ON THE GENERAL CANONICAL CORRELATION DISTRIBUTION

By A. G. Constantine and A. T. James

Division of Mathematical Statistics, C.S.I.R.O.[1]

1. Summary. The paper is divided into two parts:

A. An elementary derivation of Bartlett's results on the distribution of the canonical correlation coefficients using exterior differential forms. Briefly, our method consists of taking the original multivariate normal distribution, transforming to the canonical correlations and other variables, and then integrating out these extraneous variables.

B. A new method of calculating the conditional moments which appear in Bartlett's expansion of this distribution, based on the process of averaging over the orthogonal group. This method allows the calculation of moments of any order.

Part A

2. Introduction. Bartlett [1] obtained the general canonical correlation distribution as a multiple power series in the true canonical correlations ρ_i. In the case of more than one non-zero correlation ρ_i, the coefficients in this expansion depend on the conditional moments of the sample (ordinary) correlations s_i between the pairs of transformed variates representing the true canonical variates, when the sample canonical correlations r_i between the sample canonical variates are fixed.

Bartlett derived his results by a formal generalization of the argument used by Fisher [2] in calculating the distribution of the multiple correlation coefficient. We shall give a new proof of Bartlett's results in a concrete form more suitable for our purposes. Throughout this paper we shall use the concepts of exterior differential forms and alternating products of these forms. The definition and a discussion of these concepts may be found in James [6].

Consider a dependent vector variate with p components and an independent vector variate with $q \geqq p$ components. (Here the terms "dependent" and "independent" are to be understood in the regression sense.) If we take a sample with $n(\geqq p + q)$ degrees of freedom, we may represent it by the $p + q$ column vectors $\xi_1, \xi_2, \cdots, \xi_p$ and $\eta_1, \eta_2, \cdots, \eta_q$, each containing n components. The dependent vector is considered to be a normal variate, and we may distinguish two cases, according as the independent variate is assumed to be (a) a normal variate or (b) a set of fixed vectors in the sample space. In either case we may, without loss of generality, assume the ξ_i and η_j to be the canonical variates (see

---

Received September 23, 1957; revised March 1, 1958.

[1] Commonwealth Scientific and Industrial Research Organization, University of Adelaide, South Australia.

Hotelling [4]). This means that in case (a) the n components of each vector are standard normal variates with the joint distribution

$$\prod_{i=1}^{p}\left\{(2\pi)^{-n}(1-\rho_i^2)^{-n/2}\exp\left[\frac{-(\xi_i'\xi_i-2\rho_i\xi_i'\eta_i+\eta_i'\eta_i)}{2(1-\rho_i^2)}\right]\prod_{v=1}^{n}d\xi_{vi}\,d\eta_{vi}\right\} \cdot \prod_{j=p+1}^{q}\left\{(2\pi)^{-n/2}\exp\left[-\eta_j'\eta_j/2\right]\prod_{v=1}^{n}d\eta_{vj}\right\}. \tag{2.1}$$

In case (b), the non-central means case, we may assume the components of the ξ_i to be independently distributed with unit variance, and the η_j to be vectors lying along the first q co-ordinate axes of the sample space. $\eta_1, \cdots, \eta_p$ may also be identified with the mean vectors of $\xi_1, \cdots, \xi_p$. The joint distribution of the ξ_{vi} is therefore

$$\prod_{i=1}^{p}\left\{(2\pi)^{-n/2}\exp\left[-(\xi_i'\xi_i-2\xi_i'\eta_i+\eta_i'\eta_i)/2\right]\prod_{v=1}^{n}d\xi_{vi}\right\}. \tag{2.2}$$

We denote sample correlations between ξ_i and η_i by s_i and the sample canonical correlations between the sample canonical variates by r_i. The r_i may also be interpreted as the cosines of the critical angles between the two planes spanned by $x_1, \cdots, x_p$ and $y_1, \cdots, y_q$ respectively, where the x_i and y_j are the sample canonical variates. The distribution of the r_i for each of the two cases mentioned above will be derived in sections 3 and 4 respectively.

3. Distribution of the canonical correlation coefficients. Our starting point is the distribution (2.1). The distribution of the canonical correlations r_i will be obtained by expressing this distribution in terms of the r_i and other variables and integrating over the ranges of the latter. First of all, let us dispose of the lengths of the vectors ξ_i and η_j.

Put $\xi_i = \tau_i w_i$ and $\eta_j = \sigma_j z_j$ where τ_i and σ_j are the unit vectors along ξ_i and η_j respectively, and w_i and z_j are their lengths. Then

$$\prod_{v=1}^{n} d\xi_{vi} = w_i^{n-1}\,dw_i\,dS(\tau_i) \tag{3.1}$$

where $dS(\tau_i)$ is the element of area on the unit sphere in n-space. With an analogous expression for $\prod d\eta_{vj}$ the distribution (2.1) becomes

$$\begin{aligned}\prod_{i=1}^{p}&\left\{\frac{1}{2^{n-2}(1-\rho_i^2)^{n/2}[\Gamma(n/2)]^2}\right.\\ &\left.\cdot\exp\left[-\frac{1}{2(1-\rho_i^2)}(w_i^2+z_i^2-2\rho_i s_i w_i z_i)\right](w_i z_i)^{n-1}\,dw_i\,dz_i\right\}\\ &\times\prod_{j=p+1}^{q}\frac{1}{2^{(n-2)/2}\Gamma(n/2)}\exp\left[-\tfrac{1}{2}z_j^2\right]z_j^{n-1}\,dz_j\prod_{i=1}^{p}\frac{\Gamma(n/2)}{2\pi^{n/2}}\,dS(\tau_i)\\ &\times\prod_{j=1}^{q}\frac{\Gamma(n/2)}{2\pi^{n/2}}\,dS(\sigma_j),\end{aligned} \tag{3.2}$$

where $s_i = \tau_i'\sigma_i$ (see section 2). The constants have been split up to make the latter factors probability distributions.

The integrals of the factors containing z_j for $j = p + 1, \cdots, q$ are obviously unity. Furthermore, by expanding the factor $\exp[(1 - \rho_i^2)^{-1}\rho_i s_i w_i z_i]$ in a power series and integrating term-by-term with respect to w_i and z_i $(i = 1, \cdots, p)$ we obtain

$$\begin{aligned}(3.3)\qquad &\int_0^\infty\int_0^\infty \frac{1}{2^{n-2}(1-\rho_i^2)^{n/2}[\Gamma(n/2)]^2} \\ &\qquad\cdot \exp[-(w_i^2 + z_i^2 - 2\rho_i s_i w_i z_i)/2(1-\rho_i^2)](w_i z_i)^{n-1}\, dw_i\, dz_i \\ &\qquad\qquad = (1-\rho_i^2)^{n/2}{}_2F_1(n/2, n/2; 1/2; \rho_i^2 s_i^2) + \text{an odd function of } s_i,\end{aligned}$$

where ${}_2F_1$ is the Gaussian hypergeometric function. Later on, we shall see that the odd function of s_i vanishes in the subsequent integrations.

The next step is to express the unit column vectors τ_i and σ_j in terms of the canonical correlations r_i and the vectors x_i and y_j which determine these correlations. Let $\mathfrak{p}$ and $\mathfrak{q}$ be the p-plane and the q-plane spanned by the vectors τ_i and σ_j in n-space. Then $\mathfrak{p}$ and $\mathfrak{q}$ determine almost certainly (i.e. with probability 1) the orthonormal vectors x_i and y_i $(i = 1, \cdots, p)$ which make the critical angles between the planes, i.e. such that $x_i'y_i = r_i$, $x_i'y_j = 0$ $(i \neq j)$. Let further vectors $y_{p+1}, \cdots, y_q$ be defined as functions of $\mathfrak{p}$ and $\mathfrak{q}$ to complete an orthonormal set spanning $\mathfrak{q}$. T, Σ, X, Y will denote the matrices composed of the column vectors τ_i, σ_j, x_i, y_j, respectively. It follows that $X'X = I_p$, $Y'Y = I_q$ and $X'Y = [R \,\vdots\, 0]$ where R is the diagonal matrix with the r_i down the main diagonal. Furthermore we may write

$$(3.4)\qquad T = XA, \qquad \Sigma = YB$$

where A is a $p \times p$ and B is a $q \times q$ matrix. Then $T'T = A'A$ and $\Sigma'\Sigma = B'B$. The matrices A and B are subject only to the restriction that all their columns α_i and β_j are of unit length.

We now substitute for $\prod dS(\tau_i)$ and $\prod dS(\sigma_j)$ in (3.2), using the transformations (3.4). To avoid interrupting the continuity of the argument we shall, for the moment, only give the results of the substitution, and defer the proof until section 5. We have then from (5.4)

$$(3.5)\qquad \prod_{i=1}^{p} dS(\tau_i) = |A'A|^{(n-p)/2}\prod_{i=1}^{p} dS(\alpha_i)\, d\mathfrak{p} + {}^*(dX)\, d\mathfrak{p}$$

where $dS(\alpha_i)$ is the element of area on the unit sphere in p-space and $d\mathfrak{p}$ is the differential form representing the invariant measure on the Grassmann manifold of p-planes in n-space. The symbol ${}^*(dX)$ stands for certain differentials involving the elements of X, which, when subsequently multiplied by other differentials, will vanish. Similarly

$$(3.6)\qquad \prod_{j=1}^{q} dS(\sigma_j) = |B'B|^{(n-q)/2}\prod_{j=1}^{q} dS(\beta_j)\, d\mathfrak{q} + {}^*(dY)\, d\mathfrak{q}$$

where $dS(\beta_j)$ is the element of area on the unit sphere in q-space. Multiplying (3.5) and (3.6) we obtain

$$(3.7)\quad \prod_{i=1}^{p} dS(\tau_i) \prod_{j=1}^{q} dS(\sigma_j) = | A'A |^{(n-p)/2} \prod_{i=1}^{p} dS(\alpha_i) \, | B'B |^{(n-q)/2} \prod_{j=1}^{q} dS(\beta_j) \, d\mathfrak{p} \, d\mathfrak{q}.$$

The terms containing $*(dX)$ and $*(dY)$ vanish when multiplied by $d\mathfrak{p}\, d\mathfrak{q}$, since $d\mathfrak{p}\, d\mathfrak{q}$ is of maximum degree in $\mathfrak{p}$ and $\mathfrak{q}$ and X and Y are functions of $\mathfrak{p}$ and $\mathfrak{q}$.

The differential form $d\mathfrak{p}\, d\mathfrak{q}$ may now be expressed in terms of the r_i and other variables. Integration with respect to these latter variables yields

$$K_p K_q \phi(r_i \mid \rho_i = 0)$$

where K_p and K_q are the normalising constants of the differential forms $d\mathfrak{p}$ and $d\mathfrak{q}$ respectively, and $\phi(r_i \mid \rho_i = 0)$ is the null distribution of the r_i (see James [6]):

$$\phi(r_i \mid \rho_i = 0) = C \prod_{i=1}^{p} \{(r_i^2)^{(q-p-1)/2}(1 - r_i^2)^{(n-q-p-1)/2}\} \prod_{i<j} (r_i^2 - r_j^2) \prod_{i=1}^{p} dr_i^2$$

and

$$C = \pi^{p/2} \prod_{i=0}^{p-1} \left\{ \Gamma\left(\frac{n-i}{2}\right) \Big/ \Gamma\left(\frac{p-i}{2}\right) \Gamma\left(\frac{q-1}{2}\right) \Gamma\left(\frac{n-q-i}{2}\right) \right\}.$$

This distribution was first derived by Fisher [3], Hsu [5] and Roy [8]. The values of K_p and K_q are given by

$$K_v = \prod_{i=1}^{v} \frac{G(n-i+1)}{G(i)}, \qquad G(i) = \frac{2\pi^{i/2}}{\Gamma(i/2)}, \qquad v = p, q.$$

After this integration, the right hand side of (3.7) becomes

$$(3.8)\quad K_p \mid A'A \mid^{(n-p)/2} \prod dS(\alpha_i) K_q \mid B'B \mid^{(n-q)/2} \prod dS(\beta_j) \cdot \phi(r_i \mid \rho_i = 0),$$

showing that A, B and the r_i are independently distributed.

Substituting (3.3) and (3.8) in (3.2), we may write the distribution of the r_i as

$$(3.9)\quad \int_A \int_B \prod_{i=1}^{p} \{(1-\rho_i^2)^{n/2} {}_2F_1(n/2, n/2; 1/2; \rho_i^2 s_i^2)\} k_p \, | A'A |^{(n-p)/2} \cdot \prod_{i=1}^{p} dS(\alpha_i) k_q \, | B'B |^{(n-q)/2} \prod_{j=1}^{q} dS(\beta_j) \phi(r_i | \rho_i = 0),$$

together with the relation

$$(3.10)\quad s_i = \tau_i' \sigma_i = \alpha_{1i}\beta_{1i}r_1 + \alpha_{2i}\beta_{2i}r_{2i} + \cdots + \alpha_{pi}\beta_{pi}r_p .$$

The normalising constants k_p and k_q for the distribution of A and B are given by

$$(3.11)\quad k_v = \prod_{i=1}^{v} \frac{G(n-i+1)}{G(n)G(i)}, \qquad v = p, q.$$

In view of equation (3.10) we may now identify our distribution (3.9) with Bartlett's distribution, [1], equations (8) and (10).

If the hypergeometric functions are expanded as power series and multiplied together, the function multiplying $\phi(r_i \mid \rho_i = 0)$ is seen to be a multiple power series in the ρ_i whose coefficients depend on the expectations of monomials in the s_i with respect to the distribution

$$k_p \mid A'A \mid^{(n-p)/2} dS(\alpha_1) \cdots dS(\alpha_p) \tag{3.12}$$

of A and a similar distribution of B.

So far we have ignored the odd function of s_i appearing in the integral (3.3). However, any odd function $f(s_i)$ of s_i will have zero expectation. In fact, putting $-\alpha_i$ instead of α_i does not alter the distribution (3.12) of A, but changes s_i to $-s_i$ in view of (3.10). Therefore,

$$E[f(s_i)] = E[f(-s_i)] = E[-f(s_i)] = -E[f(s_i)]$$

and so $E[f(s_i)] = 0$. It is sufficient, therefore, to compute only moments of the form $\mu(t_1, t_2, \cdots, t_p) = E\{(s_1^2)^{t_1}(s_2^2)^{t_2} \cdots (s_p^2)^{t_p}\}$ where the expectations are taken with respect to the distributions of A and B and the r_i are held fixed. Furthermore, if we substitute in (3.9) for s_i using (3.10), the calculations are reduced to finding the moments of the α_{ij} and β_{ij}, two independent sets of variates.

Theoretically these moments could be found directly from the distributions of A and B. However, as Bartlett pointed out, this method is too difficult algebraically to be of much use, except in the case of only one non-zero ρ_i. Bartlett indicated a method whereby moments of the form $\mu(t_1, t_2)$ could be calculated, and also calculated $\mu(1, 1, 1)$ by employing various relations connecting the α-moments (see section 10). Again, both of these methods led to awkward algebra and had to be abandoned for moments of higher order, though Bartlett was able to compute $\mu(1, 1)$ $\mu(2, 1)$, $\mu(2, 2)$ and $\mu(3, 1)$. In part B of this paper we shall present a method enabling moments of any order to be computed, and shall complete the tabulation of moments up to the fourth order with $\mu(2, 1, 1)$ and $\mu(1, 1, 1, 1)$.

4. The non-central means case. Let $\mathfrak{p}$ be the random plane spanned by the vectors $\xi_1, \cdots, \xi_p$ and $\mathfrak{q}$ the fixed plane spanned by $\eta_1, \eta_2, \cdots, \eta_q$. As we saw in section 2, we may assume that the $\xi_1, \cdots, \xi_p$ are independently distributed and their components ξ_{vi} have the distribution

$$\prod_{i=1}^{p} (2\pi)^{-n/2} \exp\left[-(\xi_i' \xi_i - 2\xi_i' \eta_i + \eta_i' \eta_i)/2\right] \prod_{v=1}^{n} d\xi_{vi}. \tag{4.1}$$

Furthermore, the η_j $(j = 1, \cdots, q)$ may be taken as vectors lying along the first q co-ordinate axes in the sample space and thus having only one non-zero component each, say μ_j in the jth position.

Putting $\xi_i = \tau_i w_i$ as before, (4.1) becomes

$$\prod_{i=1}^{p} \frac{1}{2^{(n-2)/2}\Gamma(n/2)} \exp\left[-(w_i^2 - 2\mu_i s_i w_i + \mu_i^2)/2\right] w_i^{n-1}\, dw_i \times \prod_{i=1}^{p} \frac{\Gamma(n/2)}{2\pi^{n/2}}\, dS(\tau_i), \tag{4.2}$$

where $s_i = \tau_{ii}$. The integral with respect to w_i of the ith factor in the first product of (4.2) is ${}_1F_1(n/2; 1/2; \mu_i^2 s_i^2/2)e^{-\mu_i^2/2}$ + an odd function of s_i. This odd function will again vanish in subsequent integrations and may be ignored from now on.

Let X be the $n \times p$ matrix whose columns are the orthonormal vectors $x_1, x_2, \cdots, x_p$ spanning $\mathfrak{p}$ and which make the critical angles with $\mathfrak{q}$. The τ_i may be expressed as linear combinations of the x_i by putting

$$T = XA. \tag{4.3}$$

Since $X'X = I_p$ we have $T'T = A'A$. From section 5, (5.4), it follows that

$$\prod_{i=1}^{p} dS(\tau_i) = |A'A|^{(n-p)/2} \prod_{i=1}^{p} dS(\alpha_i)\, d\mathfrak{p}, \tag{4.4}$$

the differential form $*(dX)\, d\mathfrak{p}$ vanishing since X and $\mathfrak{p}$ are functions of each other.

To express $\mathfrak{p}$ in terms of the r_i, we partition X as follows:

$$X = \begin{bmatrix} Y \\ \cdots \\ Z \end{bmatrix} \tag{4.5}$$

where Y is a $q \times p$ matrix and Z is an $(n - q) \times p$ matrix. The vector $\begin{bmatrix} y_i \\ \cdots \\ 0 \end{bmatrix}$ in $\mathfrak{q}$ makes the ith critical angle with x_i in $\mathfrak{p}$. Let β_i and $\gamma_i (i = 1, \cdots, p)$ be the unit vectors along y_i and z_i, then according to [6], equation (7.10),

$$y_i = \beta_i r_i, \qquad z_i = \gamma_i \sqrt{1 - r_i^2} \tag{4.6}$$

and

$$d\mathfrak{p} = K_p \frac{1}{\prod_{i=1}^{p} G(q - i + 1)} \cdot dV(\beta) \frac{1}{\prod_{i=1}^{p} G(n - q - i + 1)} dV(\gamma)\phi(r_i \mid \rho_i = 0) \tag{4.7}$$

where K_p and $G(i)$ are defined in section 3, and $dV(\beta)$ and $dV(\gamma)$ are the invariant measures on the Stiefel manifolds of p-frames $(\beta_1, \cdots, \beta_p)$ in q-space and p-frames $(\gamma_1, \cdots, \gamma_p)$ in $(n - q)$-space. The constant has been split up to normalise these invariant measures. If we choose $q - p$ orthonormal vectors $\beta_{p+1}, \cdots, \beta_q$ orthogonal to $\beta_1, \cdots, \beta_p$ we may express $dV(\beta)$ as

$$(4.8) \qquad dV(\beta) = \prod_{i<j}^{p} \beta_j' \, d\beta_i \prod_{j=p+1}^{q} \prod_{i=1}^{p} \beta_j' \, d\beta_i .$$

Also,

$$s_i = \tau_{ii} = \sum_{j=1}^{p} x_{ij}\alpha_{ji} = \sum_{j=1}^{p} \beta_{ij} r_j \alpha_{ji}$$

If we please, we may replace β_{ij} by β_{ji} since they have the same distribution. Integrating (4.7) with respect to γ, substituting in (4.4) and then in (4.2), we obtain the distribution of the r_i as

$$(4.9) \qquad \int_A \int_B \prod_{i=1}^{p} {}_1F_1(n/2; 1/2; \tfrac{1}{2}\mu_i^2 s_i^2) e^{-\mu_i^2/2} k_p \, | A'A |^{(n-p)/2} \prod_{i=1}^{p} dS(\alpha_i) \cdot \frac{1}{\prod_{i=1}^{p} G(q - i + 1)} \beta_j' \, d\beta_i \prod_{j=p+1}^{q} \prod_{i=1}^{p} \beta_j' \, d\beta_i \phi(r_i \,|\, \rho_i = 0($$

where

$$(4.10) \qquad s_i = \alpha_{1i}\beta_{1i}r_1 + \cdots + \alpha_{pi}\beta_{pi}r_p .$$

We notice that the distribution of A is identical with its distribution in the previous case, but now the distribution of B is the invariant distribution on a Stiefel manifold and is independent of n. However, A and B are still independent.

5. Distribution of the co-ordinates of random vectors in a random plane. In relation to the rest of the paper, the purpose of this section is to derive equation (3.5) and a result at the end of section 7. However, the results will be more interesting and intelligible if discussed in terms of probabilities.

$\tau_1, \cdots, \tau_p$ are invariantly distributed unit vectors in n-space, which we write as the columns of an $n \times p$ matrix T. $\mathfrak{p}$ is the plane spanned by the τ_i. We define in $\mathfrak{p}$ a reference set of orthonormal vectors, which we write as the columns of an $n \times p$ matrix X. Thus X is a function of $\mathfrak{p}$ and

$$(5.1) \qquad X'X = I_p .$$

Let the column α_i of the $p \times p$ matrix A be the co-ordinates of τ_i relative to the reference set X:

$$(5.2) \qquad T = XA.$$

We shall show that $\mathfrak{p}$ is invariantly distributed and that A is independently distributed with density proportional to

$$(5.3) \qquad | A'A |^{(n-p)/2} \prod_{i=1}^{p} dS(\alpha_i).$$

These results are implicit in Bartlett [1]. They follow from the lemma which we shall now state and prove. For the application in section 3 we shall have to generalise the situation slightly to include the case when the reference set X is not necessarily a function of $\mathfrak{p}$ alone.

LEMMA. *If T is an $n \times p$ matrix whose columns τ_i are unit vectors, and X and A are $n \times p$ and $p \times p$ matrices satisfying* (5.1) *and* (5.2), *then*

$$\prod_{i=1}^{p} dS(\tau_i) = |A'A|^{(n-p)/2} \prod_{i=1}^{p} dS(\alpha_i)\, d\mathfrak{p} + {}^*(dX)\, d\mathfrak{p} \tag{5.4}$$

where $^*(dX)$ is a differential form in X and A, every term of which is of at least first degree in dX. If X is a function of $\mathfrak{p}$ alone, then $^*(dX)\, d\mathfrak{p} = 0$.

PROOF. Selecting a single column from the matrix equation (5.2) we have

$$\tau_i = X\alpha_i . \tag{5.5}$$

Differentiating:

$$d\tau_i = dX\alpha_i + X\, d\alpha_i . \tag{5.6}$$

As the differential form for $dS(\alpha_i)$ will be required, we introduce $p - 1$ orthonormal column vectors in p-space orthogonal to α_i. Let C_i be the $p \times p - 1$ matrix with them as columns. Then $dS(\alpha_i)$ is the alternating product of the elements in the vector $C_i'\, d\alpha_i$.

The differential form for $dS(\tau_i)$ requires $n - 1$ orthonormal vectors orthogonal to τ_i. The columns of the matrix XC_i provide $p - 1$ of them, since $C_i'X'\tau_i = C_i'X'X\alpha_i = C_i'\alpha_i = 0$. Choose the remaining $n - p$ orthonormal vectors orthogonal to the plane $\mathfrak{p}$ and let them be columns of an $n \times (n - p)$ matrix B, which is to be a function merely of p.

Premultiply (5.6) by the transpose of the partitioned matrix $[XC_i \,\vdots\, B]$:

$$\begin{bmatrix} C_i'\, X\, d\tau_i \\ B'\, d\tau_i \end{bmatrix} = \begin{bmatrix} C_i'\, X'\, dX\alpha_i + C_i'\, d\alpha_i \\ B'\, dX\alpha_i \end{bmatrix}. \tag{5.7}$$

Then, the alternating product of the differentials of the vector on the left is $dS(\tau_i)$ and hence the product of all of these for $i = 1, \cdots, p$ is the density on the left-hand side of (5.4).

The alternating product of all the differentials in the right-hand side of (5.7) for $i = 1, \cdots, p$ will give the density in the new co-ordinates. Let us deal with the vector differentials $B'\, dX\alpha_i$ first. These p vector differentials, corresponding to $i = 1, \cdots, p$, comprise the columns of the matrix $B'\, dXA$, of whose elements we therefore want the alternating product. The alternating product of the elements of a *row* of this matrix is $|A|$ times the product of the row of the elements of $B'\, dX$. There being $n - p$ rows in $B'\, dXA$, the alternating product of all its elements is then $|A|^{n-p} \prod_j \prod_i b_j'\, dx_i$. The differential form

$$\prod_j \prod_i b_j'\, dx_i$$

is the invariant measure, $d\mathfrak{p}$, on the Grassmann manifold, i.e. the uniform distribution of a p-plane in n-space (see [6]).

As the differential forms represent probability densities and must therefore be positive, we replace $| A |$ by its modulus $| A'A |^{1/2}$.

The product of the elements of the vector $C_i' \, d\alpha_i$ is $dS(\alpha_i)$. All the products involving an element of $C_i'X' \, dX\alpha_i$ we lump together in the symbol $* \, dX$. Collecting all factors we obtain (5.4). Q.E.D.

We conclude with a result that we shall need in section 7. From (5.1) and (5.2) we have $T'T = A'A$. Hence, if A has the distribution (5.3) then the moments of $A'A$ are the same as the moments of $T'T$ where T has the distribution $\prod dS(\tau_i)$.

PART B

6. Introduction. In this part of the paper we shall be concerned with the problem of calculating the conditional moments

$$\mu(t_1, t_2, \cdots, t_p) = E[(s_1^2)^{t_1}(s_2^2)^{t_2} \cdots (s_p^2)^{t_p}]$$

required for the expansion of the distribution of the canonical correlations r_i.

Recalling the results of sections 3 and 4, we saw that the expectations of monomials in the s_i^2 could be replaced by the expectations of monomials $m(A, B)$ in $\alpha_{ij}\beta_{ij}$ in view of the relation

$$s_i = \alpha_{1i}\beta_{1i}r_1 + \cdots + \alpha_{pi}\beta_{pi}r_p . \tag{6.1}$$

Furthermore, since $A = (\alpha_{ij})$ is distributed independently of $B = (\beta_{ij})$,

$$E[m(A, B)] = E[m(A)] \cdot E[m(B)]$$

where $m(A)$ and $m(B)$ are monomials in the elements of A and B respectively. Considering case (a) for the moment, we saw that the distributions of A and B were

$$k_p \, | A'A |^{(n-p)/2} \, dS(\alpha_1) \cdots dS(\alpha_p), \tag{6.2}$$

and

$$k_q \, | B'B |^{(n-q)/2} \, dS(\beta_1) \cdots dS(\beta_q)$$

respectively. Consequently, $E[m(B)]$ may be obtained from $E[m(A)]$ by simply replacing p with q.

In case (b), though the distribution of A is still given by (6.2), the distribution of B is given by (4.8), the invariant distribution on the Stiefel manifold of p-frames in q-space. We notice, however, that if we let $n \to \infty$ in case (a), then the set of random vectors $(\beta_1, \cdots, \beta_p)$ becomes a rigid p-frame, and this, of course, is exactly the situation in case (b). Hence the β-moments may be obtained from those in case (a) by letting $n \to \infty$. To summarise, then, it is sufficient to compute only the moments of the distribution (6.2).

To compute these moments by direct integration is obviously going to lead to involved algebra. However, by first averaging the monomials $m(A)$ over the orthogonal group we can considerably simplify the problem. Before proceeding further we shall briefly discuss this important process.

7. Average over the orthogonal group. The process $\mathfrak{M}$ of averaging over a group is a linear process whereby a function, defined on a space on which a group of transformations acts, is changed into a function invariant under the group. In particular, we consider the group $\mathfrak{H}$ of all orthogonal matrices H, and a matrix $A = (\alpha_{ij})$ which is transformed by the elements of $\mathfrak{H}$:

$$A \rightarrow HA \tag{7.1}$$

If $f(A)$ is a function of the elements of A, then

$$\mathfrak{M}f(A) = \int_{\mathfrak{H}} f(H^{-1}A)\, dV(H)$$

is a function invariant under the transformations (7.1). $V(H)$ is the invariant measure on the orthogonal group, normalised so that $V(\mathfrak{H}) = 1$. $\mathfrak{M}f$ is called the *average or mean value* of the function over the group. (This definition of "mean value" should not be confused with the usual statistical definition.) Since $\mathfrak{M}f$ is invariant under the orthogonal group, it must be expressible as a function of the basic invariants $\alpha_i'\alpha_j$ (see Weyl [9], pp. 52–6).

We wish to calculate the expectations of monomials $m(A)$ in the elements of A. Since the distribution (6.2) is invariant under the transformations (7.1), $E[m(A)] = E[m(H^{-1}A)]$, and hence it follows that

$$E[m(A)] = \int E[m(A)]\, dV(H) = \int E[m(H^{-1}A)]\, dV(H)$$

$$= E \int m(H^{-1}A)\, dV(H) = E[\mathfrak{M}m(A)].$$

In section 8 we shall show how to calculate $\mathfrak{M}m(A)$.

However, assuming for the moment that this has been done, we see that the problem has been reduced to the evaluation of the expectations of certain invariant functions $\phi(A'A)$, say. At this point it should be noted that the problem of the β-moments in case (b) has been completely solved. For, if we let $n \rightarrow \infty$, then $B'B = I$ with probability 1, and hence $E[m(B)] = \phi(I)$. $E[m(B)]$ can be then evaluated by the method given in James [7], pp. 374–5. However, since we require the β-moments for case (a), we may as well compute those for case (b) by letting $n \rightarrow \infty$ in the former moments, as indicated in section 6.

For the α-moments (and the β-moments for case (a)), we still have to evaluate the expectations of the invariant functions. In section 5 we have shown that the $\alpha_i'\alpha_j$ have the same distribution as quantities $\tau_i'\tau_j$ where $\tau_1, \cdots, \tau_p$ are independently uniformly distributed unit vectors in n-space. Finally, then, there remains the calculation of the moments of the $\tau_i'\tau_j$. This will be accomplished in section 9.

8. Calculation of $\mathfrak{M}m(A)$. In section 7 it was shown that

$$E[m(A)] = E[\mathfrak{M}m(A)] = E[\phi(A'A)].$$

In this section we shall show how to evaluate $\mathfrak{M}m(A)$.

Let

$$m(A) = \alpha_{i_1 j_1}^{k_1} \alpha_{i_2 j_2}^{k_2} \cdots \tag{8.1}$$

denote a monomial in the α_{ij}. Then if C is an arbitrary $p \times p$ matrix, the expansion of the function $\exp(\operatorname{tr} C'A)$ contains every monomial (8.1) multiplied by the same monomial $m(C)$ in the corresponding elements of C, and divided by $k_1 ! \, k_2 ! \cdots$. James [7] has shown that $\mathfrak{M} \exp(\operatorname{tr} C'A)$ can be expanded as a multiple power series in the elementary symmetric functions $z_1, z_2, \cdots, z_p$ of the latent roots of $C'CA'A$. Thus, if $\lambda_1, \cdots, \lambda_p$ are the latent roots of $C'CA'A$, then

$$z_1 = \sum \lambda_i = \operatorname{tr} C'CA'A,$$

$$z_2 = \sum_{i<j} \lambda_i \lambda_j = \text{sum of principal 2nd order minors of } C'CA'A, \text{ etc.},$$

and

$$\begin{aligned}
\mathfrak{M} \exp(\operatorname{tr} C'A) = {} & 1 + \frac{z_1}{2p} + \frac{z_1^2}{8p(p+2)} + \frac{z_2}{2p(p+2)(p-1)} \\
& + \frac{z_1^3}{8 \cdot 3!\, p(p+2)(p+4)} + \frac{z_1 z_2}{4p(p+2)(p+4)(p-1)} \\
& + \frac{z_3}{p(p+2)(p+4)(p-1)(p-2)} + \frac{z_1^4}{2^4 4!\, p(p+2)(p+4)(p+6)} \\
& + \frac{z_1^2 z_2}{16p(p+2)(p+4)(p+6)(p-1)} \\
& + \frac{z_2^2}{8p(p+2)(p+4)(p+6)(p-1)(p+1)} \\
& + \frac{(p+2) z_1 z_3}{2p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)} \\
& + \frac{(5p+6) z_4}{2p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)(p-3)} + \cdots .
\end{aligned} \tag{8.2}$$

Hence, $\mathfrak{M}m(A)$ can be found by equating the coefficients of $m(C)$ on both sides of (8.2).

If we write $A'A$ in the form

$$\begin{bmatrix}
1 & \alpha_1'\alpha_2 & \alpha_1'\alpha_3 & \cdots & \alpha_1'\alpha_p \\
\alpha_1'\alpha_2 & 1 & \alpha_2'\alpha_3 & \cdots & \cdot \\
\alpha_1'\alpha_3 & \alpha_2'\alpha_3 & 1 & & \cdot \\
\vdots & \vdots & \vdots & & \vdots \\
\alpha_1'\alpha_p & \alpha_2'\alpha_p & & & 1
\end{bmatrix} \tag{8.3}$$

we see that $\mathfrak{M}m(A)$ will be a linear combination of monomials in the invariants $\alpha_i'\alpha_j$. The expansion (8.2) is sufficient to compute all conditional moments up to order 4. If higher moments are required, further terms can be added to (8.2) by the use of recurrence relations derived from the differential equations given in James [7].

9. Calculation of the moments of the invariants. We are given that $\tau_1, \tau_2, \cdots, \tau_p$ are independently uniformly distributed column vectors in n-space, and we require the expectations of monomials in $\tau_i'\tau_j$. If a monomial in $\tau_i'\tau_j$ were expanded as a sum of monomials in the τ_{ij}, the expectations of each of these could be calculated and summed. However, the expansions would become very complicated. They can be avoided by the following method, which is an extension of an idea due to Bartlett [1] p. 13.

Let $e_1, e_2, \cdots, e_p$ be the unit vectors along the first p coordinate axes. Then the joint distribution of $\tau_1, \cdots, \tau_p$ is the same as that of $A_1e_1, A_2e_2, \cdots, A_pe_p$, where the A_i are random orthogonal matrices independently and invariantly distributed (see James [6]). Furthermore, the invariant functions will not be altered if they are calculated from the vectors $e_1, A_1'A_2e_2, \cdots, A_1'A_pe_p$. These vectors have the same distribution as $e_1, A_2e_2, \cdots, A_pe_p$ since $A_1'A_2, \cdots, A_1'A_p$ are still independently invariantly distributed. Again, if $A_2 = (a_{ij})$, say, the invariant functions will not be altered if we replace the vectors by

$$e_1, B_2'A_2e_2, \cdots, B_2'A_pe_p$$

where

$$B_2 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & a_{22}/b_{22} & * & \cdots & \\ 0 & a_{32}/b_{22} & * & \cdots & \\ \cdot & \cdot & & \cdots & \\ \cdot & \cdot & & \cdots & \\ \cdot & \cdot & & \cdots & \\ 0 & a_{n2}/b_{22} & & \cdots & \end{bmatrix},$$

$b_{22}^2 = 1 - a_{12}^2 = a_{22}^2 + \cdots + a_{n2}^2$, and the remaining elements are chosen so that B_2 is orthogonal. Clearly,

$$B_2'A_2e_2 = \begin{bmatrix} a_{12} \\ b_{22} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since the matrices $B_2'A_3, \cdots, B_2'A_p$ are still independently invariantly distributed we may replace the vectors by

$$e_1, B_2'A_2e_2, A_3e_3, \cdots, A_pe_p.$$

Proceeding in this way we see that we obtain the same values for the expectations of the invariants if we replace $\tau_1, \tau_2, \cdots, \tau_p$ by

$$(9.1)\qquad \begin{bmatrix}1\\0\\0\\ \cdot\\ \cdot\\ \cdot\\0\end{bmatrix}, \quad \begin{bmatrix}a_{12}\\b_{22}\\0\\ \cdot\\ \cdot\\ \cdot\\0\end{bmatrix}, \quad \begin{bmatrix}a_{13}\\a_{23}\\b_{33}\\0\\ \cdot\\ \cdot\\0\end{bmatrix}, \cdots .$$

(To avoid introducing further notation, we have denoted the third column of A_3 by the elements $a_{13}, a_{23}, \cdots, a_{n3}$, those of the fourth column of A_4 by $a_{14}, a_{24}, \cdots, a_{n4}$ etc. Then $b_{33}^2 = 1 - a_{13}^2 - a_{23}^2$, $b_{44}^2 = 1 - a_{14}^2 - a_{24}^2 - a_{34}^2$, etc.)

Example 1. As an example let us evaluate

$$E[(\alpha_1'\alpha_2)(\alpha_2'\alpha_3)(\alpha_3'\alpha_4)(\alpha_1'\alpha_4)] = E[(\tau_1'\tau_2)(\tau_2'\tau_3)(\tau_3'\tau_4)(\tau_1'\tau_4)].$$

Substituting from (9.1), this expectation is equal to

$$(9.2)\qquad E[a_{12}(a_{12}a_{13} + b_{22}a_{23})(a_{13}a_{14} + a_{23}a_{24} + b_{33}a_{34})a_{14}].$$

Now any monomial in the a_{ij}, b_{ii} containing an odd power has zero expectation since the distribution is unaltered if we replace a_{ij} by $-a_{ij}$ or b_{ii} by $-b_{ii}$. Hence, (9.2) reduces to $E(a_{12}^2a_{13}^2a_{14}^2)$. α_2, α_3 and α_4 are independently uniformly distributed unit vectors, and hence $E(a_{12}^2) = E(a_{13}^2) = E(a_{14}^2) = 1/n$. Therefore,

$$E[(\alpha_1'\alpha_2)(\alpha_2'\alpha_3)(\alpha_3'\alpha_4)(\alpha_1'\alpha_4)] = 1/n^3.$$

Example 2. $E(\Delta)$ where $\Delta = |A'A|$.
Put

$$C = \begin{bmatrix}1 & a_{12} & a_{13} & \cdots\\ 0 & b_{22} & a_{33} & \cdots\\ 0 & 0 & b_{33} & \cdots\\ \vdots & \vdots & \vdots & \end{bmatrix}.$$

Then $\Delta = |C'C| = |C|^2$, and

$$E(\Delta) = E(1\cdot b_{22}^2\cdot b_{33}^2\cdots b_{pp}^2)$$

$$= 1\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{n-p+1}{n},$$

since $E(b_{22}^2) = 1 - E(a_{12}^2) = 1 - 1/n$, etc.

10. Example of the calculation of the conditional moments. Following Bartlett, we introduce the notation[2]

$$(10.1)\qquad \begin{aligned} E(\alpha_{11}^2\alpha_{12}^2)E(\beta_{11}^2\beta_{12}^2) &= \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \\ E(\alpha_{11}^2\alpha_{22}^2)E(\beta_{11}^2\beta_{22}^2) &= \begin{pmatrix} 2 & \cdot \\ \cdot & 2 \end{pmatrix}, \quad \text{etc.} \end{aligned}$$

From equation (6.1) it is seen that the conditional moments can be expressed as linear combinations of "arrays" similar to those in (10.1). As we saw in section 6, it is sufficient to calculate the α-moments only.

To illustrate the method let us calculate the α-moment or "half-factor" corresponding to

$$\begin{pmatrix} 1 & 1 & \cdot \\ \cdot & 1 & 1 \\ 1 & \cdot & 1 \end{pmatrix},$$

i.e. $E(\alpha_{11}\alpha_{13}\alpha_{21}\alpha_{22}\alpha_{32}\alpha_{33})$.

The first step is to calculate $\mathfrak{M}m(A)$. Now,

$$C'CA'A = \begin{bmatrix} * & . \ c_{21}c_{22} + \cdots . & c_{11}c_{13} + \cdots . & \cdots \\ c_{21}c_{22} + \cdots . & * \quad . & c_{32}c_{33} + \cdots . & \cdots \\ c_{11}c_{13} + \cdots . & c_{32}c_{33} + \cdots . & * \quad . & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \cdot \begin{bmatrix} 1 & \alpha_1'\alpha_2 & \alpha_1'\alpha_3 & \cdots \\ \alpha_1'\alpha_2 & 1 & \alpha_2'\alpha_3 & \cdots \\ \alpha_1'\alpha_3 & \alpha_2'\alpha_3 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

All remaining terms in $C'C$ may be neglected as they will not contribute to $m(C)$ in the expansion (8.2).

$$\begin{aligned} \therefore z_1^3 &= 48(\alpha_1'\alpha_2)(\alpha_2'\alpha_3)(\alpha_1'\alpha_3)m(C) + \cdots, \\ z_1z_2 &= 4\{3(\alpha_1'\alpha_2)(\alpha_2'\alpha_3)(\alpha_1'\alpha_3) - (\alpha_1'\alpha_2)^2 - (\alpha_2'\alpha_3)^2 - (\alpha_1'\alpha_3)^2\}m(C) + \cdots, \\ z_3 &= 2m(C)\begin{vmatrix} 1 & \alpha_1'\alpha_2 & \alpha_1'\alpha_3 \\ \alpha_1'\alpha_2 & 1 & \alpha_2'\alpha_3 \\ \alpha_1'\alpha_3 & \alpha_2'\alpha_3 & 1 \end{vmatrix} + \cdots. \end{aligned}$$

Hence, after calculating the expectations of the invariant functions by the method of section 9, we have

$$\begin{aligned} E(z_1^3) &= \frac{48}{n^2}m(C) + \cdots \\ E(z_1z_2) &= -\frac{12(n-1)}{n^2}m(C) + \cdots \\ E(z_3) &= \frac{2(n-1)(n-2)}{n^2}m(C) + \cdots. \end{aligned}$$

[2] Actually our notation differs slightly from Barlett's. Whereas Bartlett worked in terms of rows vectors, we have worked in terms of column vectors, and hence Bartlett's α_{ij} corresponds to our α_{ji}.

Substituting in (8.2) and equating the coefficients of $m(C)$, we obtain after simplification

$$E(\alpha_{11}\alpha_{13}\alpha_{21}\alpha_{22}\alpha_{32}\alpha_{33}) = \frac{(n-p)(2n-p)}{n^2p(p+2)(p+4)(p-1)(p-2)},$$

which agrees with the value tabulated by Bartlett.

Any other α-moment can be calculated in a similar fashion. In particular, the moments tabulated by Bartlett were checked and the various terms contained in $\mu(2, 1, 1)$ and $\mu(1, 1, 1, 1)$ have been calculated and included in the appendix. Actually, only the α-moments have been tabulated. The complete value for case (a) may be obtained by multiplying the α-moment by a similar value with q replacing p. The complete value for case (b) is obtained by taking the previous value and letting $n \to \infty$ in the second half.

Incidentally, the α-moments may be checked by an independent method. For example, consider the monomial $\alpha_{11}^4\alpha_{12}^2$. If we multiply it by $\alpha_3'\alpha_3$, which is identically unity, then $E[\alpha_{11}^4\alpha_{12}^2(\alpha_3'\alpha_3)] = E[\alpha_{11}^4\alpha_{12}^2]$. But expanding the term on the left-hand side, we get

$$E[\alpha_{11}^4\alpha_{12}^2] = E[\alpha_{11}^4\alpha_{12}^2\alpha_{13}^2] + E[\alpha_{11}^4\alpha_{12}^2\alpha_{23}^2] + E[\alpha_{11}^4\alpha_{12}^2\alpha_{33}^2] + \cdots,$$

and therefore

$$\begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}4\\2\\2\end{pmatrix} + (p-1)\begin{pmatrix}4 & \cdot\\2 & \cdot\\ \cdot & 2\end{pmatrix}.$$

Similarly, by expanding $(\alpha_1'\alpha_2)^2(\alpha_1'\alpha_3)^2$, whose expectation $= 1/n^2$, we have

$$p\begin{pmatrix}4\\2\\2\end{pmatrix} + p(p-1)\begin{pmatrix}2 & 2\\2 & \cdot\\ \cdot & 2\end{pmatrix} + 4p(p-1)\begin{pmatrix}3 & 1\\1 & 1\\2 & \end{pmatrix} + 2p(p-1)\begin{pmatrix}2 & 2\\1 & 1\\1 & 1\end{pmatrix}$$

$$+ 2p(p-1)(p-2)\begin{pmatrix}2 & 1 & 1\\ \cdot & 1 & 1\\2 & \cdot & \cdot\end{pmatrix} + 4p(p-1)(p-2)\begin{pmatrix}2 & 1 & 1\\1 & 1 & \cdot\\1 & \cdot & 1\end{pmatrix}$$

$$+ p(p-1)(p-2)(p-3)\begin{pmatrix}1 & 1 & 1 & 1\\1 & 1 & \cdot & \cdot\\ \cdot & \cdot & 1 & 1\end{pmatrix} = 1/n^2.$$

REFERENCES

[1] M. S. Bartlett, "The general canonical correlation distribution," *Ann. Math. Stat.*, Vol. 18 (1947), pp. 1–17.

[2] R. A. Fisher, "The general sampling distribution of the multiple correlation coefficient," *Proc. Roy. Soc.*, Vol. A121 (1928), pp. 654–673.

[3] R. A. Fisher, "The sampling distribution of some statistics obtained from non-linear equations," *Ann. Eug.*, Vol. 9 (1939), pp. 238–249.

[4] H. Hotelling, "Relations between two sets of variates," *Biometrika*, Vol. 28 (1936) pp. 321–377.

[5] P. L. Hsu, "On the distribution of roots of certain determinantal equations," *Ann. Eug.*, Vol. 9 (1939), pp. 250–258.
[6] A. T. James, "Normal multivariate analysis and the orthogonal group," *Ann. Math Stat.*, Vol. 25 (1954), pp. 40–75.
[7] A. T. James, "A generating function for averages over the orthogonal group," *Proc. Roy. Soc.*, Vol. A229 (1955), pp. 367–375.
[8] S. N. Roy, "p-statistics or some generalisations in analysis of variance appropriate to multivariate problems," *Sankhya*, Vol. 4 (1939), pp. 381–396.
[9] H. Weyl, *The Classical Groups*, Princeton University Press, 2nd ed., 1946.

Appendix

$$\begin{pmatrix}4\\2\\2\end{pmatrix} = \frac{3(n+4)(n+6)}{n^2p(p+2)(p+4)(p+6)} \cdot \begin{pmatrix}4 & \cdot\\2 & \cdot\\\cdot & 2\end{pmatrix} = \frac{3(n+4)(np+5n-6)}{n^2p(p+2)(p+4)(p+6)(p-1)}.$$

$$\begin{pmatrix}2 & 2\\2 & \cdot\\2 & \cdot\end{pmatrix} = \frac{(n+4)(np+3n+2p-6)}{n^2p(p+2)(p+4)(p+6)(p-1)} \cdot \begin{pmatrix}3 & 1\\1 & 1\\2 & \cdot\end{pmatrix} = \frac{-3(n-p)(n+4)}{n^2p(p+2)(p+4)(p+6)(p-1)}.$$

$$\begin{pmatrix}4 & \cdot\\1 & 1\\1 & 1\end{pmatrix} = \frac{-3(n-p)(n+4)}{n^2(p+2)(p+4)(p+6)(p-1)}.$$

$$\begin{pmatrix}4 & \cdot\\\cdot & 2\\\cdot & 2\end{pmatrix} = \frac{3\{n^2(p+3)(p+5)+2n(p+1)(p+3)-8(2p+3)\}}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)}.$$

$$\begin{pmatrix}2 & 2\\2 & \cdot\\\cdot & 2\end{pmatrix} = \frac{n^2(p+3)^2+2n(p+1)(2p+3)+4(p^2-4p-6)}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)}.$$

$$\begin{pmatrix}3 & 1\\1 & 1\\\cdot & 2\end{pmatrix} = \frac{-3(n-p)(np+3n+2p)}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)} \cdot \begin{pmatrix}2 & 2\\1 & 1\\1 & 1\end{pmatrix} = \frac{-(n-p)(np-3n+8p+12)}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)}.$$

$$\begin{pmatrix}4 & \cdot & \cdot\\\cdot & 2 & \cdot\\\cdot & \cdot & 2\end{pmatrix} = \frac{3\{n^2(p^3+8p^2+13p-2)-2n(5p^2+27p+22)+8(5p+6)\}}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)}.$$

$$\begin{pmatrix}2 & 2 & \cdot\\2 & \cdot & \cdot\\\cdot & \cdot & 2\end{pmatrix} = \frac{n^2(p^3+6p^2+3p-6)+2n(p^3-19p-18)-4(3p^2-8p-12)}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)}.$$

$$\begin{pmatrix} \cdot & 2 & 2 \\ 2 & \cdot & \cdot \\ 2 & \cdot & \cdot \end{pmatrix} = \frac{n^2(p+3)(p+5) + 2n(p+1)(p+3) - 8(2p+3)}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)}.$$

$$\begin{pmatrix} 3 & 1 & \cdot \\ 1 & 1 & \cdot \\ \cdot & \cdot & 2 \end{pmatrix} = \frac{-3(n-p)(np^2 + 5np + 2n - 6p - 4)}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)}.$$

$$\begin{pmatrix} 4 & \cdot & \cdot \\ \cdot & 1 & 1 \\ \cdot & 1 & 1 \end{pmatrix} = \frac{-3(n-p)(np^2 + 7np + 14n - 8p - 16)}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)}.$$

$$\begin{pmatrix} 2 & \cdot & 2 \\ 1 & 1 & \cdot \\ 1 & 1 & \cdot \end{pmatrix} = \frac{-(n-p)(np^2 + 3np + 6n + 2p^2 - 6p - 12)}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)}.$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & \cdot \\ 2 & \cdot & \cdot \end{pmatrix} = \frac{-(n-p)(np + 3n + 2p)}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)}.$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & \cdot \\ 1 & \cdot & 1 \end{pmatrix} = \frac{-(n-p)(3p^2 - 4np + p - 6)}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)}.$$

$$\begin{pmatrix} 2 & 1 & 1 \\ \cdot & 1 & 1 \\ 2 & \cdot & \cdot \end{pmatrix} = \frac{-(n-p)(np^2 + 3np + 6n + 2p^2 - 6p - 12)}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)}.$$

$$\begin{pmatrix} 3 & 1 & \cdot \\ \cdot & 1 & 1 \\ 1 & \cdot & 1 \end{pmatrix} = \frac{3(n-p)(2np + 4n - p^2 - p - 2)}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)}.$$

$$\begin{pmatrix} 2 & 2 & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \end{pmatrix} = \frac{n^2(p^3 + 8p^2 + 13p - 2) - 2n(5p^2 + 27p + 22) + 8(5p + 6)}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)}$$

$$\begin{pmatrix} 2 & 2 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & 1 \end{pmatrix} = \frac{-(n-p)(np^2 + 7np + 14n - 8p - 16)}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)}.$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \end{pmatrix} = \frac{(n-p)(n-p-2)}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)}.$$

$$\begin{pmatrix} 2 & 1 & 1 & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 2 \end{pmatrix} = \frac{-(n-p)(np^2 + 5np + 2n - 6p - 4)}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)}.$$

$$\begin{pmatrix} 1 & 1 & \cdot & 2 \\ \cdot & 1 & 1 & \cdot \\ 1 & \cdot & 1 & \cdot \end{pmatrix} = \frac{(n-p)(2np + 4n - p^2 - p - 2)}{n^2p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)}.$$

$$\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = \frac{(n+2)(n+4)(n+6)}{n^3p(p+2)(p+4)(p+6)}, \quad \begin{pmatrix} 2 & \cdot \\ 2 & \cdot \\ 2 & \cdot \\ \cdot & 2 \end{pmatrix} = \frac{(n+2)(n+4)(np + 5n - 6)}{n^3p(p+2)(p+4)(p+6)(p-1)}.$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 2 & \cdot \\ 2 & \cdot \end{pmatrix} = \frac{-(n-p)(n+2)(n+4)}{n^3p(p+2)(p+4)(p+6)(p-1)}.$$

$$\begin{pmatrix} 2 & \cdot \\ 2 & \cdot \\ \cdot & 2 \\ \cdot & 2 \end{pmatrix} = \frac{(n+2)\{n^2(p+3)(p+5) + 2n(p+1)(p+3) - 8(2p+3)\}}{n^3(p+2)(p+4)(p+6)(p-1)(p+1)}.$$

$$\begin{pmatrix} 2 & \cdot \\ \cdot & 2 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{-(n-p)(n+2)(np+3n+2p)}{n^3p(p+2)(p+4)(p+6)(p-1)(p+1)} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{3(n-p)(n+2)(n-p-2)}{n^3p(p+2)(p+4)(p+6)(p-1)(p+1)}.$$

$$\begin{pmatrix} 2 & \cdot & \cdot \\ 2 & \cdot & \cdot \\ \cdot & 2 & \cdot \\ \cdot & \cdot & 2 \end{pmatrix} = \frac{(n+2)\{n^2(p^3+8p^2+13p-2)-2n(5p^2+27p+22)+8(5p+6)\}}{n^3p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)}.$$

$$\begin{pmatrix} 2 & \cdot & \cdot \\ 1 & 1 & \cdot \\ 1 & 1 & \cdot \\ \cdot & \cdot & 2 \end{pmatrix} = \frac{-(n-p)(n+2)(np^2+5np+2n-6p-4)}{n^3p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)}.$$

$$\begin{pmatrix} 1 & \cdot & 1 \\ 1 & 1 & \cdot \\ 1 & 1 & \cdot \\ 1 & \cdot & 1 \end{pmatrix} = \frac{(n+2)(n-p)(n-p-2)}{n^3p(p+2)(p+4)(p+6)(p-1)(p+1)}.$$

$$\begin{pmatrix} 1 & 1 & \cdot \\ \cdot & 1 & 1 \\ 1 & \cdot & 1 \\ 2 & \cdot & \cdot \end{pmatrix} = \frac{(n-p)(n+2)(2np+4n-p^2-p-2)}{n^3p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)}.$$

$$\begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & 1 & 1 \\ \cdot & 1 & 1 \\ 2 & \cdot & \cdot \end{pmatrix} = \frac{-(n-p)(n+2)(np^2+7np+14n-8p-16)}{n^3p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)}.$$

$$\begin{pmatrix} 2 & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & 2 \end{pmatrix} = \frac{n^3(p^4+7p^3+p^2-35p-6)-12n^2(p^3+6p^2+3p-6)+4n(19p^2+79p+54)-48(5p+6)}{n^3p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)(p-3)}.$$

$$\begin{pmatrix} 1 & 1 & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & 1 \end{pmatrix} = \frac{(n-p)\{n^2(p^2+5p+18) - n(p^3+5p^2+18p) + 4(2p^2+3p-6)\}}{n^3p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)(p-3)} .$$

$$\begin{pmatrix} 1 & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot \\ 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 2 \end{pmatrix} = \frac{(n-p)\{2n^2(p^2+4p) - n(p^3+4p^2+15p+18) + 6(p^2+p+2)\}}{n^3p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)(p-3)} .$$

$$\begin{pmatrix} 2 & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & 2 \end{pmatrix} = \frac{-(n-p)\{n^2(p^3+6p^2+3p-6) - 2n(5p^2+21p+18) + 8(5p+6)\}}{n^3p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)(p-3)} .$$

$$\begin{pmatrix} 1 & 1 & \cdot & \cdot \\ \cdot & 1 & 1 & \cdot \\ \cdot & \cdot & 1 & 1 \\ 1 & \cdot & \cdot & 1 \end{pmatrix} = \frac{-(n-p)\{n^2(5p+6) - n(5p^2+6p) + (p^3+p^2+2p)\}}{n^3p(p+2)(p+4)(p+6)(p-1)(p+1)(p-2)(p-3)} .$$