ON A CHARACTERIZATION OF COVARIANCES

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1. Introduction. Let $F(s, t)$, $-\infty < s, t < \infty$, be a covariance function, that is to say $F(s, t) = \overline{F(t, s)}$ and $F(s, t)$ is non-negative definite. Let $m(s)$ be any complex valued function on $-\infty < s < \infty$. It is trivial that then

$$F(s, t) + m(s)\overline{m(t)}$$

is also a covariance. However, this is no longer true if we consider instead

$$F(s, t) - m(s)\overline{m(t)}. \tag{1}$$

In this paper we obtain a set of necessary and sufficient conditions on $m(s)$ in order that (1) be a covariance under the restriction that $F(s, t)$ is a stationary covariance; i.e., $F(s, t) = F(s - t)$. We also indicate an application of the result to the problem of estimating the mean value of a stochastic process.

2. Main results.

THEOREM 1. *Let* $R(t)$ *be a continuous stationary covariance function with* $R(0)$ *finite. Let* $m(s)$ *be any function on* $-\infty < s < \infty$. *Then a necessary and sufficient condition that*

$$R(s, t) = R(s - t) - m(s)\overline{m(t)} \tag{2}$$

be a covariance function is that $m(t)$ *have the representation*

$$m(t) = \int_{-\infty}^{\infty} \exp(itx)\, d\mu, \tag{3}$$

where $\mu(\cdot)$ *is a function of bounded variation, and that, further,*

$$\int_{-\infty}^{\infty} |d\mu/dG|^2\, dG \leqq 1, \tag{4}$$

$G(\cdot)$ *being the spectral distribution corresponding to* $R(t)$, *so that*

$$R(t) = \int_{-\infty}^{\infty} \exp(itx)\, dG. \tag{5}$$

PROOF. *Necessity:* Let $R(s, t)$ given by (2) be a covariance. Then we can (see [1], p. 72) construct a Gaussian process $y(t)$, $-\infty < t < \infty$, with zero mean so that $E[y(s)\overline{y(t)}] = R(s, t)$. Now, since $R(t, t)$ must be non-negative if (2) is to yield a covariance, $m(t)$ is necessarily bounded. Letting

$$x(t) = y(t) + m(t),$$

we have $E[x(s)\overline{x(t)}] = R(s - t)$, so that the $x(t)$ process has finite first and second moments and is stationary in the wide sense. Moreover, $R(t)$ is continuous. Using the spectral representation theorem ([1], p. 527), we have

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$$x(t) = \int_{-\infty}^{\infty} \exp(it\lambda)\, dZ(\lambda),$$

where $Z(\lambda)$ has orthogonal increments with $E[|dZ(\lambda)|^2] = dG(\lambda)$. Now let $\mu(\lambda) = E[Z(\lambda) - Z(\lambda_0)]$ for some fixed λ_0. Then, for any finite sequence of non-overlapping intervals $\{a_i, b_i\}$, we have

$$|\sum_i (\mu(b_i) - \mu(a_i))|^2 \leqq E[|\sum_i (Z(b_i) - Z(b_i))|^2] = \sum_i \int_{a_i}^{b_i} dG,$$

so that $\mu(\lambda)$ is of bounded variation, and also absolutely continuous with respect to the measure dG on the Borel field of the real line. Moreover

$$E[x(t)] = m(t) = \int_{-\infty}^{\infty} \exp(it\lambda)\, d\mu(\lambda).$$

To prove (4), let $f(\cdot)$ be any function in $L_2(dG)$, the L_2 space with respect to the measure dG. Then

$$E\left[\int_{-\infty}^{\infty} f(\lambda)\, dZ(\lambda)\right] = \int_{-\infty}^{\infty} \mathrm{f}(\lambda)\, \mathrm{d}\mu(\lambda)$$

defines a linear functional on $L_2(dG)$. Denoting this functional by $L(f)$, we have

$$|L(f)|^2 \leqq \int_{-\infty}^{\infty} |f(\lambda)|^2\, dG(\lambda) = \|f\|^2,$$

$\|f\|$ being the $L_2(dG)$ norm. Hence the norm, $\|L\|$, of the functional satisfies $\|L\| \leqq 1$. Moreover, there is a $g(\cdot)$ in $L_2(dG)$ so that for every $f(\cdot)$ in $L_2(dG)$,

$$\int_{-\infty}^{\infty} f(\lambda)\, d\mu = \int_{-\infty}^{\infty} f(\lambda) g(\lambda)\, dG.$$

This implies that $g = d\mu/dG$ and since

$$\int_{-\infty}^{\infty} |g(\lambda)|^2\, dG = \|L\| \leqq 1$$

necessity follows.

Sufficiency. If $R(t, s)$ is defined by (2), and if the conditions (3) and (4) are satisfied, we have, for any finite sequence of numbers $\{a_i\}$ and any $\{t_i\}$,

$$\sum \sum a_i R(t_i, t_j) \bar{a}_j,$$

$$= \int_{-\infty}^{\infty} |\sum a_j \exp(it_j\lambda)|^2\, dG - \left| \int_{-\infty}^{\infty} \sum a_j\, d\mu/dG \exp(it_j\lambda)\, dG \right|^2,$$

where the second term, by the Schwartz inequality, is

$$\leqq \int_{-\infty}^{\infty} |\sum_j a_j \exp(it_j\lambda)|^2\, dG \int_{-\infty}^{\infty} |d\mu/dG|^2\, dG$$

$$\leqq \int_{-\infty}^{\infty} |\sum_j a_j \exp(it_j\lambda)|^2\, dG,$$

using (4). Hence $\sum\sum a_i R(t_i, t_j)\bar{a}_j \geqq 0$ as required.

If $R(s, t)$ given by (2) is also required to be stationary, a stronger result is the following corollary:

COROLLARY 1. *If* $\mathrm{R}(t)$ *is a stationary continuous covariance, a necessary and sufficient condition that*

$$R(s, t) = R(s - t) - m(s)\overline{m(t)}$$

be also a stationary covariance is that

$$m(t) = a \exp(i\lambda_0 t), \qquad \lambda_0 \text{ real}, \tag{6}$$

where the spectral distribution, $G(\cdot)$, *corresponding to* $\mathrm{R}(t)$ *has a jump (atomic part) at* λ_0 *not less than* $| a |^2$.

PROOF. We first note that the required stationarity implies that

$$m(s)\overline{m(t)} = f(s - t),$$

which in turn necessarily implies that $m(s)$ be of the form $m(s) = a \exp(i\lambda_0 s)$ since, by Theorem 1, $m(s)$ is continuous. The rest of the corollary is immediate from Theorem 1.

We have so far assumed $R(t)$ to be continuous, so that (5) holds. In the absence of (5), $m(t)$ may not have the representation (3). To see this, we have only to take a non-Lebesgue measurable character of the real line $\chi(t)$, and set $R(t) = 2\chi(t)$, $m(t) = \chi(t)$, in (2). Then $m(t)$ cannot have the form (3) or (6), since this would imply continuity, which is false.

Theorem 1 has, as may be expected, an immediate paraphrase for stochastic processes. In the terminology of Doob ([1], p. 95) a stochastic process

$$x(t), -\infty < t < \infty,$$

is stationary in the wide sense if $E[| x(t) |^2]$ is finite and

$$E[x(t)\overline{x(s)}] = R(t - s), \tag{8}$$

without any additional assumption on the mean value $E[x(t)]$. As Doob has pointed out, the usual assumption that $E[x(t)]$ be a constant, is unnatural. On the other hand, (8) does impose a restriction on the character of $E[x(t)]$, and this may be read from Theorem 1. Thus the following result may be stated.

THEOREM 2. *Let* $x(t)$ *be a stochastic process stationary in the wide sense, and continuous in the mean of order two. Let* $\mathrm{R}(t)$ *be its covariance function (given by (8)) with spectral distribution* $G(\lambda)$. *Then a necessary and sufficient condition that a function* $m(t)$, $-\infty < t < \infty$, *be the mean value of such a process is that it satisfy* (3) *and* (4).

3. Application and extensions. As an application of this result we shall consider a problem that arises in the estimation of the mean value of a stochastic process. It has been treated by Grenander [2], [3] using the special concept of the Hellinger integral. We shall, for simplicity, use the discrete parameter version, since this has no essential bearing on the problem. Moreover, since the discrete parameter versions of Theorems 1 and 2 are obvious, we shall not state them separately. Thus let

$$y_n = x_n + m\mu_n \tag{9}$$

be a time series, $-\infty < n < \infty$, where x_n is a stationary time series with (finite) covariance $R(n)$ and zero mean, and where $E[y_n] = m\mu_n$. Further, let μ_n have the form

$$\mu n = \int_{-1/2}^{1/2} \exp 2\pi i n\lambda \, d\mu.$$

It is desired to estimate the constant m from the $\{y_n\}$ series and the question is: under what conditions do we have consistent (in the mean square sense) linear unbiassed estimates for m? By this we mean that we wish to know whether it is possible to construct a sequence ζ_n of random variables of the form

$$\zeta_n = \sum_{-n}^{n} c_k^n y_k ,$$

where the coefficients c_k^n are to be so chosen that $E[\zeta_n] = m$, and where further, the sequence ζ_n converges in the mean of order two to m. An answer to this question is given by Theorem 3.

THEOREM 3. *A necessary and sufficient condition that consistent (in the mean square sense) linear unbiassed estimates for* m *in* (9) *exist is that* $\{m\mu_n\}$ *not be a member of the class of sequences which can serve as the mean value of a wide sense stationary time series with covariance* $R(n)$, *for any non-zero value of* m.

PROOF. *Necessity.* Suppose, contrariwise, that, for some m_0 not equal to zero, the sequence $\{m_0\mu_n\}$ can be the mean value of a series with covariance $R(n)$. Then paraphrasing Theorem 2 to the discrete parameter case, we must have

$$m_0\mu_n = m_0 \int_{-1/2}^{1/2} \exp 2\pi i n\lambda \, d\mu,$$

where

$$\int_{-1/2}^{1/2} | d\mu/dG |^2 \, dG \leqq 1/m_0^2 < \infty,$$

with

$$R(n) = \int_{-1/2}^{1/2} \exp (2\pi i n\lambda) \, dG.$$

Next, let $m_n^* = \sum_{-n}^{n} c_k^n y_k$ be any linear unbiassed estimate for m. Then we must have $m = \sum_{-n} c_k^n E[y_k] = m\sum_{-n}^{n} c_k^n \mu_k$, so that, if

$$P_n(\lambda) = \sum c_k^n \exp (2ik\pi\lambda),$$

then $\int_{1/2}^{1/2} P_n(\lambda) d\mu = 1$. However,

$$\left| \int_{-1/2}^{1/2} P_n(\lambda) \, d\mu \right|^2 \leqq \int_{-1/2}^{1/2} | P_n(\lambda) |^2 \, dG \int_{-1/2}^{1/2} | d\mu/dG |^2 \, dG.$$

Hence

$$\int_{-1/2}^{1/2} |P_n(\lambda)|^2 \, dG \geqq 1 \Big/ \left[\int_{-1/2}^{1/2} |d\mu/dG|^2 \, dG \right].$$

Or,

$$\text{var } m_n^* \geqq m_0^2 > 0.$$

Thus no consistent linear unbiassed estimate is possible.

Sufficiency. Suppose $m\mu_n$ cannot be the mean value of a time series with covariance $R(n)$. This can happen only in one of two following ways:

(i) $d\mu$ is absolutely continuous with respect to dG, but

$$\int_{-1/2}^{1/2} |d\mu/dG|^2 \, dG = +\infty.$$

(ii) $d\mu$ is not absolutely continuous with respect to dG. First, suppose (ii) is true. Then we can find a Borel set B on which

$$\int_B dG = 0 \quad \text{and} \quad \int_B d\mu \neq 0.$$

Under these conditions, it is possible to construct a sequence of polynomials $P_n(\lambda)$ in $\exp(2\pi i\lambda)$ such that

$$\int_{-1/2}^{1/2} |P_n(\lambda)|^2 \, dG \to \int_B dG = 0,$$

and

$$\int_{-1/2}^{1/2} P_n(\lambda) \, d\mu = 1.$$

But with each such polynomial, $P_n(\lambda) = \sum c_k^n \exp(2\pi ik\lambda)$, we can associate the linear unbiassed estimate $m_n^* = \sum c_k^n y_k$, whose variance tends to zero, proving the existence of consistent linear unbiassed estimates, as required.

Next, suppose alternative (i) holds. Then $g(\lambda) = d\mu/dG$ is Borel measurable, and

$$\int_{-1/2}^{1/2} |g(\lambda)| \, dG < \infty \tag{10}$$

$$\int_{-1/2}^{1/2} |g(\lambda)|^2 \, dG = +\infty. \tag{11}$$

In view of (10), for any polynomial $P(\lambda)$ in $\exp(2\pi i\lambda)$, we have

$$\int_{-1/2}^{1/2} |P(\lambda)| \, |g(\lambda)| \, dG < \infty,$$

so that

$$\int_{-1/2}^{1/2} P(\lambda)\, d\mu = \int_{-1/2}^{1/2} P(\lambda) g(\lambda)\, dG \tag{12}$$

defines a linear functional on the polynomials $P(\lambda)$ which form a dense sub-space of $L_2(dG)$. Now to show the existence of consistent linear unbiassed estimates for m, it is enough to show that we can find a sequence of polynomials $P_n(\lambda)$ in exp $(2\pi i\lambda)$ such that

$$\int_{-1/2}^{1/2} P_n(\lambda)\, d\mu \to c \neq 0$$

and

$$\int_{-1/2}^{1/2} |P_n(\lambda)|^2\, dG \to 0.$$

However, if this is not true, (12) would define a bounded linear functional on $L_2(dG)$ (because of continuity on a dense sub-space) thus contradicting (11).

It would appear that the basic result (Theorem 1) is capable of extension to the harmonizable covariances of Loève [4]. In this connection, it may be noted that $R(s, t)$ in Theorem 1 is easily verified to be harmonizable.

REFERENCES

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