# ASYMPTOTIC EXPANSIONS FOR THE SMIRNOV TEST AND FOR THE RANGE OF CUMULATIVE SUMS[1]

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**Summary.** Let $z_n$ denote the position at time $n$ of a particle describing a one-dimensional random walk, such that the increments $\zeta_n = z_n - z_{n-1}$ $(n = 1, 2, \cdots)$ are independent random variables, assuming only the values $+1$ and $-1$, each with probability $\frac{1}{2}$. Of considerable importance in many applications is the conditional probability

$$p_n(i, j, c) = P(z_n = j, 0 < z_m < c, m = 1, \cdots, n \mid z_0 = i);$$

here, $i, j, c, n$ denote positive integers. In section 1, an asymptotic development for $p_n(i, j, c)$ is given; for each positive integer $m$, it yields an approximation to $p_n(i, j, c)$ with error smaller than $Cn^{-m}$ where $C$ is independent of $i, j, c$ and $n$. As a simple application, an asymptotic development for the binomial coefficient $\binom{n}{s}$ is derived by letting $i, j, c$ tend to infinity in such a manner that $j - i = 2s - n$.

As a second application, an asymptotic expansion is derived for the joint distribution of the extrema of the difference between the empirical distributions of two samples of size $n$.

The above asymptotic development for $p_n(i, j, c)$ is obtained by applying the central Lemma 4 to an exact formula for $p_n(i, j, c)$. In Section 5, using this formula, an exact formula is obtained for the distribution of the range $R_n$ of the $n + 1$ numbers $z_0, \cdots, z_n$. Applying Lemma 4 to it, a complete asymptotic expansion for the distribution of $R_n$ is derived.

**1. Main result.** Consider a random walk $z_0, z_1, \cdots$ of independent increments $\zeta_n = z_n - z_{n-1}$, such that

$$P(\zeta_n = +1) = P(\zeta_n = -1) = \tfrac{1}{2}, \qquad (n = 1, 2, \cdots).$$

In the sequel, $n, i, j, c$ always denote integers with $n \geqq 0, 0 < i < c, 0 \leqq j \leqq c$. Let $p_n(i, j, c)$ denote the conditional probability, given $z_0 = i$, that $z_n = j$ and $0 < z_m < c$ for $m = 0, 1, \cdots, n$. Observe that $p_n(i, j, c) = 0$ unless the integers $j - i$ and $n$ are of the same parity.

It is well-known that

$$p_n(i,j,c) = 2^{-n} \sum_{k=-\infty}^{\infty} \left[ \binom{n}{(n + j - i + 2kc)/2} - \binom{n}{(n + j + i + 2kc)/2} \right], \tag{1}$$

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if $n + i + j$ is even. Moreover,

$$p_n(i, j, c) = (2/c) \sum_{k=1}^{c-1} \sin k\pi i/c \sin k\pi j/c\, (\cos k\pi/c)^n. \tag{2}$$

A simple proof of (1) and (2) is as follows. Let $c$ and $i$ be fixed; then the function $p_n(i, j, c)$ is uniquely determined by the obvious relations

$$p_{n+1}(i, j, c) = [p_n(i, j-1, c) + p_n(i, j+1, c)]/2, \qquad (0 < j < c),$$

$p_n(i, 0, c) = p_n(i, c, c) = 0$ and $p_0(i, j, c) = 1$, if $i = j$, $= 0$ if $i \neq j$. But it is easily verified that the function defined by the right hand side of (2), (or (1), respectively), satisfies all these relations.

If $s(k)$ denotes the $k$th term in the right hand side of (2) we have $s(c - k) = (-1)^{i+j+n}s(k)$; moreover, $s(c/2) = 0$ if $c$ is even. Hence, (2) may be written as

$$p_n(i, j, c) = (4/c) \sum_{k=1}^{[(c-1)/2]} \sin k\pi i/c \sin k\pi j/c\, (\cos k\pi/c)^n \tag{3}$$

if $i + j + n$ is even, ($p_n(i, j, c) = 0$, otherwise). Using (3), we shall derive an asymptotic development for $p_n(i, j, c)$ with a remainder $O(n^{-m-\frac{1}{2}})$ holding uniformly with respect to all the parameters $i, j$ and $c$, ($m = 1, 2, \cdots$).

More precisely, let

$$A_{\nu-1} = \frac{2^{2\nu}(2^{2\nu} - 1)}{(2\nu)\,(2\nu)!} B_\nu, \qquad (\nu = 1, 2, \cdots), \tag{4}$$

where $B_\nu > 0$ denotes the $\nu$th Bernoulli number, ($B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, $B_4 = 1/30$, $B_5 = 5/66, \cdots$; $A_0 = 1/2$, $A_1 = 1/12$, $A_2 = 1/45$, $A_3 = 17/2520$, $A_4 = 61/28350$, $A_5 = 691/935550$). Further, let

$$A_{\mu h} = \sum{}' A_1^{\nu_1} \cdots A_\mu^{\nu_\mu} (\nu_1! \cdots \nu_\mu!)^{-1}, \tag{5}$$

where the summation is extended over all the sets $(\nu_1, \cdots, \nu_\mu)$ of non-negative integers which satisfy

$$\nu_1 + \nu_2 + \cdots + \nu_\mu = h; \qquad \nu_1 + 2\nu_2 + \cdots + \mu\nu_\mu = \mu.$$

Thus, $A_{00} = 1$ and $A_{\mu h} = 0$ if $h > \mu$. Further, for $\mu \geqq 1$, $A_{\mu 0} = 0$, $A_{\mu 1} = A_\mu$, $A_{\mu\mu} = (12)^{-\mu}/\mu!$; also $A_{32} = A_1 A_2 = 1/540$.

Finally, let

$$H_{2r}^*(x) = \left(\frac{d}{dx}\right)^{2r} e^{-x^2/2} = \bar{H}_{2r}(x)\, e^{-x^2/2}, \tag{6}$$

($r = 0, 1, \cdots$). For instance, $\bar{H}_0 = 1$, $\bar{H}_2 = x^2 - 1$, $\bar{H}_4 = x^4 - 6x^2 + 3$, $\bar{H}_6 = x^6 - 15x^4 + 45x^2 - 15$. In general,

$$\bar{H}_{2r}(x) = (2r)! \sum_{\nu=0}^{r} (-2)^{-\nu} x^{2(r-\nu)}/(\nu!(2r - 2\nu)!).$$

We can now state the main result concerning $p_n(i, j, c)$.

THEOREM 1. *Let*

$$g_r = 4(\alpha/\pi)^{\frac{1}{2}} \sum_{k=1}^{\infty} \sin k\pi i/c \sin k\pi j/c \, e^{-\alpha k^2}(2\alpha k^2)^r, \tag{7}$$

*where, for brevity,*

$$\alpha = \pi^2 n/(2c^2).$$

*A formula equivalent to* (7) *is*

$$g_r = (-1)^r \sum_{k=-\infty}^{\infty} [H_{2r}^*(n^{-\frac{1}{2}}(j - i + 2kc)) - H_{2r}^*(n^{-\frac{1}{2}}(j + i + 2kc))], \tag{8}$$

*if* $r = 0, 1, \cdots$. *Finally, let*

$$u_m = p_n(i, j, c) - (2/\pi)^{\frac{1}{2}} \sum_{\mu=0}^{m-1} n^{-\mu-\frac{1}{2}} \sum_{h=0}^{\mu} (-1)^h A_{\mu h} g_{\mu+h}, \tag{9}$$

($m = 1, 2, \cdots$). *Then, for each integer* $m \geqq 1$ *and each constant* $K > 0$, *there exists a number* $M > 0$, *depending on* $m$ *and* $K$, *but independent of* $i, j, c, n$, *such that*

$$| u_m | \leqq M(e^{-Kn^{\frac{1}{3}}} + n^{-m-\frac{1}{2}}e^{-\alpha}(1 + \alpha^{2m+\frac{1}{2}})), \tag{10}$$

*for each choice of the integers* $i, j, c, n$ *with* $i + j + n$ *even*, $0 < i < c$, $0 \leqq j \leqq c$, $n > 0$.

**2. Auxiliary results. Proof of Theorem 1.**

LEMMA 1. *Let* $-\log \cos w^{\frac{1}{2}} = w/2 + w^2/12 + \cdots$ *denote the analytic function for* $| w | < \pi^2/4$, *which assumes real and positive values for* $w$ *real*, $0 < w < \pi^2/4$, *and let*

$$\varphi(w) = (-\log \cos w^{\frac{1}{2}} - w/2)w^{-2}. \tag{1}$$

*Then* $\varphi(w) \geqq 0$ *for* $w$ *real and positive*, $w < \pi^2/4$. *Moreover, we have the Taylor expansion*

$$e^{u\varphi(w)} = \sum_{\mu=0}^{\infty} \sum_{h=0}^{\mu} A_{\mu h} u^h w^{\mu-h}, \tag{2}$$

*holding for* $| w | < \pi^2/4$ *and arbitrary* $u$, *where the* $A_{\mu h}$ *are as defined above.*

PROOF. Let $A_j > 0$ be defined by (1.4), especially, $A_0 = \frac{1}{2}$. Integrating the well-known expansion $\tan x = \sum_{\nu=0}^{\infty} (2\nu + 2)A_\nu x^{2\nu+1}$, ($| x | < \pi/2$), we obtain $-\log \cos x = x^2/2 + \sum_{\nu=1}^{\infty} A_\nu x^{2\nu+2}$. Hence, from (1), $\varphi(w) = w^{-1}\sum_{\nu=1}^{\infty} A_\nu w^\nu$, ($| w | < \pi^2/4$). The above assertions now easily follow.

Observe that, from (1), formula (1.3) may be written as

$$\begin{aligned} p_n(i, j, c) = \frac{2}{c} \sum_{k=1}^{[(c-1)/2]} \\ \cdot \left(\cos \frac{k\pi(j - i)}{c} - \cos\left(\frac{k\pi(j + i)}{c}\right)\right) e^{-\alpha k^2} \psi\left(-\frac{4}{n}(\alpha k^2)^2, \frac{2}{n}\alpha k^2\right), \end{aligned} \tag{3}$$

where $\alpha = \pi^2 n/(2c^2)$ and

$$(4) \qquad \psi(u, w) = e^{u\varphi(w)} = \sum_{\mu=0}^{\infty} \sum_{h=0}^{\mu} A_{\mu h} u^h w^{\mu-h}, \qquad (|w| < \pi^2/4).$$

The proofs not being any more difficult, and in view of the proof in Section 5, we shall determine the asymptotic behavior (for small values of $|\sigma|$ and $|\tau|$) of more general sums of the type

$$\sum_{1 \leqq k \leqq \lambda} \cos kx \, e^{-\beta k^2} f(\sigma(\beta k^2)^p, \tau(\beta k^2)^q).$$

Here, $f(u, w)$ denotes a fixed analytic function for $|u| < u_0$, $0 < |w| < w_0$, $(u_0 > 0, w_0 > 0)$, admitting the expansion

$$(5) \qquad f(u, w) = w^{-s} \sum_{\mu=0}^{\infty} \sum_{h=0}^{\mu} B_{\mu h} u^h w^{\mu-h},$$

$(|u| < u_0, 0 < |w| < w_0)$, where $s$ denotes an integer and the $B_{\mu h}$ are complex constants.

LEMMA 2. *Let $m$ denote a fixed non-negative integer, and let*

$$R(m) = f(u, w) - w^{-s} \sum_{\mu=0}^{m} \sum_{h=0}^{\mu} B_{\mu h} u^h w^{\mu-h}.$$

*Then to each pair of positive constants $u_1$ and $w_1$ with $u_1 < u_0$, $w_1 < w_0$ there corresponds a constant $M$, independent of $u$, $w$, such that*

$$|w^s R(m)| \leqq M(|u|^m + |w|^m),$$

*whenever $|u| \leqq u_1$, $|w| \leqq w_1$.*

PROOF. Let $|u| \leqq u_1$, $|w| \leqq w_1$ and put $\theta = \operatorname{Max}(|u|/u_1, |w|/w_1)$, $\theta \leqq 1$. We have

$$|w^s R(m)| = \Big|\sum_{\mu=m}^{\infty} \sum_{h=0}^{\mu} B_{\mu h} u^h w^{\mu-h}\Big| \leqq \theta^m \sum_{\mu=m}^{\infty} \sum_{h=0}^{\mu} |B_{\mu h}| u_1^h w_1^{\mu-h} = K\theta^m.$$

LEMMA 3. *To each real number $r \neq -\frac{1}{2}$ there corresponds a constant $M$, such that, for each choice of the positive numbers $\beta$ and $\lambda$,*

$$(6) \qquad \begin{aligned} \sum_{k \geqq \lambda} e^{-\beta k^2} (\beta k^2)^r &\leqq M\beta^{-\frac{1}{2}} e^{-\beta\lambda^2}(1 + (\beta\lambda^2)^{r+\frac{1}{2}}), && \text{if } r > -\tfrac{1}{2}, \\ &\leqq M\beta^{-\frac{1}{2}} e^{-\beta\lambda^2}(\beta\lambda^2)^{r+\frac{1}{2}}, && \text{if } r < -\tfrac{1}{2}. \end{aligned}$$

PROOF. Let $S(\beta, \lambda)$ denote the left hand side of (6). In the proof we may assume that $\lambda \geqq 1$. For, suppose the lemma has been proved for this case. Then, for $0 < \lambda < 1$ and $r < -\frac{1}{2}$,

$$S(\beta, \lambda) = S(\beta, 1) \leqq M\beta^{-\frac{1}{2}} e^{-\beta} \beta^{r+\frac{1}{2}} \leqq M\beta^{-\frac{1}{2}} e^{-\beta\lambda^2} (\beta\lambda^2)^{r+\frac{1}{2}}.$$

On the other hand, let $0 < \lambda < 1$ and $r > -\frac{1}{2}$. Then $S(\beta, \lambda) = S(\beta, 1) \leqq M\beta^{-\frac{1}{2}} e^{-\beta}(1 + \beta^{r+\frac{1}{2}})$. If $2^{-\frac{1}{2}} \leqq \lambda < 1$ we have

$$e^{-\beta}(1 + \beta^{r+\frac{1}{2}}) \leqq e^{-\beta}(1 + (2\beta\lambda^2)^{r+\frac{1}{2}}) \leqq 2^{r+\frac{1}{2}} e^{-\beta\lambda^2}(1 + (\beta\lambda^2)^{r+\frac{1}{2}}).$$

Further, for $0 < \lambda < 2^{-\frac{1}{2}}$, $e^{-\beta}(1 + \beta^{r+\frac{1}{2}}) \leqq e^{-\beta\lambda^2}e^{-\beta/2}(1 + \beta^{r+\frac{1}{2}}) \leqq Ke^{-\beta\lambda^2}$, if $K$ denotes the maximum value of the function $e^{-\beta/2}(1 + \beta^{r+\frac{1}{2}})$, $\beta > 0$.

Thus, let $\lambda \geqq 1$. The function $f(x) = e^{-\beta x^2}(\beta x^2)^r$, $(x > 0)$, is decreasing if $r \leqq 0$ and, for $r > 0$, has a unique maximum at $x_0 = (r/\beta)^{\frac{1}{2}}$, where $f(x_0) = C$, $C$ denoting a constant independent of $\beta$. Further, if $r > 0$, $f(x)$ is increasing for $0 < x \leqq x_0$ and decreasing for $x \geqq x_0$. Hence, letting

$$I = \int_\lambda^\infty e^{-\beta x^2}(\beta x^2)^r\, dr, \tag{7}$$

we have

$$\begin{aligned} S(\beta, \lambda) &\leqq I + C, && \text{if } r > 0 \text{ and } \lambda < x_0, \\ &\leqq I + e^{-\beta\lambda^2}(\beta\lambda^2)^r, && \text{if } r \leqq 0 \text{ or } \lambda \geqq x_0. \end{aligned} \tag{8}$$

If $r > 0$ and $\lambda < x_0 = (r/\beta)^{\frac{1}{2}}$ we have, from $\lambda \geqq 1$, that $\beta \leqq \beta\lambda^2 < r$, hence, $C \leqq Ce^{r-\beta\lambda^2}(\beta/r)^{-\frac{1}{2}}$. In any case, from $\lambda \geqq 1$, $e^{-\beta\lambda^2}(\beta\lambda^2)^r \leqq \beta^{-\frac{1}{2}}e^{-\beta\lambda^2}(\beta\lambda^2)^{r+\frac{1}{2}}$. Finally, letting $\beta x^2 = y$ in (7), $I = \beta^{-\frac{1}{2}}J(\beta\lambda^2)/2$, where

$$J(w) = \int_w^\infty e^{-y}y^{r-\frac{1}{2}}\, dy. \tag{9}$$

It follows from (8) that it suffices to prove the existence of an absolute constant $M$, such that, for $w > 0$,

$$\begin{aligned} J(w) &\leqq Me^{-w}(1 + w^{r+\frac{1}{2}}), && \text{if } r > -\tfrac{1}{2}, \\ &\leqq Me^{-w}w^{r+\frac{1}{2}}, && \text{if } r < -\tfrac{1}{2} \end{aligned} \tag{10}$$

Letting $y = w(1 + z)$ in (9), we have $J(w) = e^{-w}w^{r+\frac{1}{2}} \int_0^\infty e^{-wz}(1 + z)^{r-\frac{1}{2}}\, dz$. This proves (10) when either $r < -\frac{1}{2}$ or $r > -\frac{1}{2}$, $w \geqq c$, $c$ denoting a fixed positive constant. Finally, if $r > -\frac{1}{2}$, $w \leqq c$, we have

$$J(w) \leqq \Gamma(r + \tfrac{1}{2}) \leqq \Gamma(r + \tfrac{1}{2})e^{c-w}.$$

LEMMA 4. *Consider the sum*

$$S = \sum_{1 \leqq k \leqq \lambda} \cos kx\, e^{-\beta k^2} f(\sigma(\beta k^2)^p, \tau(\beta k^2)^q), \tag{11}$$

*where $\sigma$ and $\tau$ denote complex numbers, $x$ a real number, $\lambda$, $\beta$ positive real numbers, $p$, $q$ non-negative integers; ($S = 0$ if $\lambda < 1$). Further, let $B_{\mu h}$, $s$, $u_0$, $w_0$ be as in* (5), *and*

$$S_m = \sum_{\mu=0}^{m-1} \sum_{h=0}^{\mu} B_{\mu h}\sigma^h\tau^{\mu-h-s} \sum_{k=1}^{\infty} \cos kx\, e^{-\beta k^2}(\beta k^2)^{ph+q(\mu-h-s)}. \tag{12}$$

Assertion: *to each choice of the integer $m > 0$ and the positive numbers $u_1 < u_0$, $w_1 < w_0$ there corresponds a constant $M > 0$, independent of $\lambda$, $x$, $\beta$, $\sigma$, $\tau$, such that*

$$\begin{aligned} | S - S_m | \leqq M\beta^{-\frac{1}{2}} | \tau |^{-s}\{&e^{-\beta\lambda^2}(\beta\lambda^2)^{-qs+\frac{1}{2}} \\ &+ e^{-\beta} | \sigma |^m(1 + \beta^{pm-qs+\frac{1}{2}}) + e^{-\beta} | \tau |^m(1 + \beta^{qm-qs+\frac{1}{2}})\}, \end{aligned}$$

*for each choice of the parameters* $\lambda$, $x$, $\beta$, $\sigma$ *and* $\tau$, *satisfying* $\lambda > 0$, $\beta > 0$, $\beta\lambda^2 \geqq 1$ *and*

$$(13) \qquad |\sigma|(\beta\lambda^2)^p \leqq u_1, \qquad |\tau|(\beta\lambda^2)^q \leqq w_1.$$

*Finally, the same assertion holds true if in* (11) *and* (12) *the summation variable* $k$ *is restricted to the odd integers.*

PROOF. In the following, $M$ denotes a positive constant, independent of $\lambda$ $x$, $\beta$, $\sigma$, $\tau$, *not necessarily the same* constant on each occasion. From Lemma 2, using (13), for $1 \leqq k \leqq \lambda$,

$$f(\sigma(\beta k^2)^p, \tau(\beta k^2)^q) = \sum_{\mu=0}^{m-1}\sum_{h=0}^{\mu} B_{\mu h}\sigma^h\tau^{\mu-h-s}(\beta k^2)^{ph+q(\mu-h-s)} + a_{mk}$$

with $|a_{mk}| \leqq M|\tau|^{-s}(|\sigma|^m(\beta k^2)^{pm-qs} + |\tau|^m(\beta k^2)^{qm-qs})$. Hence, from $|\cos kx| \leqq 1$, (whether or not $k$ is restricted to the odd positive integers), $|S - S_m| \leqq T_1 + T_2$, where $T_1 = M|\tau|^{-s}\sum_{k=1}^{\infty} e^{-\beta k^2}(|\sigma|^m(\beta k^2)^{pm-qs} + |\tau|^m(\beta k^2)^{qm-qs})$, and $T_2 = \sum_{\mu=0}^{m-1}\sum_{h=0}^{\mu}|B_{\mu h}\sigma^h\tau^{\mu-h-s}|\sum_{k>\lambda} e^{-\beta k^2}(\beta k^2)^{ph+q(\mu-h-s)}$ From $\beta\lambda^2 \geqq 1$ and Lemma 3,

$$T_2 \leqq M\beta^{-\frac{1}{2}}e^{-\beta\lambda^2}\sum_{\mu=0}^{m-1}\sum_{h=0}^{\mu}|\sigma^h\tau^{\mu-h-s}|(\beta\lambda^2)^{ph+q(\mu-h-s)+\frac{1}{2}}$$
$$\leqq M\beta^{-\frac{1}{2}}e^{-\beta\lambda^2}|\tau|^{-s}(\beta\lambda^2)^{-qs+\frac{1}{2}},$$

from (13). Further, from Lemma 3, applied with $\lambda = 1$,

$$T_1 \leqq M\beta^{-\frac{1}{2}}e^{-\beta}|\tau|^{-s}(|\sigma|^m(1 + \beta^{pm-qs+\frac{1}{2}}) + |\tau|^m(1 + \beta^{qm-qs+\frac{1}{2}}),$$

yielding the stated assertion.

LEMMA 5. *For* $\beta > 0$, $r = 0, 1, \cdots$, *we have*

$$\delta_0^r + 2\sum_{k=1}^{\infty}\cos kx\, e^{-\beta k^2}(2\beta k^2)^r = (-1)^r(\pi/\beta)^{\frac{1}{2}}\sum_{k=-\infty}^{\infty} H_{2r}^*\left(\frac{x + 2\pi k}{\sqrt{2\beta}}\right),$$

*where* $H_{2r}^*$ *is defined by* (1.6).

PROOF. In view of $H_0^*(y) = e^{-y^2/2}$ and $\delta_0^0 = 1$, the special case $r = 0$ is equivalent to a well-known identity for theta-functions. Differentiating $2r$ times with respect to $x$, the general result immediately follows.

PROOF OF THEOREM 1. Let $i$, $j$, $n$, $c$ denote integers, $0 < i < c$, $0 \leqq j \leqq c$, $n > 0$, $i + j + n$ even. Then $p_n(i, j, c)$ is given by (3) and (4), where

$$(14) \qquad \alpha = \pi^2 n/(2c^2).$$

Further, let $m$ denote a given positive integer, $K$ a given positive constant. It suffices to prove (1.10), (with $M$ depending only on $m$ and $K$), under the additional restriction that

$$(15) \qquad n \geqq K^2 \geqq 1$$

(for, letting afterwards $K = 1$, the general result immediately follows). Let

$\lambda > 0$ be defined by

$$(16) \qquad \alpha\lambda^2 = Kn^{\frac{1}{2}},$$

hence,

$$(17) \qquad (\pi\lambda/c)^2 = 2n^{-1}\alpha\lambda^2 = 2n^{-\frac{1}{2}}K \leqq 2 < \pi^2/4,$$

thus, $\lambda < c/2$. From Lemma 1 and (4),

$$0 \leqq \psi\left(-\frac{4}{n}(\alpha k^2)^2, \frac{2}{n}\alpha k^2\right) \leqq 1, \qquad 0 < k < c/2,$$

thus, the contribution to the right hand side of (3) of the terms with $k > \lambda$ is at most equal to $(4/c)(c/2)e^{-\alpha\lambda^2} = 2e^{-Kn^{\frac{1}{2}}}$. Consequently, from (3) and (14),

$$(18) \qquad \begin{aligned} p_n(i, j, c) &= O(e^{-Kn^{\frac{1}{2}}}) \\ &\quad + (2/\pi)(2\alpha/n)^{\frac{1}{2}}(S(\pi(j-i)/c) - S(\pi(j+i)/c)), \end{aligned}$$

where $S(x) = \sum_{1 \leqq k \leqq \lambda} \cos kx\, e^{-\alpha k^2}\psi(-(4/n)(\alpha k^2)^2, (2/n)\alpha k^2)$. In order to estimate the latter sum, we apply Lemma 4 with $\lambda$ as above, $\beta = \alpha$, $\sigma = -4/n$, $\tau = 2/n$, $p = 2$, $q = 1$, $f(u, w) = \psi(u, w)$, $s = 0$, $u_1 = 4K^2$, ($u_0$ arbitrary, $u_0 > u_1$), $w_1 = 2 < \pi^2/4 = w_0$. Then (13) holds, from (17) and $(4/n)(\alpha\lambda^2)^2 = 4K^2 = u_1$. Moreover, $\alpha\lambda^2 = Kn^{\frac{1}{2}} \geqq K \geqq 1$. Hence, using Lemma 1, Lemma 4 yields that, (for real values $x$),

$$(19) \qquad |S(x) - S_m(x)| \leqq M\alpha^{-\frac{1}{2}}(n^{\frac{1}{4}}e^{-Kn^{\frac{1}{2}}} + n^{-m}e^{-\alpha}(1 + \alpha^{2m+\frac{1}{2}})),$$

where $M$ denotes a constant depending only on $K$ and $m$. Here,

$$(20) \qquad S_m(x) = \sum_{\mu=0}^{m-1}\sum_{h=0}^{\mu} A_{\mu h}(-4/n)^h(2/n)^{\mu-h}\sum_{k=1}^{\infty} \cos kx e^{-\alpha k^2}(\alpha k^2)^{\mu+h}.$$

Theorem 1 is an immediate consequence of (18), (19), (20) and Lemma 5, the latter implying the equivalence of (1.7) and (1.8).

**3. Asymptotic expansion of the binomial coefficient.** Let $p_n(i, j)$ denote the conditional probability, given $z_0 = i$, that $z_n = j$, $z_\nu > 0$ for $\nu = 0, 1, \cdots, n$, thus,

$$(1) \qquad p_n(i, j) = \lim_{c \to \infty} p_n(i, j, c).$$

From (1.8), $\lim_{c\to\infty}(-1)^r g_r = H_{2r}^*(n^{-\frac{1}{2}}(j - i)) - H_{2r}^*(n^{-\frac{1}{2}}(j + i))$, hence, from Theorem 1 and (1), if $i > 0$, $j \geqq 0$, $n + i + j$ even,

$$(2) \qquad \begin{aligned} p_n(i, j) = (2/\pi)^{\frac{1}{2}}\sum_{\mu=0}^{m-1}(-1)^\mu n^{-\mu-\frac{1}{2}}\sum_{h=0}^{\mu} A_{\mu h}[H_{2\mu+2h}^*(n^{-\frac{1}{2}}(j - i)) \\ - H_{2\mu+2h}^*(n^{-\frac{1}{2}}(j + i))] + u_m \end{aligned}$$

where

$$(3) \qquad |u_m| \leqq Mn^{-m-\frac{1}{2}},$$

$M$ denoting a constant independent of $i$, $j$, $n$. Further,

$$(4) \qquad p_n(i,j) = 2^{-n}\left[\binom{n}{(n+j-i)/2} - \binom{n}{(n+j+i)/2}\right],$$

(e.g., from (1) and (1.1)), if $i > 0$, $j \geqq 0$, $n + i + j$ even. Keeping $n$ fixed and letting $i, j$ tend to infinity, such that $n + j - i = 2s$, $s$ an integer, we have from (2), (3) and (4),

$$(5) \quad 2^{-n}\binom{n}{s} = (2/\pi)^{\frac{1}{2}}\sum_{\mu=0}^{m-1}(-1)^{\mu}n^{-\mu-\frac{1}{2}}\sum_{h=0}^{\mu}A_{\mu h}\,H^{*}_{2\mu+2h}(n^{-\frac{1}{2}}(2s-n)) + O(n^{-m-\frac{1}{2}}),$$

the remainder holding uniformly in $s$ and $n$. An alternative proof of (5) might be obtained by starting with Stirling's formula or from an application of a general theorem of C. G. Esseen (*Acta Mathematica*, Vol. 77 (1945), p. 63).

**4. The Smirnov test with equal sample sizes.** Let $x_1, \cdots, x_n, y_1, \cdots, y_n$ denote $2n$ independent observations on a real random variable having a continuous distribution. Further, let

$$F_1(s) = \sum_{x_i \leqq s} 1/n, \qquad F_2(s) = \sum_{y_i \leqq s} 1/n$$

denote the empiric distributions of the samples $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$, respectively. Finally, let

$$(1) \qquad P_n(a, b) = \text{Prob}\,(-a/n < F_2(s) - F_1(s) < b/n \text{ for all } s),$$

where $a$, $b$ denote positive integers or $+\infty$. It is not difficult to show, cf. Gnedenko and Korolyuk [4], that, irrespective of the underlying distribution,

$$(2) \qquad 2^{-2n}\binom{2n}{n}P_n(a, b) = p_{2n}(a, a, a+b),$$

where $p_n(i, j, c)$ is precisely the quantity studied in the previous sections. Hence, from (1.1),

$$\binom{2n}{n}P_n(a,b) = \sum_{k=-\infty}^{\infty}\left[\binom{2n}{n+kc} - \binom{2n}{n+a+kc}\right],$$

where $c = a + b$, a result due to Gnedenko and Rvačeva [5]. Moreover, from (2) and (1.3),

$$(3) \qquad 2^{-2n}\binom{2n}{n}P_n(a, b) = (4/c)\sum_{k=1}^{[(c-1)/2]}(\sin k\pi a/c)^2 \cos k\pi/c)^{2n},$$

where $c = a + b$, especially,

$$(4) \qquad 2^{-2n}\binom{2n}{n}P_n(a, a) = (2/a)\sum_{k=1}^{[a/2]}(\cos\,(k - \tfrac{1}{2})\pi/a)^{2n}.$$

Massey [6] gave a table of $P_n(a, a)$ for $n \leqq 40$, $a \leqq 13$, (in his notation, $a =$

$k + 1$). In computing this table from (4), the resulting series would contain at most six terms, most of which are neglegibly small for (say) $n \geqq 10$.

Applying Theorem 1 to (2), one obtains an asymptotic development for $P_n(a, b)$. In fact, let

$$(5) \qquad g_r = 4(\alpha/\pi)^{\frac{1}{2}} \sum_{k=1}^{\infty} \sin k\pi a/c \sin k\pi b/c \, e^{-\alpha k^2}(2\alpha k^2)^r,$$

where $c = a + b$, $\alpha = \pi^2 n/c^2$; an equivalent formula is

$$(6) \qquad (-1)^r g_r = \sum_{k=-\infty}^{\infty} \{H_{2r}^*(2kc(2n)^{-\frac{1}{2}}) - H_{2r}^*((2a + 2kc)(2n)^{-\frac{1}{2}})\},$$

(both formulae being especially simple if $a = b$). Then, for each fixed integer $m \geqq 1$, there exists a constant $M$, independent of $a$, $b$, $n$, such that

$$(7) \qquad \left| 2^{-2n} \binom{2n}{n} P_n(a, b) - (2/\pi)^{\frac{1}{2}} \sum_{\mu=0}^{m-1} (2n)^{-\mu-\frac{1}{2}} \sum_{h=0}^{\mu} (-1)^h A_{\mu h} \, g_{\mu+h} \right| \leqq M n^{-m-\frac{1}{2}}$$

holds true for each choice of the positive *integers* $a$, $b$, $n$. Moreover, from Stirling's formula, for $n$ large,

$$(8) \qquad 2^{2n} \binom{2n}{n}^{-1} \sim (\pi n)^{\frac{1}{2}}(1 + 1/(8n) + 1/(128n^2) - 5/(1024n^3) + \cdots).$$

Combining (7) and (8), one obtains an asymptotic expansion of $P_n(a, b)$ in powers of $1/n$ with a remainder $O(n^{-m})$ holding uniformly with respect to the integers $a$ and $b$, ($m = 1, 2, \cdots$). For instance, the special case $m = 4$ yields

$$(9) \qquad \begin{aligned} P_n(a, b) = {} & g_0 + (3g_0 - g_2)/(24n) + (\tfrac{9}{2}g_0 - 3g_2 - \tfrac{16}{5}g_3 + \tfrac{1}{2}g_4)/(24n)^2 \\ & + (-\tfrac{135}{2}g_0 - \tfrac{9}{2}g_2 - \tfrac{48}{5}g_3 - \tfrac{711}{70}g_4 + \tfrac{16}{5}g_5 - \tfrac{1}{6}g_6)/(24n)^3 + O(n^{-4}). \end{aligned}$$

The weaker result

$$P_n(a, b) = g_0 + (3g_0 - g_2)/(24n) + o(n^{-1})$$

is due to Gnedenko [3].

Finally, from (2), (3.1), (3.2), we have the expansion

$$2^{-2n} \binom{2n}{n} P_n(a, \infty) \sim (2/\pi)^{\frac{1}{2}} \sum_{\mu=0}^{\infty} (-1)^{\mu}(2n)^{-\mu-1/2} \sum_{h=0}^{\mu} A_{\mu h}\{H_{2\mu+2h}^*(0) - H_{2\mu+2h}^*(2an^{-\frac{1}{2}})\}.$$

Using (8), one obtains results of the type

$$(10) \qquad P_n(a, \infty) = 1 - e^{-a^2/n}\{1 + a^2(1 - a^2/(3n))/(2n^2)\} + O(n^{-2}),$$

the remainder holding uniformly in $a$. Here, (10) contains a result due to Gnedenko [3].

*Remark.* The reader should note that (9) holds only for positive *integer* values of $a$ and $b$. For instance, from (9) and (5),

$$(11) \qquad P_n(a, a) = G_0(\pi^2 n/(4a^2)) + O(n^{-1}), \qquad (a = 1, 2, \cdots),$$

where $G_0(x) = 4(x/\pi)^{\frac{1}{2}}(e^{-x} + e^{-9x} + e^{-25x} + \cdots)$. Suppose that one wants to choose the integer $a_\beta$ such that $P_n(a_\beta, a_\beta)$ is close to a given number $\beta$, (say, $\beta = .95$). From existing tables, one can find $x_0$ such that $G_0(x_0) = \beta$. Now, for the reasonable choice of $a_\beta$ as the smallest integer $\geqq \pi/2(n/x_0)^{\frac{1}{2}}$, one can only say that $\pi^2 n/(4a^2) = x_0 + O(n^{-\frac{1}{2}})$, thus, from (11), $P_n(a_\beta, a_\beta) = \beta + O(n^{-\frac{1}{2}})$. On the other hand, if $n$ is large and $a_\beta$ has been chosen, (say) in the above manner, formula (9) will yield an excellent approximation to $P_n(a_\beta, a_\beta)$.

**5. The range of cumulative sums.** Let $\zeta_1, \zeta_2, \cdots$ be independent random variables, each assuming only the values $+1$ and $-1$ with equal probability. Further, let $R_n$ denote the range of the cumulative sums $z_0, \cdots, z_n$, $(z_m = \zeta_1 + \cdots + \zeta_m, z_0 = 0)$. Thus, $R_n = U_n + V_n$, where

$$-U_n = \text{Min}(z_0, \cdots, z_n), \qquad V_n = \text{Max}(z_0, \cdots, z_n).$$

Note that $R_n$, $U_n$ and $V_n$ assume only the values 0, 1, $\cdots$, $n$. In this section, by applying Lemma 4 to the exact formula (1) below, we shall obtain a complete asymptotic expansion for the distribution of $R_n$. For each positive integer $m$, it yields an approximation to $P(R_n < r)$ with error smaller than $Cn^{-m-\frac{1}{2}}$, $C$ denoting a constant independent of $n$ and $r$.

LEMMA 6. *We have*

$$(1) \qquad P(R_n < r) = A_{r+1}(n) - A_r(n), \qquad (r = 1, 2, \cdots),$$

*where*

$$(2) \qquad A_c(n) = (1/c) \sum_{k=1}^{c-1} (1 - (-1)^k) \cot^2 k\pi/(2c) (\cos k\pi/c)^n.$$

PROOF. From the definition of $p_n(i, j, c)$ in Section 1, (replacing $z_n$ by $z_n' = a + z_n$), we have, for positive integers $a$, $b$,

$$P(U_n < a, V_n < b, z_n = j - a) = p_n(a, j, a + b).$$

Further, $p_n(a, j, a + b) = 0$ if $j \leqq 0$ or $j \geqq a + b$, hence,

$$P(U_n < a, V_n < b) = \sum_{j=1}^{a+b-1} p_n(a, j, a + b).$$

Moreover, for $r = 1, 2, \cdots$,

$$P(U_n + V_n < r)$$
$$= \sum_{a=1}^{r} \{P(U_n < a, V_n < r - a + 1) - P(U_n < a, V_n < r - a)\}.$$

From $U_n + V_n = R_n$, $P(U_n < r, V_n < 0) = 0$, it follows that (1) holds with

$A_c(n) = \sum_{i=1}^{c-1}\sum_{j=1}^{c-1} p_n(i, j, c)$. Using (1.2), the latter formula easily implies (2).

REMARK. A formula equivalent to (2) is

$$A_c(n) = 2^{-n+1} \sum_{0 \leqq m \leqq n/2} \binom{n}{m} f_c(n - 2m).$$

Here, $f_c(x)$ is defined by $f_c(0) = (c - 1)/2$, $f_c(h) = c - 2h$, $(h = 1, \cdots, c - 1)$, $f_c(c) = -c + 1$, $f_c(sc + h) = (-1)^s f_c(h)$, $(s = 1, 2, \cdots; h = 1, \cdots, c)$. We omit the proof.

Transforming in (2) the terms with $k > c/2$ to the summation variable $k' = c - k$, we have

$$(3) \quad A_c(n) = (1/c) \sum_{k=1}^{[(c-1)/2]} \{(1 - (-1)^k) \cot^2 k\pi/(2c) + (1 - (-1)^{c-k}) \tan^2 k\pi/(2c)\} (\cos k\pi/c)^n.$$

Applying Lemma 4 to (3), one may derive the asymptotic expansion of $A_c(n)$ for large $n$. For convenience, we shall restrict ourselves to the case that, in (3), $n$ is an odd positive integer, $c$ an even positive integer, thus, from (3),

$$A_c(n) = (8/c) \sum_{\substack{k=1 \\ k \equiv 1 (\text{mod } 2)}}^{c/2-1} \operatorname{cosec}^2 k\pi/c \, (\cos k\pi/c)^{n+1}.$$

In view of (2.1), the latter formula may be written as

$$(4) \quad A_c(n - 1) = (8/c) \sum_{\substack{k=1 \\ k \equiv 1 (\text{mod } 2)}}^{c/2-1} e^{-\alpha k^2} f\left(-\frac{4}{n} (\alpha k^2)^2, \frac{2}{n} \alpha k^2\right),$$

($c$ and $n$ even), where

$$(5) \quad \alpha = \pi^2 n/(2c^2)$$

and

$$(6) \quad f(u, w) = \operatorname{cosec}^2 w^{\frac{1}{2}} \, e^{u\varphi(w)}.$$

Here, from Lemma 1,

$$(7) \quad e^{u\varphi(w)} = \sum_{\mu=0}^{\infty} \sum_{h=0}^{\mu} A_{\mu h} u^h w^{\mu-h}, \qquad (|w| < \pi^2/4).$$

Further, differentiating the well-known Taylor expansion of $\cot z$ about 0,

$$(8) \quad \operatorname{cosec}^2 w^{\frac{1}{2}} = w^{-1} \sum_{\nu=0}^{\infty} C_\nu w^\nu, \qquad (|w| < \pi^2),$$

where $C_0 = 1$ and

$$(9) \quad C_\nu = (2\nu - 1)2^{2\nu} B_\nu/(2\nu)! \qquad (\nu = 1, 2, \cdots),$$

$B_\nu$ denoting the $\nu$th Bernoulli number, $(B_1 = 1/6, \cdots; C_1 = 1/3, C_2 = 1/15,$

$C_3 = 2/189$, $C_4 = 1/675, \cdots$ ). Hence, from (6), (7) and (8), for $|w| < \pi^2/4$ and arbitrary values $u$,

$$f(u, w) = w^{-1} \sum_{\mu=0}^{\infty} \sum_{h=0}^{\mu} B_{\mu h} u^h w^{\mu-h}, \tag{10}$$

where

$$B_{\mu h} = \sum_{\nu=0}^{\mu-h} C_\nu A_{\mu-\nu,h} > 0; \tag{11}$$

here, $B_{\mu\mu} = A_{\mu\mu}$, $B_{\mu 0} = C_\mu$ (from $A_{00} = 1$, $A_{\mu 0} = 0$ for $\mu > 0$), thus, $B_{00} = 1$, $B_{10} = 1/3$, $B_{11} = 1/12$, $B_{20} = 1/15$, $B_{21} = 1/20$, $B_{22} = 1/288$.

THEOREM 2. *Let*

$$G_r = G_r(\alpha) = 4(\alpha/\pi)^{\frac{1}{2}} \sum_{\substack{k=1 \\ k \equiv 1 \pmod 2}}^{\infty} e^{-\alpha k^2}(2\alpha k^2)^r, \tag{12}$$

*where $\alpha$ is given by* (5). *For $r = 0, 1, \cdots$ an equivalent formula is*

$$G_r = (-1)^r \sum_{k=-\infty}^{\infty} (-1)^k H_{2r}^*(k\pi(2\alpha)^{-\frac{1}{2}}). \tag{13}$$

*Then, for each positive integer $m$ and each positive constant $K$, there exists a constant $M > 0$, not depending on $c$ or $n$, such that*

$$T_m = A_c(n-1) - (8/\pi)^{\frac{1}{2}} \sum_{\mu=0}^{m-1} n^{-\mu+\frac{1}{2}} \sum_{h=0}^{\mu} (-1)^h B_{\mu h} G_{\mu+h-1} \tag{14}$$

*satisfies*

$$|T_m| \leqq M(e^{-Kn^{\frac{1}{4}}} + n^{-m+\frac{1}{2}} e^{-\alpha}(1 + \alpha^{2m-\frac{1}{2}})), \tag{15}$$

*for each choice of the even positive integers $n$ and $c$.*

PROOF. Let $n$, $c$ denote even positive integers, thus, (4) holds true. Further, let $m \geqq 1$ be a given integer, $K > 0$ a given constant, $K_1$ a fixed constant $> K$. Without loss of generality, we may assume that $n^{\frac{1}{4}} \geqq K_1 \geqq 1$. Let $\lambda > 0$ be defined by $\alpha\lambda^2 = K_1 n^{\frac{1}{2}}$. Then

$$(\pi\lambda/c)^2 = \frac{2}{n} \alpha\lambda^2 = 2n^{-\frac{1}{2}} K_1 \leqq 2 < \pi^2/4, \tag{16}$$

thus, $\lambda < c/2$. From Lemma 1, $\varphi(w) \geqq 0$ for $0 \leqq w \leqq \pi^2/4$, hence, from (6), the contribution to the right hand side of (4) of the terms with $k > \lambda$ is at most equal to $(8/c)$ $(c/4)$ $\operatorname{cosec}^2$ $((2/n)\alpha\lambda^2)^{\frac{1}{2}} e^{-\alpha\lambda^2} \leqq (\pi^2/2)((2/n)K_1 n^{\frac{1}{2}})^{-1} e^{-K_1 n^{\frac{1}{4}}} = O(e^{-Kn^{\frac{1}{4}}})$. Hence, from (4) and (5),

$$A_c(n-1) = O(e^{-Kn^{\frac{1}{4}}}) + (8/\pi)(2\alpha/n)^{\frac{1}{2}} S, \tag{17}$$

where

$$S = \sum_{\substack{1 \leq k \leq \lambda \\ k \equiv 1 \pmod 2}} e^{-\alpha k^2} f\left(-\frac{4}{n}(\alpha k^2)^2, \frac{2}{n} \alpha k^2\right).$$

We now apply Lemma 4 with $\beta = \alpha$, $\sigma = -4/n$, $\tau = 2/n$, $p = 2$, $q = 1$, $s = 1$, $u_1 = 4K_1^2$, $(u_0 > u_1$ arbitrary), $w_1 = 2 < \pi^2/4 = w_0$. Then (2.13) holds, from (16) and $(4/n)(\alpha\lambda^2)^2 = 4K_1^2 = u_1$. Moreover, $\alpha\lambda^2 = K_1 n^{\frac{1}{2}} \geqq K_1 \geqq 1$. Hence, in view of (10), Lemma 4 yields

$$(18) \qquad | S - S_m | \leqq M\alpha^{-\frac{1}{2}} n(n^{-\frac{1}{4}} e^{-K_1 n^{\frac{1}{2}}} + n^{-m} e^{-\alpha}(1 + \alpha^{2m-\frac{1}{2}})),$$

$M$ denoting a constant independent of $\alpha$ and $n$. Here,

$$(19) \qquad S_m = \sum_{\mu=0}^{m-1} \sum_{h=0}^{\mu} B_{\mu h}(-4/n)^h (2/n)^{\mu-h-1} \sum_{\substack{k=1 \\ k \equiv 1 \pmod 2}}^{\infty} e^{-\alpha k^2} (\alpha k^2)^{\mu+h-1}.$$

Thus, if $G_r$ is defined by (12), (15) is an immediate consequence of (17), (18) and (19). That (12) and (13) are equivalent for $r = 0, 1, \cdots$, follows by subtracting the asserted relation of Lemma 5 with $x = \pi$ from that with $x = 0$.

*Remark.* In view of (3) and $\tan^2 w^{\frac{1}{2}} = w + 2w^2/3 + \cdots$, it is easily seen that, for $m \leqq 2$, the estimate (15) holds for all positive integers $n$ and $c$.

Let us introduce the distribution function

$$(20) \qquad F_n(r) = P(R_n < r) + P(R_n = r)/2$$

and the quasi-frequency function

$$(21) \qquad f_n(r) = P(R_n = r - 1)/4 + P(R_n = r)/2 + P(R_n = r + 1)/4.$$

From (1),

$$(22) \qquad 2F_{n-1}(c) = A_{c+2}(n - 1) - A_c(n - 1)$$

and

$$(23) \qquad 4f_{n-1}(c + 1) = A_{c+4}(n - 1) - 2A_{c+2}(n - 1) + A_c(n - 1).$$

Hence, applying Theorem 2, one obtains an asymptotic development for the quantities $F_{n-1}(c)$ and $f_{n-1}(c + 1)$, $c$ and $n$ denoting even positive integers.

In order to simplify these expansions, we introduce

$$(24) \qquad \gamma_\mu(\alpha) = \sum_{h=0}^{\mu} (-1)^h B_{\mu h} G_{\mu+h-1}(\alpha),$$

$(\mu = 0, 1, \cdots)$, where $G_r(\alpha)$ is defined by (12) or (13), (the latter only for $r \geqq 0$), $\alpha$ ranging through the positive real numbers. From Theorem 2, for each integer $m \geqq 1$,

$$(25) \qquad A_c(n - 1) = (8/\pi)^{\frac{1}{2}} \sum_{\mu=0}^{m-1} \gamma_\mu(\pi^2 n/(2c^2)) n^{-\mu+\frac{1}{2}} + O(n^{-m+\frac{1}{2}})$$

if $n$ and $c$ are even positive integers, the remainder holding uniformly in $c$. From (12) and Lemma 3, for each integer $r$,

$$(26) \qquad G_r(\alpha) = O(e^{-\alpha}(1 + \alpha^{r+\frac{1}{2}})), \qquad (\alpha > 0).$$

Moreover, from (12),

$$\frac{dG_r}{d\alpha} = -(2\alpha)^{-1}(G_{r+1} - (2r+1)G_r). \tag{27}$$

Hence, letting $\alpha = \pi^2 n/(2c^2)$ and $D = \partial/\partial c$,

$$DG_r(\alpha) = c^{-1}(G_{r+1} - (2r+1)G_r), \tag{28}$$

thus,

$$D^2 G_r = c^{-2}(G_{r+2} - (4r+5)G_{r+1} + (2r+1)(2r+2)G_r). \tag{29}$$

In general,

$$D^s G_r(\alpha) = c^{-s} \sum_{\nu=0}^{s} a_{s\nu}(r) G_{r+\nu}(\alpha), \tag{30}$$

($\alpha = \pi^2 n/(2c^2)$, $s = 0, 1, \cdots$), where the $a_{s\nu}(r)$ are certain constants, independent of $n$ and $c$, which may be computed from the recursion relation $a_{s\nu}(r) = a_{s-1,\nu-1}(r) - (2r + 2\nu + s)a_{s-1,\nu}(r)$, ($a_{s\nu}(r) = 0$ if $\nu < 0$ or $\nu > s$).

It follows from (24) and (30), that

$$D^s \gamma_\mu(\pi^2 n/(2c^2)) = \gamma_{\mu s}(\pi^2 n/(2c^2)) n^{-s/2}, \qquad (s = 0, 1, \cdots), \tag{31}$$

where

$$\gamma_{\mu s}(\alpha) = (2\alpha/\pi^2)^{s/2} \sum_{h=0}^{\mu} \sum_{\nu=0}^{s} (-1)^h B_{\mu h} a_{s\nu}(\mu + h - 1) G_{\mu+h+\nu-1}(\alpha). \tag{32}$$

Here, the functions $\gamma_{\mu s}(\alpha)$ are explicitly known, for instance, from $B_{00} = 1$, (12) and (28),

$$\gamma_{01}(\alpha) = (2/\pi)^{\frac{1}{2}} \sum_{k=0}^{\infty} e^{-\alpha(2k+1)^2}(2\alpha + (2k+1)^{-2}). \tag{33}$$

Further, from (13) and (29),

$$\gamma_{02}(\alpha) = 2 \sum_{k=1}^{\infty} (-1)^{k-1} e^{-(k\pi)^2/(4\alpha)} k^2. \tag{34}$$

Observe that, from (26) and (32), $\gamma_{\mu s}(\alpha)$ is a bounded function of $\alpha$, $\alpha > 0$, whenever $s \geqq 1$. Hence, from (31), letting

$$f_\mu(n, c) = \gamma_\mu(\pi^2 n/(2c^2)),$$

we have, for each positive integer $q$, and $\Delta > 0$,

$$f_\mu(n, c + \Delta) - f_\mu(n, c) = \sum_{s=1}^{q-1} n^{-s/2} \gamma_{\mu s}(\pi^2 n/(2c^2)) \Delta^s/s! + O(\Delta^q n^{-q/2}), \tag{35}$$

the remainder holding uniformly in $c$. Finally, letting $q = 2m - 2\mu$, (22), (23) and (25) easily imply the following result.

THEOREM 3. *Let $F_n(r)$, $f_n(r)$ be defined by* (20) *and* (21). *Then, for each positive integer m, there exists a constant M, independent of n and c, such that*

$$(36)\qquad \left| F_{n-1}(c) - (2/\pi)^{\frac{1}{2}} \sum_{\mu=0}^{m-1} \sum_{s=1}^{2m-2\mu-1} n^{-(2\mu+s-1)/2}\gamma_{\mu s}(\pi^2 n/(2c^2))2^s/s! \right| \leqq Mn^{-m+\frac{1}{2}},$$

*and*

$$(37)\qquad \left| f_{n-1}(c+1) - (2/\pi)^{\frac{1}{2}} \sum_{\mu=0}^{m-2} \sum_{s=1}^{2m-2\mu-1} n^{-(2\mu+s-1)/2}\gamma_{\mu s}(\pi^2 n/(2c^2))2^s(2^{s-1}-1)/s! \right| \leqq Mn^{-m+\frac{1}{2}},$$

*for each choice of the even positive integers n and c, where $\gamma_{\mu s}(\alpha)$ is defined by* (32). *Here, for each $\mu$, $s \geqq 1$, $\gamma_{\mu s}(\alpha)$ is a bounded function of $\alpha$, $\alpha > 0$.*

Note that, from the remark following the proof of Theorem 2, (36) and (37) hold for each choice of the positive integers $n$ and $c$, provided $m \leqq 2$. From (36), applied with $m = 2$, we have $F_{n-1}(c) = (8/\pi)^{\frac{1}{2}}(\gamma_{01}(\alpha) + n^{-\frac{1}{2}}\gamma_{02}(\alpha) + n^{-1}(\gamma_{11}(\alpha) + 2\gamma_{03}(\alpha)/3) + O(n^{-\frac{3}{2}})$, with $\alpha = \pi^2 n/(2c^2)$, especially, from (33),

$$(38)\qquad F_{n-1}(c) = (8/\pi^2) \sum_{k=0}^{\infty} e^{-\pi^2(2k+1)^2 n/2c^2}(\pi^2 n/c^2 + (2k+1)^{-2}) + O(n^{-\frac{1}{2}}).$$

Further, from (37), applied with $m = 2$,

$$f_{n-1}(c+1) = (8/\pi)^{\frac{1}{2}}(n^{-\frac{1}{2}}\gamma_{02}(\alpha) + 2n^{-1}\gamma_{03}(\alpha)) + O(n^{-\frac{3}{2}}),$$

with $\alpha = \pi^2 n/(2c^2)$, especially, from (34),

$$(39)\qquad f_{n-1}(c-1) = 8(2\pi n)^{-\frac{1}{2}} \sum_{k=1}^{\infty} (-1)^{k-1} e^{-(kc)^2/(2n)} k^2 + O(n^{-\frac{1}{2}}).$$

As was shown by Feller [2], cf. also Darling and Siegert ([1], p. 638), the slightly weaker result, obtained by replacing in (39) the remainder $O(n^{-\frac{1}{2}})$ by $o(1)$, holds whenever the $\zeta_n$ are independently and identically distributed random variables, $E(\zeta_n) = 0$, $\text{Var}(\zeta_n) = 1$.

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