ON A GENERALISATION OF THE KRONECKER PRODUCT DESIGNS[1]

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1. Summary and introduction. The use of incomplete block designs for estimating and judging the significance of the difference of treatment effects is now standard. Fisher and Yates [2] have provided a complete table of balanced incomplete block designs (BIB) for the value of $r, k \leqq 10$. In this paper a method of constructing a special class of partially balanced incomplete block designs (PBIB) from the known BIB designs is given. Vartak [6] and Sillito [4] have used the Kronecker product of matrices to construct statistical designs. Their method and the method given in this paper differ only in the fact that in their method two distinct elements of a matrix are replaced by two distinct matrices, while in the method considered in this paper, two or more distinct elements of a matrix are replaced by different matrices. All the PBIB designs considered in this paper are with three associate classes, and the rectangular association scheme for $v = pq$ treatments are written in p rows and q columns so that treatments in the same row are the first associates, treatments in the same column are the second associates and the rest are the third associates.

2. Notation and some definitions. Throughout this paper, parameters of a BIB design will be denoted by v^*, b^*, r^*, k^*, λ^*. The parameters of a PBIB will be denoted by $v, b, r, k, \lambda_1, \cdots, \lambda_m, n_1, \cdots, n_m, P_{ij}^k$, as defined by Bose and Nair [1] and later generalised by Nair and Rao [3].

The following three designs, with the given incidence matrices and parameters will be included in the BIB's in this paper.

(a) A null design with the incidence matrix, $0(v^*, b^*)$ [the $v^* \times b^*$ matrix with all the elements equal to zero]. Parameters: $v^*, b^*, r^* = k^* = \lambda^* = 0$.

(*b*) A randomised block design with the incidence matrix $E(v^*, b^*)$ [the $v^* \times b^*$ matrix with all the elements equal to unity]. Parameters: $v^*, b^*, r^* = \lambda^* = b^*, k^* = v^*$.

(c) A design with the incidence matrix $I(v^*)$ [the $v^* \times v^*$ identity matrix]. Parameters: $v^* = b^*, r^* = k^* = 1, \lambda^* = 0$.

Definition 2.1. Two BIB designs with incidence matrices N_1 and N_2 and parameters $v_1^*, b_1^*, r_1^*, k_1^*, \lambda_1^*$ and $v_2^*, b_2^*, r_2^*, k_2^*, \lambda_2^*$, respectively will be called associable designs, if and only if $b_1^* = b_2^*, v_1^* = v_2^*$ and the design $N_{12} = [N_1/N_2]$ formed by joining the corresponding blocks of the two designs has the following properties:

(i) The ith treatment of N_1 occurs with the ith treatment of N_2 exactly μ_{12} times for all $i = 1, 2, \cdots, v_1^*$.

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(ii) The ith treatment of N_1 occurs with the jth treatment of N_2 exactly η_{12} times for all $i \neq j; i, j = 1, 2, \cdots, v_1^*$.

The following results are direct consequences of the above definition:

A BIB design (N_1) with parameters $v^*, b^*, r^*, k^*, \lambda^*$ is associable with

(i) a null design $\{N_2 = 0(v^*, b^*)\}$ and $\mu_{12} = 0 = \eta_{12}$

(ii) a randomised block design $\{N_3 = E(v^*, b^*)\}$ and $\mu_{13} = r^* = \eta_{13}$,

(iii) its complementary design $\{N_4 = E(v^*, b^*) - N_1\}$ and $\mu_{14} = 0$ $\eta_{14} = r^* - \lambda^*$, and

(iv) itself (N_1) and $\mu_{11} = r^*$ $\eta_{11} = \lambda^*$.

Definition 2.2. Let $A = a_{ij}, i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$, be a matrix whose elements take the S values $1, 2, \cdots, S$. We shall call matrix A a balanced matrix in S integers if and only if the following conditions are satisfied:

(i) The number of times the integer p occurs in a column is the same for all the columns and is equal to α_p, say.

(ii) The number of times the integer p occurs in a row is the same for all the rows and is equal to β_p, say.

(iii) The number of times the combination $\binom{p}{q}$ or $\binom{q}{p}$ occurs in a pair of rows is the same for all the pairs of rows and is equal to γ_{pq}, say. ($\binom{p}{q}$ and $\binom{q}{p}$ will be considered as identical combinations).

From Definition 2.2 we can easily prove the following lemmas:

Lemma 2.1. *If, in a balanced matrix A in S integers, some of the integers, say h, are replaced by 1 and the remaining $(S - h)$ integers by 0, then the matrix A will be converted into an incidence matrix of a BIB design.*

Lemma 2.2. *If A is a matrix whose elements take the S values $1, 2, \cdots, S$ and if any one of the integers is replaced by 1 and the remaining $S - 1$ by 0, or any two of the integers are replaced by 1 and remaining $S - 2$ are replaced by 0 and if, in all these $S + \binom{S}{2}$ ways, it is found that the matrix A is converted into an incidence matrix of a BIB design, then the matrix A must be a balanced matrix in S integers.*

Corollary to Lemmas 2.1 and 2.2. *The necessary and sufficient condition for a matrix A to be balanced is that there exist S BIB designs $N_1, N_2, \cdots, N_S$ such that $\sum N_i = E(m, n)$, $(N_i + N_j)$ is also an incidence matrix of a BIB design for all $i \neq j$ and $A = \sum_{i=1}^{S} iN_i$.*

Lemma 2.3. (Generalisation of Lemma 2.1). *If there is a balanced matrix A in S_2 integers, and these S_2 integers are divided in S_1 $(S_1 < S_2)$ groups such that each group contains at least one integer and all the elements of a group are replaced by an identical integer, then the matrix A will become a balanced matrix in S_1 integers.*

It is obvious that the incidence matrix of a BIB is a balanced matrix in two integers. Further, from the corollary to Lemmas 2.1 and 2.2 for the existence of a balanced matrix in S integers with parameters as given in Definition 2.2, it is necessary that several BIB's with $v^* = m$, $b^* = n$ and for all the values of $k^* = \alpha_i$, or $\alpha_i + \alpha_j$, $j, i = 1, 2, \cdots, S$ exist separately. The existing BIB's satisfying the above conditions and v, $b \leqq 15$ for $S \geqq 3$ are as follows:

	$v^* = m$	$b^* = n$	α's
(1)	3	3, 6, 9, 12 or 15.	$\alpha_1 = \alpha_2 = \alpha_3 = 1$
(2)	4	12	any values such that $\sum \alpha_i = 4$
(3)	5	10	any values such that $\sum \alpha_i = 5$
(4)	6	15	$\alpha_1 = \alpha_2 = \alpha_3 = 2$
(5)	7	7, 14	$\alpha_1 = \alpha_2 = 3, \alpha_3 = 1$
(6)	9	12	$\alpha_1 = \alpha_2 = \alpha_3 = 3$
(7)	11	11	$\alpha_1 = \alpha_2 = 5, \alpha_3 = 1$
(8)	15	15	$\alpha_1 = \alpha_2 = 7, \alpha_3 = 1$

The balanced matrices with the above parameters are constructible and are given in the appendix. Any other balanced matrix with $m, n \leqq 15$ and $S \geqq 3$ does not exist because the corresponding BIB's do not exist.

3. Construction of PBIB's. Using associable BIB's and a balanced matrix we can construct PBIB designs with three associate classes as given by the following theorem.

THEOREM 3.1. *Let there be S BIB designs with incidence matrices $N_1, N_2, \cdots, N_S$. Let the parameters of the ith design be v^*, b^*, r_i^*, k_i^*, and λ_i^*. Let the pth design be associable with the qth design with the parameters μ_{pq} and η_{pq} for all $p, q = 1, 2, \cdots, S$. Now let there be an $m \times n$ balanced matrix A in S integers $1, 2, \cdots, S$ with parameters α_p, β_p, γ_{pq} as defined in 2.2. Now if we replace the integer p in matrix A by the matrix N_p ($p = 1, 2, \cdots, S$), then matrix A will be converted into an incidence matrix of a PBIB with the following parameters:*

$$v = m \cdot v^*,$$

$$b = n \cdot b^*,$$

$$r = \sum_{p=1}^{S} \beta_p \gamma_p^*,$$

$$k = \sum_{p=1}^{S} \alpha_p k_p^*,$$

$$n_1 = v^* - 1, \qquad n_2 = m - 1, \qquad n_3 = n_1 n_2,$$

$$\lambda_1 = \sum_{p=1}^{S} \beta_p \lambda_p^*, \qquad \lambda_2 = \sum_{p \geqq q=1}^{S} \gamma_{ph} \mu_{pq}, \qquad \lambda_3 = \sum_{p \geqq q=1}^{S} \gamma_{pq} \eta_{pq};$$

$$P_{ij}^1 = \begin{bmatrix} n_1 - 1 & 0 & 0 \\ & 0 & n_2 \\ & & (n_1 - 1)n_2 \end{bmatrix}, \qquad P_{ij}^2 = \begin{bmatrix} 0 & 0 & n_1 \\ & n_2 - 1 & 0 \\ & & (n_2 - 1)n_1 \end{bmatrix},$$

$$P_{ij}^3 = \begin{bmatrix} 0 & 1 & n_1 - 1 \\ & 0 & n_2 - 1 \\ & & (n_1 - 1)(n_2 - 1) \end{bmatrix}.$$

PROOF: Expressions for v, b, r, k are quite obvious and need no proof. Others can be proved as follows:

From the method of construction, it is obvious that we have m groups of v^* treatments each corresponding to a column of matrix A. The treatments belonging to the same group will be called the first associates. The ith treatment of one group and the ith treatment of another group will be called the second associates. The ith treatment of one group and the jth treatment $(i \neq j)$ of another group will be called the third associates. Expressions for n_i's and P^k_{ij}'s immediately follow from this association scheme.

Now in any row of the matrix A the integer p occurs β_p times, therefore N_p also occurs β_p times. In N_p any pair of treatments occurs together λ_p^* times. Hence any pair of treatments belonging to the same group occurs exactly $\sum_{p=1}^{s} \beta_p\lambda_p^*$ times. Therefore

$$\lambda_1 = \sum_{p=i}^{s} \beta_p \lambda_p^*.$$

Similarly, for any pair of rows of the matrix A the combination $\binom{p}{q}$ or $\binom{q}{p}$ occurs exactly γ_{pq} times. Hence, the ith treatment of one group and the ith treatment of another group together $\sum \gamma_{pq}\mu_{pq}$ times, and the ith treatment of one group and the jth treatment $(i \neq j)$ of another group occur together exactly $\sum \gamma_{pq}\, \eta_{pq}$ times. Hence

$$\lambda_2 = \sum_{p \geq q=1}^{s} \gamma_{pq}\, \mu_{pq}$$

and

$$\lambda_3 = \sum_{p \geq q=1}^{s} \gamma_{pq}\, \eta_{pq}.$$

This completes the proof of Theorem 3.1

As an illustration let us take the matrix N_1 as the incidence matrix of a BIB with the parameters v^*, b^*, r^*, k^*, λ^*, N_2 as the incidence matrix of its complementary design $(N_2 = E(v^*, b^*) - N_1)$, $N_3 = 0(v^*, b^*)$ and

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Then

$$\mu_{12} = 0, \qquad \mu_{13} = 0, \qquad \mu_{23} = 0,$$
$$\eta_{12} = r^* - \lambda^*, \qquad \eta_{13} = 0, \qquad \eta_{23} = 0.$$

Hence on substituting N_p in place of p in matrix A, $(p = 1, 2, 3)$, we obtain the incidence matrix of a PBIB with the parameters

$$v = 3v^*, \qquad b = 3b^*, \qquad r = b^*, \qquad k = v^*,$$
$$\lambda_1 = b^* - 2r^* + 2\lambda^*, \qquad \lambda_2 = 0, \qquad \lambda_3 = r^* - \lambda^*, \qquad \text{etc.}$$

4. Some particular cases. Let a balanced matrix A in the two integers 0, 1, be the incidence matrix of a BIB with the parameters v_1^*, b_1^*, r_1^*, k_1^*, λ_1^*, and let N_1 be the incidence matrix of a BIB with the parameters v_2^*, b_2^*, r_2^*, k_2^*, λ_2^*, then by taking N_0 as $0(v_2^*, b_2^*)$, $E(v_2^*, b_2^*)$ and $E(v_2^*, b_2^*) - N_1$ respectively we have the following three cases:

i) $N_0 = 0(v_2^*, b_2^*)$, we obtain a PBIB with the parameters

$$v = v_1^* v_2^*, \qquad b = b_1^* b_2^*, \qquad r = r_1^* r_2^*, \qquad k = k_1^* k_2^*,$$

$$\lambda_1 = r_1^* \lambda_2^*, \qquad \lambda_2 = r_2^* \lambda_1^*, \qquad \lambda_3 = \lambda_1^* \lambda_2^*.$$

ii) $N_0 = E(v_2^*, b_2^*)$, we obtain a PBIB with the parameters

$$v = v_1^* v_2^*, \qquad b = b_1^* b_2^*, \qquad r = r_2^* r_2^* + (b_1^* - r_1^*) b_2^*,$$

$$k = k_1^* k_2^* + (v_1^* - k_1^*) v_2^*,$$

$$\lambda_1 = r_1^* \lambda_2^* + (b_1^* - r_1^*) b_2^*,$$

$$\lambda_2 = (b_1^* - 2r_1^* + \lambda_1^*) b_2^* + 2(r_1^* - \lambda_1^*) r_2^* + \lambda_1^* r_2^*,$$

$$\lambda_3 = (b_1^* - 2r_1^* + \lambda_1^*) b_2^* + 2(r_1^* - \lambda_1^*) r_2^* + \lambda_1^* \lambda_2^*.$$

iii) $N_0 = E(v_2^*, b_2^*) - N_1$, we obtain a PBIB with the parameters

$$v = v_1^* v_2^*, \qquad b = b_1^* b_2^*,$$

$$r = r_1^* r_2^* + (b_1^* - r_1^*)(b_2^* - r_2^*),$$

$$k = k_1^* k_2^* + (v_1^* - k_1^*)(v_2^* - k_2^*),$$

$$\lambda_1 = r_1^* \lambda_2^* + (b_1^* - r_1^*)(b_2^* - 2r_2^* + \lambda_2^*),$$

$$\lambda_2 = r_2^* \lambda_1^* + (b_2^* - r_2^*)(b_1^* - 2r_1^* + \lambda_1^*),$$

$$\lambda_3 = (b_2^* - 2r_2^* + \lambda_2^*)(b_1^* - 2r_1^* + \lambda_1^*) + 2(r_2^* - \lambda_2^*)(r_1^* - \lambda_1^*) + \lambda_1^* \lambda_2^*.$$

It will be noticed that case (i) above is exactly identical with the Kronecker Product of BIB's as given by M. N. Vartak [6]; the same method also gives some of the designs by D. A. Sprott [5].

The method employed by G. P. Sillito [4] to construct BIB is identical with the case (iii) above with the conditions $A = N_1$ and $b_1^* = 4(r_1^* - \lambda_1^*)$. Also, of the two PBIB's with $v = b = pq$ given by R. C. Bose [1], one is a particular case of (iii) above with $A = I(p)$ and $N_1 = E(q, q) - I(q)$ and the other is identical with (ii) above on taking $A = E(p, p) - I(p)$ and $N_1 = I(q)$.

Furthermore it should be noted that if there are two designs of this type with the same number of treatments, the same association scheme for three associate classes, and the same block size, we can construct a new design of this type, just by adjoining the two designs.

Lastly if in a design $\lambda_1 = \lambda_3$ or $\lambda_2 = \lambda_3$, the design is reduced to a group divisible design, and if $\lambda_1 = \lambda_2 = \lambda_3$, the design is reduced to a BIB.

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APPENDIX

The following is the complete list of the balanced matrices with $m, n \leqq 15$ and $S \geqq 3$.

Balanced matrices with $m = 3$, $n = 6$, 9 or 12 and $m = 7$, $n = 14$ can be obtained by replicating the solutions (1) and (5) Balanced matrices with $m = 4$, $S = 3$ and $m = 5$, $S = 3, 4$ can be obtained from the solutions (2) and (3) by the help of Lemma 2.4.

(1) $m = n = S = 3$.

1	2	3
2	3	1
3	1	2

(2) $m = S = 4$, $n = 12$.

1	2	3	4	1	2	3	4	1	2	3	4
2	1	4	3	3	4	1	2	4	3	2	1
3	4	1	2	4	3	2	1	2	1	4	3
4	3	2	1	2	1	4	3	3	4	2	1

(3) $m = S = 5$, $n = 10$.

1	2	3	4	5	1	2	3	4	5
2	3	4	5	1	4	5	1	2	3
3	4	5	1	2	2	3	4	5	1
4	5	1	2	3	5	1	2	3	4
5	1	2	3	4	3	4	5	1	2

(4) $m = 6$, $n = 15$, $S = 3$.

1	1	1	1	1	2	2	2	2	2	3	3	3	3	3
1	2	2	2	2	2	3	3	3	3	1	1	1	1	3
2	1	2	3	3	3	1	2	1	3	1	2	2	3	1
2	3	3	2	1	3	2	1	3	1	2	1	3	2	1
3	2	3	1	3	1	1	3	2	1	2	3	2	1	2
3	3	1	3	2	1	3	1	1	2	3	2	1	2	2

(5) $m = n = 7$, $S = 3$.

1 3 2 2 1 2 1

First row;
matrix is cyclic.

(6) $m = 9$, $n = 12$, $S = 3$.

1	2	3	1	2	3	1	2	3	1	2	3
1	2	3	2	3	1	2	3	1	3	1	2
1	2	3	3	1	2	3	1	2	2	3	1
2	3	1	1	2	3	2	3	1	2	3	1
2	3	1	2	3	1	3	1	2	1	2	3
2	3	1	3	1	2	1	2	3	3	1	2
3	1	2	1	2	3	3	1	2	3	1	2
3	1	2	2	3	1	1	2	3	2	3	1
3	1	2	3	1	2	2	3	1	1	2	3

(7) $m = n = 11, \quad S = 3.$

3 1 2 1 1 1 2 2 2 1 2

First row;
matrix is cyclic

(8) $m = n = 15, \quad S = 3.$

3	1 1 1 1 1 1 1	2 2 2 2 2 2 2
2	3 2 2 1 2 1 1	2 1 2 2 1 1 1
2		
2		
2		
2		
2		
2		
1	1 2 2 2 1 1 2	3 1 1 2 1 2 2
1		
1		
1		
1		
1		
1		

Each 7 $\times$ 7 block is cyclic.

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