NOTES

BARTLETT DECOMPOSITION AND WISHART DISTRIBUTION

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1. Introduction. In a recent paper Wijsman [1] has presented a method of deriving the Bartlett decomposition and the Wishart distribution by using orthogonal matrices depending upon certain random vectors. The method used in the paper was simple compared to other methods in literature. The present paper gives a similar method, depending on orthogonalization of vectors, and is simpler and direct. This method explicitly gives the χ^2 variables and normal variables in the decomposition of the Wishart distribution in a straightforward way.

The device of writing the Wishart matrix as a product of a triangular matrix and its transpose has been used before; see [4] and [6]. In [5], this is done and it is shown that the elements of this matrix are independent chi and normal variables. However, the present method seems simpler in that it leads to the variables via transformations rather than via densities and Jacobians.

2. Notation and results. The same notation as in Wijsman's [1] paper is used. Let x_{it} $(i = 1, \cdots, k; t = 1, \cdots, n)$ be independent $N(0, 1)$ variables, forming the $k \times n$ matrix M^x_{kn}, the ith row of which is denoted by X'_i $(i = 1, \cdots, k)$, and let

$$A^x_{kn} = [a_{ij}] = M^x_{kn}(M^x_{kn})'. \tag{1}$$

Obviously $a_{ij} = X'_iX_j$ $(i, j = 1, \cdots, k)$. Orthogonalizing the vectors $X_1, \cdots, X_k$, we get vectors

$$Y_i = X_i - b'_{i1}Y_1 - b'_{i2}Y_2 - \cdots - b'_{i,i-1}Y_{i-1} \qquad i = 1, \cdots, k \tag{2}$$

where b'_{ir} $(i = 2, \cdots, k; r = 1, \cdots, i - 1)$ are so chosen that

$$Y'_iY_j = 0, \qquad i \neq j; \qquad i, j = 1, \cdots, k. \tag{3}$$

It is therefore easy to see that $b'_{ir} = Y'_rX_i/Y'_rY_r$. Let

$$b_{ir} = \frac{Y'_r X_i}{(Y'_r Y_r)^{1/2}} = (Y'_r Y_r)^{1/2}b'_{ir}. \qquad \begin{matrix} i = 2, \cdots, k \\ r = 1, \cdots, i-1. \end{matrix} \tag{4}$$

On account of (3), it can be seen that

$$X'_i X_i = Y'_i Y_i + \sum_{r=1}^{i-1} b^2_{ir} \tag{5}$$

and

$$X'_i X_j = \sum_{r=1}^{i-1} b_{ir} b_{jr} + b_{ji}(Y'_i Y_i)^{1/2} \quad \text{for} \quad j > i \qquad i, j = 1, \cdots, k.$$

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This can be written, in matrix form as

$$A_{in}^{x} = B_i B_i' \qquad i = 1, \cdots, k, \tag{6}$$

where B_i is the triangular matrix

$$\begin{bmatrix} (Y_1' Y_1)^{1/2} & 0 & 0 & \cdots & 0 \\ b_{21} & (Y_2' Y_2)^{1/2} & 0 & \cdots & 0 \\ b_{31} & b_{32} & (Y_3' Y_3)^{1/2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{i1} & b_{i2} & b_{i3} & \cdots & (Y_i' Y_i)^{1/2} \end{bmatrix}. \tag{7}$$

Taking determinants on both sides of (6), we have

$$|A_{in}^{x}| = (Y_1'Y_1)(Y_2'Y_2) \cdots (Y_i'Y_i) \qquad i = 1, \cdots, k \tag{8}$$

and

$$Y_i' Y_i = \frac{|A_{in}^{x}|}{|A_{i-1n}^{x}|}, \qquad i = 1, \cdots, k \tag{9}$$

where by convention $A_{0n}^{x} = 1$.

Consider the conditional distribution of b_{ir} $(r = 1, \cdots, i-1)$ when $X_1, \cdots, X_{i-1}$ are fixed. The b_{ir} are $i - 1$ orthogonal (on account of (3)) linear functions of the independent $N(0, 1)$ variables $x_{i1}, x_{i2}, \cdots, x_{in}$ and

$$E(b_{ir}) = 0$$

$$V(b_{ir}) = \frac{Y_r' Y_r}{Y_r' Y_r} = 1.$$

Therefore the b_{ir} $(r = 1, \cdots, i-1)$ are independent $N(0, 1)$ and by Fisher's lemma (See [2]),

$$Y_i' Y_i = X_i' X_i - \sum_{r=1}^{i-1} b_{ir}^2$$

is a χ^2 variable with $n - (i - 1)$ degrees of freedom. This is independent of the normal distribution of b_{ir}. Since the fixed variables $X_1, \cdots, X_{i-1}$ do not appear in the distribution of $Y_i'Y_i$ and b_{ir} $(r = 1, \cdots, i-1)$, and since the result is true for every i, we get the following result:

$Y_i'Y_i$ $(i = 1, \cdots k)$ *are independent* χ^2 *variables with* $n - (i - 1)$ *degrees of freedom, and* b_{ir} $(i = 2, \cdots, k;\ r = 1, \cdots, i-1)$ *are independent* $N(0, 1)$ *variables, and all these variables are independently distributed.*

The density factor in the joint distribution of the $\frac{1}{2}k(k+1)$ variables $Y_i'Y_i$ and b_{ir} is therefore

$$C_{kn} \exp\left[-\frac{1}{2}\sum_{i=2}^{k}\sum_{r=1}^{i-1} b_{ir}^2\right] \prod_{i=1}^{k} \{(Y_i' Y_i)^{(n-i-1)/2} \exp[-\frac{1}{2}Y_i' Y_i]\} \tag{10}$$

where

$$C_{kn} = 2^{nk/2}\pi^{k(k-1)/4}\prod_{i=1}^{k}\Gamma\left(\frac{n-i+1}{2}\right). \tag{11}$$

This density factor, can be written, on account of (5) and (8), as

$$C_{kn}\exp\left[-\tfrac{1}{2}\operatorname{tr} A_{kn}^{x}\right]\,|A_{kn}^{x}|^{(n-k-1)/2}\prod_{i=1}^{k}(Y_i'Y_i)^{(k-i)/2}. \tag{12}$$

If we transform from the variables $Y_i'Y_i$ ($i = 1, \cdots, k$) and b_{ir} ($i = 2, \cdots, k$; $r = 1, \cdots, i-1$) to the $k(k+1)/2$ distinct elements a_{ij} of $A_{k,n}^{x}$, by using the transformation as given by (6), the Jacobian of transformation is (see [3] Theorem 4.1)

$$\prod_{i=1}^{k}(Y_i'Y_i)^{(i-k)/2}$$

Consequently the density factor in the distribution of the elements of $A_{k,n}^{x}$ is

$$C_{kn}|A_{kn}^{x}|^{(n-k-1)/2}\exp\left[-\tfrac{1}{2}\operatorname{tr} A_{kn}^{x}\right]. \tag{13}$$

REMARKS. The $\frac{1}{2}k(k-1)$ normal variables, k χ^2 variables and the triangular matrix in the Bartlett decomposition, as mentioned by Wijsman [1], are directly given by (6).

Further, from (8), (9) and the distribution of $Y_i'Y_i$ as derived above, it follows immediately that

(i) $$\frac{|A_{in}^{x}|}{|A_{i-1n}^{x}|} \quad \text{are independent } \chi^2 \text{ variables}$$

with $n-(i-1)$ degrees of freedom ($i = 1, \cdots, k$). From this, the results of Lemma 2 and 3 in Wijsman's [1] paper follow very easily.

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