ON LINEAR ASSOCIATIVE ALGEBRAS CORRESPONDING TO ASSOCIATION SCHEMES OF PARTIALLY BALANCED DESIGNS

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1. Introduction. Given v objects $1, 2, \cdots, v$, a relation satisfying the following conditions is said to be an association scheme with m classes:

(a) Any two objects are either 1st, or 2nd, $\cdots$, or mth associates, the relation of association being symmetrical, i.e., if the object α is the ith associate of the object β, then β is the ith associate of α.

(b) Each object α has n_i ith associates, the number n_i being independent of α.

(c) If any two objects α and β are ith associates, then the number of objects which are jth associates of α, and kth associates of β, is p_{jk}^i and is independent of the pair of ith associates α and β.

The numbers v, n_i $(i = 1, 2, \cdots, m)$ and p_{jk}^i $(i, j, k = 1, 2, \cdots, m)$ are the parameters of the association scheme.

If we have an association scheme with m classes and given parameters, then we get a partially balanced design with r replications and b blocks if we can arrange the v objects into b sets (each set corresponding to a block) such that

(i) each set contains k objects (all different);
(ii) each object is contained in r sets;
(iii) if two objects α and β are ith associates, then they occur together in λ_i sets, the number λ_i being independent of the particular pair of ith associates α and β.

Partially balanced designs were introduced in experimental studies by Bose and Nair [5], and have recently come into fairly general practical use. The concept of the association scheme, though inherent in Bose and Nair's definition, was explicitly introduced by Bose and Shimamoto [6], as an aid to the classification and analysis of partially balanced designs.

2. Association schemes as concordant graphs. An association scheme with v objects and m classes may be visualized as follows:

Let the objects be points. Suppose we have m colors $C_1, C_2, \cdots, C_m$. If two objects are ith associates we connect them by a segment of the ith color. The points together with the segments of the ith color form a linear graph which will

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be regular of degree n_i as a result of property (b). We may say that the n graphs together are *concordant*[2] when properties (a) and (c) are also satisfied, the meanings of these being as follows:

a) Every pair of points is connected by a single segment of one of the m colors. The graphs are non-oriented.

b) If any two points α and β are connected by a segment of the ith color, then the number of points which are connected to α by a segment of color C_j and to β by a segment of color C_k, is p^i_{jk} and is independent of the particular pair of points chosen.

Equivalently p^i_{jk} is the number of 2-chains directed from α to β and consisting of segments of colors C_j and C_k in that order. Clearly the p^i_{jk} are closely related to the number of triangles in the graph formed of segments of colors C_i, C_j, C_k. Properties (a), (b) and (c) are just enough to specify the number of segments of each color on each point, and the number of triangles of each combination of colors on each segment. The total number of segments, the total number of 2-chains, and the total number of triangles in the graph are then readily determined. Methods based on the incidence matrices of the graphs [16] can be used with (3.6) to enumerate certain chains of more than two segments. The arrangement in these graphs of all configurations involving two points or three points shows a striking regularity which does not extend to configurations having more than three points. It can be shown by examples that the points of the graph of color C_i may not all lie on the same number of complete 4-points, and that two association schemes with the same parameter values may give graphs differing in the total number of complete 4-points. This shows that the structure of concordant graphs is not determined completely by properties (a) to (c).

3. Association matrices. We define[3]

$$B_i = (b^\beta_{\alpha i}) = \begin{bmatrix} b^1_{1i} & b^2_{1i} & \cdots & b^v_{1i} \\ \cdots & \cdots & \cdots & \cdots \\ b^1_{vi} & b^2_{vi} & \cdots & b^v_{vi} \end{bmatrix},$$

where

$b^\beta_{\alpha i} = 1,$ if the objects α and β are ith associates (or connected by a segment of the ith graph);

$= 0,$ otherwise.

B_i is a symmetric matrix, in which each row total and each column total is n_i.

Let each object be the zeroth associate of itself and of no other treatment.

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[2] Not to be confused with *chromatic* graphs, in which points, not segments, are colored. For a general discussion of linear graphs, see [11].

[3] The convention will be adopted here of using a superscript as the column index of a matrix, the first subscript as the row index, and the second subscript as the index of the matrix itself. This choice is dictated by the notation already established for the parameters p^i_{jk}.

Then

$$\begin{aligned} B_0 &= I_v, && \text{the } v \times v \text{ identity matrix,} \\ n_0 &= 1, && \\ p^0_{ij} &= n_i, && \text{if } i = j, \\ &= 0, && \text{otherwise,} \\ p^i_{0k} &= 1, && \text{if } i = k, \\ &= 0, && \text{otherwise,} \\ \lambda_0 &= r, && \text{for designs.} \end{aligned}$$

The following identities are known [5] and can be proved easily by combinatorial methods. Proofs based on the matrices B_i will be given in Section 5.

$$(3.1) \qquad \begin{gathered} \sum_{i=0}^{m} n_i = v, \\ \sum_{j=0}^{m} p^i_{jk} = n_k, \\ p^i_{jk} = p^i_{kj}, \\ n_i p^i_{jk} = n_j p^j_{ik} = n_k p^k_{ij}. \end{gathered}$$

Further the following two identities hold for designs:

$$(3.2) \qquad \begin{gathered} bk = vr, \\ \sum_{i=0}^{m} n_i \lambda_i = rk. \end{gathered}$$

Among the numbers

$$b^\beta_{\alpha 0}, \qquad b^\beta_{\alpha 1}, \cdots, \qquad b^\beta_{\alpha m}$$

only one is unity, i.e., $b^\beta_{\alpha i}$ if α and β are ith associates. Hence

$$(3.3) \qquad B_0 + B_1 + \cdots + B_m = J_v,$$

where J_v is the $v \times v$ matrix each of whose elements is unity.

It also follows that the linear form

$$(3.4) \qquad c_0 B_0 + c_1 B_1 + \cdots + c_m B_m$$

is equal to the zero matrix if and only if

$$c_0 = c_1 = \cdots = c_m = 0;$$

hence the linear functions of $B_0, B_1, \cdots, B_m$ form a vector space with basis $B_0, B_1, \cdots, B_m$.

LEMMA 3.1.

$$\sum_{\gamma=1}^{v} b_{\alpha j}^{\gamma} b_{\gamma k}^{\beta} = p_{jk}^{0} b_{\alpha 0}^{\beta} + \cdots + p_{jk}^{i} b_{\alpha i}^{\beta} + \cdots + p_{jk}^{m} b_{\alpha m}^{\beta}. \tag{3.5}$$

The objects α and β are zeroth associates if $\alpha = \beta$; otherwise they are either 1st, 2nd, $\cdots$ or mth associates from condition (a) of Section 1. Suppose they are ith associates. Both terms of the product $b_{\alpha j}^{\gamma} b_{\gamma k}^{\beta}$ are unity if and only if γ is the jth associate of α as well as kth associate of β. Hence from condition (c) Section 1, the left-hand side of (3.5) is p_{jk}^{i}. Again since α and β are ith associates $b_{\alpha l}^{\beta}$ is unity if $l = i$ and is zero otherwise. Hence the right-hand side of (3.5) is also equal to p_{jk}^{i}. This proves the Lemma.

We now note that the left-hand side of (3.5) is the element in the αth row and βth column of the product $B_j B_k$, and $b_{\alpha l}^{\beta}$ is the element in the αth row and βth column of B_l ($l = 0, 1, \cdots, m$). Thus[4]

$$B_j B_k = p_{jk}^{0} B_0 + p_{jk}^{1} B_1 + \cdots + p_{jk}^{m} B_m. \tag{3.6}$$

The product of two matrices of the form (3.4), where the c_i are scalars, may be expressed as a linear combination of terms of the form $B_j B_k$ and will reduce to the form (3.4). The set of matrices of this form is therefore closed under multiplication. It is clear that it forms an Abelian group under addition. Thus the linear functions of $B_0, B_1, \cdots, B_m$ form a ring with unit element, which will be a linear associative algebra if the coefficients c_i range over a field. Multiplication is also commutative. This statement and the equivalent statement $p_{jk}^{i} = p_{kj}^{i}$ will be shown in Section 5 to follow from (3.6) and the symmetry of B_i.

Linear associative algebras have of course been extensively studied and are treated, for example, in [13]. The properties of most importance in the present study are easily established, and brief proofs will now be given for the sake of completeness.

We first find the consequences of the associative law of matrix multiplication:

$$\begin{aligned} B_i(B_j B_k) &= B_i \sum_u p_{jk}^{u} B_u \\ &= \sum_u p_{jk}^{u} B_i B_u \\ &= \sum_{u,t} p_{jk}^{u} p_{iu}^{t} B_t. \end{aligned}$$

Also

$$\begin{aligned} (B_i B_j) B_k &= \left(\sum_u p_{ij}^{u} B_u\right) B_k \\ &= \sum_u p_{ij}^{u} B_u B_k \\ &= \sum_{u,t} p_{ij}^{u} p_{uk}^{t} B_t. \end{aligned}$$

[4] The fundamental formula (3.6) was first obtained by W. A. Thompson [17], [19] and was independently discovered by the second author [15]. Other results of Section 3 were included in a set of lectures [2] at the University of Frankfurt by the first author. Some of these were independently obtained in another form by the second author. When the two authors learned of each other's work, they decided to collaborate in a joint paper.

From the independence of $B_0, B_1, \cdots, B_m$,

$$\sum_u p_{ij}^{u} p_{uk}^{t} = \sum_u p_{jk}^{u} p_{iu}^{t}. \tag{3.7}$$

In these equations the summation over u runs from 0 to m and the remaining indices are arbitrary but fixed,

$$0 \leqq i, j, k, t \leqq m$$

Now let us define $\mathcal{P}_k$ by[5]

$$\mathcal{P}_k = (p_{ik}^{j}) = \begin{bmatrix} p_{0k}^{0} & p_{0k}^{1} & \cdots & p_{0k}^{m} \\ p_{1k}^{0} & p_{1k}^{1} & \cdots & p_{1k}^{m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{mk}^{0} & p_{mk}^{1} & \cdots & p_{mk}^{m} \end{bmatrix}, \qquad k = 1, 0, \cdots, m.$$

Now the left side of (3.7) is the element in the ith row and tth column of $\mathcal{P}_j\mathcal{P}_k$. Also the element in the ith row and tth column of $\mathcal{P}_u$ is p_{iu}^{t}, so that the right side of (3.7) is the element in the ith row and tth column of

$$p_{jk}^{0}\mathcal{P}_0 + p_{jk}^{1}\mathcal{P}_1 + \cdots + p_{jk}^{m}\mathcal{P}_m.$$

Hence we have

$$\mathcal{P}_j\mathcal{P}_k = p_{jk}^{0}\mathcal{P}_0 + p_{jk}^{1}\mathcal{P}_1 + \cdots + p_{jk}^{m}\mathcal{P}_m. \tag{3.8}$$

Thus, the $\mathcal{P}$'s multiply in the same manner as the B's. Since $p_{0k}^{i} = 1$ if $k = i$ and 0 otherwise, the 0th row of $\mathcal{P}_k$ contains a 1 in column k and 0's in other positions, which is enough to show that if

$$c_0\mathcal{P}_0 + c_1\mathcal{P}_1 + \cdots + c_m\mathcal{P}_m = 0,$$

then

$$c_0 = c_1 = \cdots = c_m = 0;$$

i.e., $\mathcal{P}_0, \mathcal{P}_1, \cdots, \mathcal{P}_m$ are linearly independent. They thus form the basis for a vector space and combine in the same way as the B's under addition, as well as under multiplication. They provide a regular representation in

$$(m + 1) \times (m + 1)$$

matrices of the algebra given by the B's, which are $v \times v$ matrices. In particular, $\mathcal{P}_0 = I_{m+1}$.

Since the B's are commutative, the $\mathcal{P}$'s are commutative. In general they are not incidence matrices and are not symmetric. $\mathcal{P}_k$ does not have equal row totals, but has the same equal column totals n_k as B_k. In analogy with (3.3), all elements of row j of $\sum_k \mathcal{P}_k$ are equal to n_j. Let

$$B = c_0B_0 + c_1B_1 + \cdots + c_mB_m$$

[5] It should be noted that these matrices differ from matrices $P_k = (p_{ij}^{k})$ which were defined in several earlier papers ([4], [5], [6]) but do not have the same algebraic properties.

be any element of our algebra, and let $f(\lambda)$ be a polynomial. Then we can express

$$f(B) = l_0B_0 + l_1B_1 + \cdots + l_mB_m .$$

If

$$\mathcal{P} = c_0\mathcal{P}_0 + c_1\mathcal{P}_1 + \cdots + c_m\mathcal{P}_m$$

is the representation of B, then

$$f(\mathcal{P}) = l_0\mathcal{P}_0 + l_1\mathcal{P}_1 + \cdots + l_m\mathcal{P}_m .$$

Let $f(\lambda)$ be the minimum function of B and $\phi(\lambda)$ the minimum function of $\mathcal{P}$. Then $f(\lambda)$ is the monic polynomial of least degree for which

$$f(B) = 0.$$

$$f(B) = 0 \rightarrow l_0 = l_1 = \cdots = l_m = 0 \rightarrow f(\mathcal{P}) = 0;$$

i.e., $f(\lambda)$ is divisible by $\phi(\lambda)$.

Similarly $\phi(\lambda)$ is divisible by $f(\lambda)$. Since both are monic polynomials,

$$f(\lambda) = \phi(\lambda).$$

That is, B and $\mathcal{P}$ have the same distinct characteristic roots, and every matrix B has at most $m + 1$ distinct characteristic roots, which are solutions of the minimum equation of $\mathcal{P}$.

4. Applications to combinatorial problems. Association matrices will be used to derive some results first obtained in [9] by a longer method.

The incidence matrix $N = (n_{ij})$ of a design is defined by

$$\begin{aligned} n_{ij} &= 1, \quad \text{if treatment } i \text{ occurs in block } j, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Then

$$B = NN' = rB_0 + \lambda_1B_1 + \cdots + \lambda_mB_m ,$$

$$\mathcal{P} = r\mathcal{P}_0 + \lambda_1\mathcal{P}_1 + \cdots + \lambda_m\mathcal{P}_m .$$

Also

$$C = r\left(1 - \frac{1}{k}\right)B_0 - \frac{\lambda_1}{k}B_1 - \cdots - \frac{\lambda_m}{k}B_m ,$$

where C is the coefficient matrix in the normal equations for estimating the treatment effects after adjusting for the block effects [6]. Clearly C is a symmetric matrix. If e is a characteristic root of C, then $k(r - e)$ is a characteristic root of B. It is known that C has rank $v - 1$ for a connected design [1]. In this case,[6] therefore, 0 is a simple root of C and rk is a simple root of B, a fact which could also be shown directly as follows.

[6] Connectedness was assumed implicitly in [9].

The elements of B or NN' are non-negative, and for connected designs B is irreducible. Also it is easy to verify that the sum of the elements in any row or column of B is $\sum n_i\lambda_i = rk$. Hence

$$B^* = \frac{1}{rk} B = \frac{1}{rk} NN'$$

is a stochastic matrix, which shows that unity is a simple root of B^* and is greater than all the other roots [7]. Thus rk is a simple root of B. The results of Section 3 show that rk is a root of $\mathcal{P}$ and exceeds the other roots. If this root is removed from $| \mathcal{P} - I\theta | = 0$, then for the case $m = 2$, the other two characteristic roots of $\mathcal{P}$ will be roots of a quadratic equation which reduces to

$$(r - \theta)^2 + [(\lambda_1 - \lambda_2)(p_{12}^2 - p_{12}^1) - (\lambda_1 + \lambda_2)](r - \theta) + [(\lambda_1 - \lambda_2)(\lambda_2 p_{12}^1 - \lambda_1 p_{12}^2) + \lambda_1\lambda_2] = 0,$$

on using the identities (3.1) and (3.2). The roots are given by

$$(4.1)\qquad \begin{aligned} r - \theta_1 &= \tfrac{1}{2}[(\lambda_1 - \lambda_2)(-\gamma - \sqrt{\Delta}) + (\lambda_1 + \lambda_2)], \\ r - \theta_2 &= \tfrac{1}{2}[(\lambda_1 - \lambda_2)(-\gamma + \sqrt{\Delta}) + (\lambda_1 + \lambda_2)], \end{aligned}$$

where

$$\gamma = p_{12}^2 - p_{12}^1, \qquad \beta = p_{12}^1 + p_{12}^2, \qquad \Delta = \gamma^2 + 2\beta + 1.$$

Therefore,

$$| NN' - I_v\theta | = (rk - \theta)(\theta_1 - \theta)^{\alpha_1}(\theta_2 - \theta)^{\alpha_2}.$$

To determine the multiplicities α_1 and α_2 we note that

$$\operatorname{Tr} I_v = 1 + \alpha_1 + \alpha_2 = v,$$
$$\operatorname{Tr} NN' = rk + \alpha_1\theta_1 + \alpha_2\theta_2 = vr.$$

Solving and using (4.1),

$$(4.2)\qquad \begin{aligned} \alpha_1 &= \frac{n_1 + n_2}{2} - \frac{(n_1 - n_2) + \gamma(n_1 + n_2)}{2\sqrt{\Delta}}, \\ \alpha_2 &= \frac{n_1 + n_2}{2} + \frac{(n_1 - n_2) + \gamma(n_1 + n_2)}{2\sqrt{\Delta}}. \end{aligned}$$

Thus the multiplicities α_1 and α_2 of the roots of NN' are determined in terms of the parameters of the design. It is striking that, being independent of r and λ_i, these multiplicities are the same for all designs having a given association scheme. This is an instance of some general properties of α_i to be established in Section 6.

For a design to exist, α_1 and α_2 must be integral. The condition this imposes on the parameters appearing in (4.2) has been used in studies of the existence and non-existence of designs [3], [8], [9], [12], [15].

5. Applications of algebraic properties of association matrices. In this section we assume only that B_i ($i = 0, \cdots, m$), are symmetric incidence matrices satisfying

$$B_0 = I_v, \tag{5.1}$$

$$\sum_{i=0}^{m} B_i = J_v, \tag{5.2}$$

$$B_j B_k = \sum_{i=0}^{m} p_{jk}^i B_i, \tag{5.3}$$

for some set of constants p_{jk}^i. All of the properties of the algebra except commutativity follow immediately, including its representation in terms of the matrices $\mathcal{P}_k = (p_{ik}^j)$. Also, p_{jk}^i are elements of products of incidence matrices and must be non-negative integers. From

$$B_k = B_0 B_k = \sum_i p_{0k}^i B_i$$

we deduce the special values

$$p_{0k}^i = 1, \qquad \text{if } i = k,$$
$$= 0, \qquad \text{if } i \neq k.$$

The diagonal element in row t, column t of $B_j B_k'$ may be interpreted as the number of positions occupied by 1's in row t of B_j as well as in row t of B_k. (5.2) shows that if $k \neq j$ this element is zero. If $k = j$ it is equal to the number of 1's in row t of B_j. The expansions of $B_j B_k' = B_j B_k$ and $B_j B_j' = B_j^2$ then show that

$$p_{jk}^0 = 0, \qquad j \neq k,$$

and that p_{jj}^0 is equal to each row total of B_j. These row totals must therefore be equal. As a matter of notation set

$$p_{jj}^0 = n_j .$$

Row totals in (5.2) show

$$\sum n_j = v.$$

Also

$$\left(\sum_j B_j\right) B_k = J_v B_k = n_k J_v = \sum_i n_k B_i,$$

and

$$\sum_j (B_j B_k) = \sum_j \sum_i p_{jk}^i B_i = \sum_i \left(\sum_j p_{jk}^i\right) B_i .$$

Hence comparing coefficients,

$$\sum_j p_{jk}^i = n_k .$$

We now show that commutativity follows from symmetry of B_i.

$$B_k B_j = B_k' B_j' = (B_j B_k)' = (\sum_i p_{jk}^i B_i)'$$
$$= \sum_i p_{jk}^i B_i' = \sum_i p_{jk}^i B_i = B_j B_k.$$

As a consequence,

$$p_{jk}^i = p_{kj}^i.$$

We also deduce

$$\mathcal{P}_j \mathcal{P}_k = \mathcal{P}_k \mathcal{P}_j.$$

Equating the elements in the sth row and tth column of $\mathcal{P}_j \mathcal{P}_k$ and $\mathcal{P}_k \mathcal{P}_j$,

$$\sum_i p_{sj}^i p_{ik}^t = \sum_i p_{sk}^i p_{ij}^t. \tag{5.4}$$

This relation is equivalent to (3.7). Taking $t = 0$ we get

$$\sum_i p_{sj}^i p_{ik}^0 = \sum_i p_{sk}^i p_{ij}^0,$$
$$n_k p_{sj}^k = n_j p_{sk}^j.$$

We have now shown that all the known identities (3.1) follow from the properties of the algebra which were stated at the beginning of this section. However the relation

$$\mathcal{P}_j \mathcal{P}_k = \mathcal{P}_k \mathcal{P}_j$$

leads to new identities when $m > 2$.

To prove a new identity in the case $m = 3$, set $j = s = 1$, $k = t = 2$ in (5.4), giving

$$n_1 + p_{11}^1 p_{12}^2 + p_{11}^2 p_{22}^2 + p_{11}^3 p_{32}^2 = p_{12}^1 p_{11}^2 + p_{12}^2 p_{21}^2 + p_{12}^3 p_{31}^2.$$

We remark that when $m = 3$, other choices of j, k, s, t lead to relations equivalent to this one. The use of a smaller number of parameters will make it easier to recognize equivalent expressions and will be helpful in simplifying the identity. A fairly symmetric set of parameters is the following:

$$\begin{aligned}
&n_1, \quad n_2, \quad n_3,\\
a_{12} &= n_1 p_{22}^1 = n_2 p_{12}^2,\\
a_{31} &= n_3 p_{11}^3 = n_1 p_{13}^1,\\
a_{23} &= n_2 p_{33}^2 = n_3 p_{23}^3,\\
x &= n_1 p_{23}^1 = n_2 p_{13}^2 = n_3 p_{12}^3.
\end{aligned}$$

Known identities can be used to express all p_{jk}^i in terms of these parameters, whereupon the above identity reduces to

$$
\begin{aligned}
(5.5)\qquad & x^2\left(\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}\right)+x\left(\frac{a_{12}}{n_1}+\frac{a_{23}}{n_2}+\frac{a_{31}}{n_3}-n_1-n_2-n_3-1\right) \\
& +\left(\frac{a_{31}\,a_{12}}{n_1}+\frac{a_{12}\,a_{23}}{n_2}+\frac{a_{23}\,a_{31}}{n_3}-n_1\,a_{23}-n_2\,a_{31}-n_3\,a_{12}+n_1\,n_2\,n_3\right)=0.
\end{aligned}
$$

Thus when n_1, n_2, n_3, a_{12}, a_{23}, a_{31} are given, x must satisfy a quadratic equation. This is a new relation, since known identities (3.1) do not determine x in terms of the other chosen parameters. An example will illustrate this. Let

$$n_i = 8, \qquad a_{ij} = 24.$$

Then sets of p^i_{jk} which satisfy (3.1) are obtained for

$$x = 8,\ 16,\ 24,\ 32 \text{ or } 40.$$

However, (5.5) becomes

$$3x^2/8 - 16x + 152 = 0$$

and has no integral solutions, showing that the parameter values $n_i = 8$, $a_{ij} = 24$ are impossible.

An equivalent and perhaps easier way to impose the new necessary conditions on a given set of parameters is to form the matrix products $\mathcal{P}_j\mathcal{P}_k$ and $\mathcal{P}_k\mathcal{P}_j$ and require that they be identical.

The property of symmetry in the matrices B_i was used in the proof of commutativity in the algebra, which has been of key importance in the proofs of several of the foregoing identities. The fact that the elements of B_i are 0's and 1's has been used in determining the special form of the p^i_{jk} values but has not been vital in the algebra or the identities relating p^i_{jk}. The simple example

$$B_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

shows that matrices with elements other than 0's and 1's may have the same algebraic behavior as association matrices and may lead to the same identities. This shows the necessity of the word "incidence" in the following lemma, which summarizes several results of this section.

LEMMA 5.1. *If B_i, $i = 0, 1, \cdots, m$ are symmetric incidence matrices satisfying*

$$B_0 = I_v,$$

$$\sum_{i=0}^{v} B_i = J_v,$$

$$B_j\,B_k = \sum_{i=0}^{m} p^i_{jk}\,B_i$$

for some set of constants p^i_{jk}, then B_i are the association matrices of an association scheme satisfying (a) *to* (c) *of Section* 1.

This lemma provides a useful algebraic method of verifying whether a given association relation satisfies the conditions of partial balance.

Algebraic sufficiency conditions may be used for designs as well as association schemes. It is easy to verify that an incidence matrix N is the matrix of a PBIB design if and only if N has equal column totals and

$$NN' = rB_0 + \lambda_1 B_1 + \cdots + \lambda_m B_m$$

for some m and some numbers $r, \lambda_1, \cdots, \lambda_m$, where $B_0, B_1, \cdots, B_m$ satisfy the conditions of Lemma 5.1. An application of this Lemma will be made in the proof of the next theorem.

Given an association scheme $\mathfrak{A}$ with more than m classes, let the indices of the associate classes be arranged into disjoint sets $S_0 = \{0\}, S_1, \cdots, S_m$. Define a new association relation $\mathfrak{B}$ in which associate classes correspond to sets S_i, two treatments being defined as ith associates in $\mathfrak{B}$ if and only if the associate class of the two treatments in $\mathfrak{A}$ corresponds to one of the indices in set S_i.

Association relations obtained in this way do not in general satisfy the conditions of partial balance. Lemma 4.1 of [18] states necessary and sufficient conditions for partial balance in the case $S_1 = \{1, 2\}$, $S_i = \{i + 1\}$, $i \geqq 2$, i.e., the case in which just two classes are combined. Iteration may give schemes in which several classes have been combined. However, examples are known [15] in which a combination of 3 or more classes will give a new scheme with partial balance while every combination of 2 classes fails, so that the iterative procedure is impossible. The following generalization is therefore nontrivial.

THEOREM 5.1. *Given an association scheme $\mathfrak{A}$ with v treatments and parameter values $q^{\alpha}_{\beta\gamma}$, let an association relation $\mathfrak{B}$ with v treatments have classes $0, 1, \cdots, m$ determined by disjoint sets $S_0 = \{0\}, S_1, \cdots, S_m$ of indices of $\mathfrak{A}$. In order for $\mathfrak{B}$ to satisfy the conditions of partial balance it is n.a.s. that there exist constants p^i_{jk} such that*

$$\sum_{\beta \varepsilon S_j} \sum_{\gamma \varepsilon S_k} q^{\alpha}_{\beta\gamma} = p^i_{jk} \tag{5.6}$$

uniformly for $\alpha \varepsilon S_i$, and for $i, j, k = 0, 1, \cdots, m$; in this case $\mathfrak{B}$ has parameter values p^i_{jk}.

PROOF. We denote incidence matrices of $\mathfrak{A}$ by A_α and of $\mathfrak{B}$ by B_i. From the definition of $\mathfrak{B}$,

$$B_i = \sum_{\alpha \varepsilon S_i} A_\alpha .$$

Lemma 5.1 will now be applied.

Clearly B_i are symmetric incidence matrices, $B_0 = I_v$ and $\sum_i B_i = J_v$; in order for $\mathfrak{B}$ to have partial balance it is thus n.a.s. that there exist constants p^i_{jk} such that

$$B_j B_k = \sum_{i=0}^{m} p^i_{jk} B_i .$$

Substituting,

$$\begin{aligned} B_j B_k &= (\sum_{\beta \varepsilon S_j} A_\beta)(\sum_{\gamma \varepsilon S_k} A_\gamma) \\ &= \sum_{\beta \varepsilon S_j} \sum_{\gamma \varepsilon S_j} A_\beta A_\gamma = \sum_{\beta \varepsilon Sj} \sum_{\gamma \varepsilon S_k} \sum_{\alpha} q_{\beta\gamma}^{\alpha} A_\alpha \\ &= \sum_{\alpha} (\sum_{\beta \varepsilon S_j} \sum_{\gamma \varepsilon S_k} q_{\beta\gamma}^{\alpha}) A_\alpha . \end{aligned}$$

Also

$$\begin{aligned} \sum_{i=0}^{m} p_{jk}^{i} B_i &= \sum_{i=0}^{m} p_{jk}^{i} \sum_{\alpha \varepsilon S_i} A_\alpha \\ &= \sum_{i=0}^{m} \sum_{\alpha \varepsilon S_i} p_{jk}^{i} A_\alpha ; \end{aligned}$$

the coefficient of A_α in this expression has the same value p_{jk}^{i} for every $\alpha \varepsilon S_i$. Equating to the coefficient of A_α in the previous equation we obtain (5.6) as the n.a.s. conditions on the parameter values $q_{\beta\gamma}^{\alpha}$ of $\mathcal{B}$, completing the proof of the theorem.

6. Characteristic roots of matrices in the algebras. The procedure used in Section 4 to determine the multiplicities α_1 and α_2 is readily generalized to association schemes with m classes. If θ is a characteristic root of $B = \sum_{i=0}^{m} c_i B_i$, where c_i belong to the field of real (or complex) numbers, then θ^n is a characteristic root of B^n. Also, the trace of any matrix is equal to the sum of its characteristic roots. This leads to a system of equations in the roots θ_u of $\mathcal{P} = \sum_{i=0}^{m} c_i \mathcal{P}_i$ and the multiplicities α_u of the same roots of B. θ_0 will designate $\sum_{i=0}^{m} c_i n_i$, the common value of the row totals of B.

$$\begin{aligned} \alpha_0 + \alpha_1 + \cdots + \alpha_m &= \operatorname{Tr} I, \\ \alpha_0\theta_0 + \alpha_1\theta_1 + \cdots + \alpha_m\theta_m &= \operatorname{Tr} B, \\ \alpha_0\theta_0^2 + \alpha_1\theta_1^2 + \cdots + \alpha_m\theta_m^2 &= \operatorname{Tr} B^2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_0\theta_0^m + \alpha_1\theta_1^m + \cdots + \alpha_m\theta_m^m &= \operatorname{Tr} B^m. \end{aligned} \tag{6.1}$$

Equations of this form were used in [9] but were limited to the cases $m \leqq 4$ because of the difficulty of computing Tr B^n with methods then available. (3.6) may be used to express B^n in the form

$$B^n = c_{0,n}B_0 + c_{1,n}B_1 + \cdots + c_{m,n}B_m .$$

Then, since B_0 is the only B_i with non-zero diagonal elements,

$$\operatorname{Tr} B^n = vc_{0,n} .$$

The right members of the equations are therefore easily computed. The coefficients of α_u form the Vandermonde matrix with determinant

$$\prod_{0 \leq j < k \leq m} (\theta_k - \theta_j).$$

The system will therefore have a unique solution if and only if the $m + 1$ roots θ_u are distinct. It will be shown in Corollary 6.2 that this will be the case for at least some choice of c_i.

The solutions α_i must be non-negative integers. If they can be expressed in terms of the parameters c_i and p_{jk}^i this requirement will provide necessary conditions which the parameters must satisfy in order for matrix B to exist. An explicit solution of (6.1) requires a general solution of the equation $| \theta I - \mathcal{P} | = 0$, which may be difficult to obtain for $m > 2$, but one observation may be made at once. If the basis matrices B_i exist, then matrix B will exist for arbitrary values of c_i, with characteristic roots which obviously occur with integral multiplicities. This indicates that the integral nature of α_u must be independent of c_i and dependent only on p_{jk}^i. Theorem 6.3 will show that this holds not only for the integral property but for the exact values α_u. This is somewhat surprising in view of the form of (6.1), since the values θ_u and Tr B^n depend strongly on c_i. The other theorems of this section will give further insight into the nature of the roots θ_u and multiplicities α_u, as well as simplifying their computation. Results related to some of these have been obtained independently and by a different approach in [10].

It was pointed out in Section 4 that $\alpha_0 = 1$ for the matrix NN' if it is irreducible (which is the case when the design is connected). The same theorems for stochastic matrices [7] apply to any B with non-negative coefficients c_i. In particular any matrix B_i which is irreducible has n_i as a simple root. It follows from theorems (6.1) through (6.3), which we now proceed to prove, that θ_0 is a simple root (i.e., $\alpha_0 = 1$) for any set of coefficients c_i for which

$$B = \sum_{i=0}^{m} c_i B_i$$

is irreducible.

THEOREM 6.1. *Let the characteristic roots of* $\mathcal{P}_i$ *be* z_{ui}, $u = 0, 1, \cdots, m$. *Then for a suitable ordering of* z_{ui} *for each* i, *the characteristic roots of the matrix*

$$\mathcal{P} = \sum_{i=0}^{m} c_i \mathcal{P}_i$$

are given by

$$\theta_u = \sum_{i=0}^{m} c_i z_{ui}, \qquad u = 0, 1, \cdots, m. \tag{6.2}$$

PROOF. The matrices $\mathcal{P}_0, \cdots, \mathcal{P}_m$ are pairwise commutative. Frobenius' Theorem ([14], Theorem 16.1) then states that for a suitable ordering of the characteristic roots of z_{ui} of each $\mathcal{P}_i$, and for any rational function

$$f(x_0, \cdots, x_m)$$

the roots of

$$f(\mathcal{P}_0, \cdots, \mathcal{P}_m)$$

are given by

$$f(z_{u0}, \cdots, z_{um}), \qquad u = 0, 1, \cdots, m.$$

Also, the ordering of the roots is the same for every rational function f. The required theorem follows by taking

$$f(x_0, \cdots, x_m) = \sum_{i=0}^{m} c_i x_i .$$

Corollary 6.1. *The distinct characteristic roots θ_u of*

$$B = \sum_{i=0}^{m} c_i B_i$$

are given by

$$\theta_u = \sum_{i=0}^{m} c_i z_{ui}, \qquad u = 0, 1, \cdots, m.$$

The problem of finding θ_u is therefore solved if the values z_{ui} can be found and ordered. When they are ordered as specified by Theorem 6.1, we define the matrix

$$Z = (z_{ui}).$$

Since z_{ui} are the characteristic roots of real symmetric matrices B_i, Z is a real matrix.

Theorem 6.2. *$Z = (z_{ui})$ is non-singular.*

Proof. Let

$$y_0, y_1, \cdots, y_m$$

be a real solution of the system of homogeneous equations

$$\sum_{i=0}^{m} z_{ui} y_i = 0, \qquad u = 0, 1, \cdots, m. \tag{6.3}$$

This system has coefficient matrix Z and will have a non-zero solution if and only if Z is singular. Since Z is real there is no loss of generality in taking y_i real. By Corollary 6.1 the characteristic roots of the matrix

$$B = \sum_{i=0}^{m} y_i B_i$$

are given by the left side of (6.3) and are therefore all equal to zero. The sum of all products of roots taken s at a time is thus equal to zero, $s = 1, 2, \cdots, v$; this sum is equal to the generalized trace $\mathrm{Tr}_s B$, the sum of all $s \times s$ principal minor determinants of B. B is symmetric with diagonal elements y_0 and other elements $y_1, \cdots, y_m$. This follows by noting that among the incidence matrices $B_0, B_1, \cdots, B_m$ there is one and only one, say B_i, for which the element in the tth row and uth column is unity, whereas for B_j, $j \neq i$, the corresponding element is zero. Hence B will have y_i in positions which correspond to unities of B_i. In particular the diagonal elements of B will all be y_0. Therefore

$$\mathrm{Tr}_1 B = vy_0 = 0,$$

giving

$$y_0 = 0.$$

Any element y_i ($i = 1, 2, \cdots, m$) contributes $y_0^2 - y_i^2$ or $-y_i^2$ to $\mathrm{Tr}_2 B$. Since each row of B_i sums up to n_i, the number of unities above the diagonal in B_i s $vn_i/2$. This is also the number of y_i's above the diagonal in B. Hence

$$\mathrm{Tr}_2 B = -\frac{v}{2}(n_1 y_1^2 + n_2 y_2^2 + \cdots + n_m y_m^2) = 0.$$

Since $v, n_1, \cdots, n_m$ are positive integers and $y_1, y_2, \cdots, y_m$ are real it follows that

$$y_1 = y_2 = \cdots = y_m = 0.$$

Therefore (6.3) has no non-zero solution and Z is non-singular.

COROLLARY 6.2. *Given a set of association matrices $B_0, B_1, \cdots, B_m$, any ordered set of real (or complex) numbers $\theta_0, \cdots, \theta_m$ is the ordered set of distinct characteristic roots of*

$$B = c_0B_0 + c_1B_1 + \cdots + c_mB_m$$

for a unique set of real (or complex) coefficients c_i. In particular, matrices B exist with $m + 1$ distinct roots.

PROOF. For arbitrary $\theta_0, \cdots, \theta_m$ the system (6.2) can be solved uniquely for $c_0, \cdots, c_m$.

THEOREM 6.3. *If*

$$B = c_0B_0 + c_1B_1 + \cdots + c_mB_m$$

is an element of an algebra with the association matrices B_i as basis, then

$$|\theta I - B| = \prod_{u=0}^{m} (\theta - \theta_u)^{\alpha_u},$$

where α_u are independent of $c_0, \cdots, c_m$.

PROOF. Let S be an element of the algebra which has $m + 1$ distinct characteristic roots. Then S does not satisfy any polynomial equation with degree less than $m + 1$.

$$\begin{aligned}
S_0 &= B_0 = I,\\
S &= b_{01}B_0 + b_{11}B_1 + \cdots + b_{m1}B_m,\\
S^2 &= b_{02}B_0 + b_{12}B_1 + \cdots + b_{m2}B_m,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
S^m &= b_{0m}B_0 + b_{1m}B_1 + \cdots + b_{mm}B_m.
\end{aligned}$$

Since S does not satisfy any equation of degree m or less, these equations must be independent and can be solved to give each B_i as a linear expression in S^j with constant coefficients. Hence any arbitrary element B can be written

$$B = d_0 I + d_1 S + \cdots + d_m S^m.$$

If ϕ is a characteristic root of S, then the corresponding characteristic root of B will be

$$\theta = d_0 + d_1 \phi + \cdots + d_m \phi^m.$$

All of the roots θ_μ of B may be obtained in this way by using all of the roots of S. If a root ϕ_μ has multiplicity α_μ, then the corresponding α_μ roots of B will be equal. That is, the roots θ_μ of an arbitrary matrix B have the same multiplicities α_μ as the corresponding roots ϕ_μ of the fixed matrix S and are therefore independent of the coefficients c_i occurring in B.

This completes the proof but an additional remark should be made. The element B may be such that distinct roots ϕ lead to the same value $\hat{\theta}$, whose multiplicity $\hat{\alpha}$ will be equal to the sum of two or more α_μ. In general, if M is a subset of the set $0, 1, \cdots, m$, $\theta_\mu = \hat{\theta}$ for $\mu \in M$, and $\theta_\mu \neq \hat{\theta}$ for $\mu \notin M$, then

$$\hat{\alpha} = \textstyle\sum_{\mu \in M} \alpha_\mu .$$

The statement of the theorem is correct whether θ_μ are distinct or not.

If the roots z_{ui} are obtained separately for each $\mathcal{P}_i$, it may not be immediately clear what ordering of them is required by Theorem 6.1. However, each z_{ui} is a root of B_i with multiplicity α_u. If the multiplicities are known, a suitable ordering will then be determined by any ordering of the α_u if the α_u are distinct, and partially determined if they are not all distinct. Theorem 6.5 will give another technique for ordering the roots. Theorem 6.4 reveals another significance of the distinctness or equality of the α_u.

THEOREM 6.4. *If t and only t of the multiplicities α_u are equal, then for each i the corresponding roots z_{ui} satisfy a monic polynomial equation with integral coefficients and degree t. In particular, if any α_u is distinct from the other multiplicities, the corresponding roots z_{ui} are rational integers.*

PROOF. The term m-polynomial will denote "monic polynomial with integral coefficients." The characteristic polynomial of a matrix with integral coefficients is an m-polynomial. Denote the characteristic polynomials of a basis matrix B_i and its representation $\mathcal{P}_i$ by

$$f_i(\theta) = |\theta I - B_i| = \prod_{u=0}^{m} (\theta - z_{ui})^{\alpha_u},$$

$$\phi_i(\theta) = |\theta I - \mathcal{P}_i| = \prod_{u=0}^{m} (\theta - z_{ui}).$$

For a particular root z_{ui}, let $g(\theta)$ be the m-polynomial of lowest degree s with z_{ui} as a zero. $g(\theta)$ is irreducible over the rational field. It is determined uniquely by any of its zeros and any m-polynomial which has any of its zeros is divisible by $g(\theta)$ and has all of its zeros ([13], Sec. 38). Therefore $f_i(\theta)$ and $\phi_i(\theta)$ are divisible by $g(\theta)$, which must be the product of s of the linear factors of $\phi_i(\theta)$. Moreover, the corresponding multiplicities must all have the same value $\alpha_{u'}$; otherwise, after a certain number of successive divisions of $f_i(\theta)$ by $g(\theta)$ the quotient would be an m-polynomial which has some of the zeros of $g(\theta)$ but not all. In short, $f_i(\theta)$ contains $[g(\theta)]^{\alpha_{u'}}$ as a factor and at least s of the multiplicities are equal. It may happen that the set of distinct irreducible factors with multiplicity $\alpha_{u'}$ includes others along with $g(\theta)$. The product of the factors in the set will be the polynomial of degree t described in this theorem, where t is the sum of the degree s of $g(\theta)$ and the degrees of any other factors in the set. Clearly $s \leqq t$. If $t = 1$, then $s = 1$ and $g(\theta) = \theta - z_{u'i}$. Since $g(\theta)$ has integral coefficients, it follows in this case that $z_{u'i}$ is an integer, $i = 0, \cdots, m$.

Theorems 6.1 and 6.4 are illustrated in the case of $m = 2$ associate classes by expressions (4.1) for the roots θ_1 and θ_2, the roles of c_0, c_1, c_2 being played by r, λ_1, λ_2. Although in general the roots of a quadratic equation are irrational functions of the coefficients and although λ_1 and λ_2 occur several times in the coefficients, the roots in this case are linear polynomials in r, λ_1, λ_2, with coefficients that are rational if and only if the integer Δ is a perfect square. It is shown in [9] that if $\alpha_1 \neq \alpha_2$ it is in fact necessary that Δ be a perfect square, implying that the roots are rational. The additional fact that they are integers is not obvious from (4.1). It is further shown in [9] that if $\alpha_1 = \alpha_2$, it is possible that Δ will not be a square and that the roots will be irrational. This is precisely the case in known designs of cyclic type.

THEOREM 6.5. *For fixed* $u = 0, \cdots, m$, *the roots* z_{ui} *satisfy the relations*

$$z_{uj}\, z_{uk} = \sum_{i=0}^{m} p_{jk}^{i}\, z_{ui}. \tag{6.4}$$

PROOF. The relation is proved by applying Frobenius' theorem to both sides of (3.8).

It is interesting to note the amount of simplification that has now been made in the study of a matrix of the algebra, for example the matrix

$$NN' = rB_0 + \lambda_1 B_1 + \cdots + \lambda_m B_m$$

of a design. The characteristic equation of this matrix is of degree v. The regular representation introduced in Section 3 reduces its solution to the solution of an equation of degree $m + 1$. The theorems of this section show that the characteristic roots are linear combinations of r, λ_1, $\cdots$, λ_m and that the multiplicities are entirely independent of these parameters, depending only on the association scheme. The coefficients of r, λ_1, $\cdots$, λ_m are z_{ui}, the characteristic roots of the matrices $\mathcal{P}_i$, which also depend only on the association schemes. In some cases the z_{ui} can be shown to be integers and in any case they satisfy the system of quadratic equations (6.4). Once z_{ui} values are found for some of the matrices $\mathcal{P}_1, \cdots, \mathcal{P}_m$, the equations (6.4) may be particularly useful, not only permitting an easy determination of the remaining z_{ui}, but giving them in the order required by Frobenius' theorem and used in Theorem 6.1.

The matrix $Z = (z_{ui})$ seems deserving of further study. As an indication of its usefulness we make the following remark:

$$\sum_{u=0}^{m} \alpha_u z_{ui} = \operatorname{Tr} B_i = v, \qquad i = 0,$$

$$= 0, \qquad i = 1, 2, \cdots, m.$$

This is equivalent to the system of equations

$$(\alpha_0, \alpha_1, \cdots, \alpha_m)Z = (v, 0, 0, \cdots, 0)$$

providing an alternative to (6.1) for determining α_u.

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