\ &= \mu - \int_0^d [F_{k-1}(v) - F_k(v)]\,dv.\end{aligned}$$

Therefore, by (20) and (21)

$$P_k = \mu^{-1}\int_0^d [F_{k-1}(v) - 2F_k(v) + F_{k+1}(v)]\,dv \qquad (k \geqq 1)$$

$$P_0 = 1 - \mu^{-1}\int_0^d [1 - F(v)]\,dv$$

10. The Albert and Nelson generalization. Let $p \varepsilon [0, 1]$. Define

$$Y^{(p)} = \begin{cases} Y \text{ with probability } p \\ 0 \text{ with probability } 1 - p = q \end{cases}$$

which has c.d.f. G_p where $G_p(0) = p$, $G_p(x) = q + pG(x)$ for $x > 0$. Albert and Nelson [1] suggested as a generalization of the Types I and II counter models, the model in which the deadtime caused by an incoming pulse is Y or $Y^{(p)}$ according as the pulse is registered or not. Formally, define $n_0 = 0$ (a.s.) and $j \geqq 1$

$$(22)\qquad n_j = \min\{k \varepsilon I^+ : S_i + Y_i^{(p)} \leqq S_k\ (n_{j-1} < i < k),\ S_{n_{j-1}} + Y \leqq S_k\}$$

where as usual the subscript on $Y_i^{(p)}$ denotes identically and independent random variables with c.d.f. G_p. The purpose of this section is to show that the c.d.f. H of the secondary renewal process, $Z_j = S_{n_j} - S_{n_{j-1}}$ $(j \geqq 0)$ obtained for this generalization is in fact completely solved once the general Type II problem is solved, and in this sense this generalization is a very slight one.

Let $Z^{(p)}$ be the secondary renewal process of a Type II counter model in which the deadtime r.v. is $Y^{(p)}$. Let $H^{(p)}$ denote its c.d.f., $\Phi^{(p)}$ its characteristic function and $N^{(p)}(x) = \sum_{j=0}^\infty H_j^{(p)}(x)$. The distribution function of the Z-renewal process may then be given by

THEOREM 5. *For all* $z \geqq 0$

$$H(z) = \int_0^z \int_{0-}^{z-x} [1 - H^{(p)}(z-x-y)]\,G(x+y-)\,dN^{(p)}(y)\,dF(x)$$

and for $s > 0$

$$\Phi(s) = [1 - \Phi^{(p)}(s)] \int_{0-}^{\infty} \int_{0}^{\infty} e^{-s(x+y)} G(x + y-)\, dF(x)\, dN^{(p)}(y). \tag{23}$$

This theorem is proven in the same way as Theorem 4 upon noticing that in evaluating Pr $(B \mid C)$, one need only consider a Type II model with r.v.'s X and $Y^{(p)}$. Thus, although at first glance one might suspect that this more general model would offer difficulties peculiar to itself, it is seen that a solution of the corresponding Type II problem automatically provides a solution of the general problem. For $p = 1$, this model reduces to the Type II model and for $p = 0$, to the Type I model, as can be seen by comparing the definition of the N-process, (22), for these two values of p.

EXAMPLE. In [5], Takacs works out the special case of the Albert and Nelson model in which Y is constant a.e. As a further example, we evaluate here the case in which $G(x) = 1 - e^{-\lambda x}$ $(x \geqq 0)$. As was pointed out above, it will be sufficient to solve the Type II problem in which the deadtime, $Y^{(p)}$, has c.d.f. $G^{(p)}(x) = 1 - pe^{-\lambda x}$ $(x \geqq 0)$ and zero elsewhere. For this case we have by (14),

$$\begin{aligned}\lambda^{(p)}(s) &= \int_{0-}^{\infty} \int_{0}^{\infty} e^{-s(x+y)}[1 - pe^{-\lambda(x+y)}]\, dF(x)\, dN^{(p)}(y) \\ &= \int_{0-}^{\infty} e^{-sy}[\varphi(s) - pe^{-(s+\lambda)y}\varphi(s + \lambda)]\, dN^{(p)}(y) \\ &= \frac{\varphi(s)}{1 - \Phi^{(p)}(s)} - p\,\frac{\varphi(s + \lambda)}{1 - \Phi^{(p)}(s + \lambda)}.\end{aligned}$$

Therefore

$$\lambda^{(p)}(s) = \frac{\varphi(s) - q\varphi(s + \lambda)}{1 - \varphi(s)} - p\,\frac{\varphi(s + \lambda)}{1 - \varphi(s)}\,\lambda^{(p)}(s + \lambda)$$

since $1 - \Phi^{(p)}(s) = [1 + \lambda^{(p)}(s)]^{-1}$. Since this relation holds for all $s > 0$, we obtain by recursion that for all $n \geqq 1$

$$\begin{aligned}\lambda^{(p)}(s) &= \sum_{j=0}^{n} (-p)^j \frac{\varphi_j - p\varphi_{j+1}}{1 - \varphi_j} \prod_{k=0}^{j-1} \frac{\varphi_{k+1}}{1 - \varphi_k} \\ &\quad + (-p)^{n+1} \prod_{k=0}^{n} \frac{\varphi_{k+1}}{1 - \varphi_k}\, \lambda^{(p)}(s + \lambda + n\lambda)\end{aligned}$$

where for convenience we have set $\varphi_j = \varphi(s + j\lambda)$. Since $\varphi(s) \to 0$ and $\lambda(s) \to 0$ as $s \to \infty$ we finally obtain

$$\begin{aligned}\lambda^{(p)}(s) &= \lim_{N\to\infty} \frac{1}{\varphi(s)} \sum_{j=0}^{N} (-p)^j \prod_{k=0}^{j} \frac{\varphi_k}{1 - \varphi_k} \\ &\quad + \sum_{j=1}^{N+1} (-p)^j \prod_{k=0}^{j} \frac{\varphi_{k+1}}{1 - \varphi_k} \\ &= \frac{1}{\varphi(s)} \sum_{j=0}^{\infty} (-p)^j \prod_{k=0}^{j} \frac{\varphi_k}{1 - \varphi_k}.\end{aligned} \tag{24}$$

Thus $\Phi^{(p)}(s) = \lambda^{(p)}(s)[1 + \lambda^{(p)}(s)]^{-1}$, the solution to the Type II model in which the deadtime distribution is $G^{(p)}(x) = 1 - pe^{-\lambda x}$ $(x \geqq 0)$, is determined. From Theorem 5, in particular equation (23), one obtains

$$\Phi(s) = \varphi(s) - \varphi(s + \lambda) \frac{1 - \Phi^{(p)}(s)}{1 - \Phi^{(p)}(s + \lambda)}$$

$$= \varphi(s) - \varphi(s + \lambda) \frac{1 + \lambda^{(p)}(s + \lambda)}{1 + \lambda^{(p)}(s)}$$

Upon substitution of (24) into this expression, one obtains the solution to the Albert and Nelson model with exponential deadtime. When $p = 1$, (23) yields the solution to the corresponding Type II problem with exponential deadtime as given explicitly by Takacs [5] and implicitly by Pollaczek [10].

Acknowledgement. The author wishes to express his appreciation to Professor S. Karlin for many profitable discussions and suggestions, and to the referee for his helpful comments.

REFERENCES

[1] G. E. ALBERT AND L. NELSON, "Contributions to the statistical theory of counter data," *Ann. Math. Stat.*, vol. 24 (1953), pp. 9–22.

[2] W. FELLER, "On probability problems in the theory of counters," *Courant Anniversary Volume* (1948), pp. 105–15.

[3] L. TAKACS, "On a probability problem arising in the theory of counters," *Proc. Cambridge Phil. Soc.*, vol. 52 (1956), pp. 488–498.

[4] K. J. ARROW, S. KARLIN AND H. SCARF, *Mathematical studies in inventory production processes*, to be published by Stanford University Press, (1958).

[5] L. TAKACS, "On the sequence of events, selected by a counter from a recurrent process of events," *Akad Nauk. U.S.S.R., Probability Theory and its applications*, vol. 1 (1956), pp. 90–102.

[6] W. L. SMITH, "On renewal theory, counter problems, and quasi-Poisson processes," *Proc. Cambridge Phil. Soc.*, vol. 53 (1957), pp. 175–193.

[7] W. L. SMITH, "Asymptotic renewal theorems," *Proc. Roy. Soc. of Edinburgh*, Ser. A, vol. 64, pp. 9–48.

[8] W. L. SMITH, "Regenerative stochastic processes," *Proc. Roy. Soc.*, Ser. A, vol. 232 (1955), pp. 6–31.

[9] W. FELLER, *Introduction to probability theory and its applications*, vol. 1, J. Wiley and Sons, 1951.

[10] F. POLLACZEK, "Sur la theorie stochastique des compteurs," *C. R. Acad. Sci. Paris*, vol. 238 (1954), pp. 766–7.

[11] H. CHERNOFF AND J. F. DALY, "The distribution of shadows," *Journal of Math. and Mechanics*, vol. 6, No. 4 (1957), pp. 567–584.