A NOTE ON MULTIPLE INDEPENDENCE UNDER MULTIVARIATE NORMAL LINEAR MODELS

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1. Introduction. S. N. Roy and Bargmann [3] used S. N. Roy's union-intersection method as the basis for providing tests and confidence intervals in the following cases:

i) $\mathbf{y}' = (y_1, \cdots, y_p) \sim N(\boldsymbol{\mu}', \Sigma)$, $H_o: \sigma_{ij} = 0, i \neq j$.
ii) $\mathbf{y}' \sim N(\boldsymbol{\mu}', \Sigma)$, but $\mathbf{y}'$ is partitioned into k sets or blocks or sizes $p_1, \cdots, p_k$. $H_o: \Sigma_{ij} = 0, i \neq j$, where Σ_{ij} is the covariance matrix between blocks i and j.

J. Roy [1] considered the following additional cases:

iii) $Y: n \times p$, $(y_{1j}, \cdots, y_{pj}) \sim N(—, \Sigma)$, $j = 1, \cdots, n$, $EY = A\theta$. $A: n \times m$ has rank $r \leqq n - p$ and is known, θ is unknown. Let $\Phi = B\theta$ be estimable, $B: t \times m$. $H_o: \Phi = 0$.
iv) $(y_1, \cdots, y_p) \sim N(\boldsymbol{\mu}', \Sigma)$. $H_o: \Sigma = \Sigma_0$ (specified).
v) $(y_1, \cdots, y_p) \sim N(\boldsymbol{\mu}', \Sigma_1)$, $(x_1, \cdots, x_p) \sim N(\boldsymbol{\nu}', \Sigma_2)$, $H_o: \Sigma_1 = \Sigma_2$.

In this note we shall consider the following modification of (iii):

vi) $Y: n \times p$, $(y_{1j}, \cdots, y_{pj}) \sim N(—, \Sigma)$, $j = 1, \cdots, n$, $EY = A\theta$ (as in (iii)). $H_o: \sigma_{ij} = 0, i \neq j$.

2. Step-down procedure to test H_o in (vi). In the notation of [1], denote the ith columns of the matrices Y and θ by $\mathbf{y}_i$ and $\boldsymbol{\theta}_i$ respectively and write

$$Y_i = [\mathbf{y}_1, \cdots, \mathbf{y}_i], \qquad \theta_i = [\boldsymbol{\theta}_1, \cdots, \boldsymbol{\theta}_i]$$

and $\Sigma_i = (\sigma_{jk})$, $j, k = 1, \cdots, i$.

If Y_i is fixed, the n elements of $\mathbf{y}_{i+1}$ are distributed independently and normally with the same variance σ_{i+1}^2 and expectations given by

$$E(\mathbf{y}_{i+1} \mid Y_i) = A\boldsymbol{\eta}_{i+1} + Y_i\boldsymbol{\beta}_i, \tag{1}$$

where $\boldsymbol{\beta}_i': 1 \times i$ is the row vector,

$$\boldsymbol{\beta}_i' = (\sigma_{1,i+1}; \cdots; \sigma_{i,i+1})\Sigma_i^{-1}, \tag{2}$$

and $\boldsymbol{\eta}_{i+1}: m \times 1$ is the column vector given by

$$\boldsymbol{\eta}_{i+1} = \boldsymbol{\theta}_{i+1} - \theta_i\boldsymbol{\beta}_i, \qquad i = 1, \cdots, p - 1. \tag{3}$$

We note that H_o is true if and only if the hypothesis $H_i: \boldsymbol{\beta}_i = 0$ holds for all $i = 1, \cdots, p - 1$. Now the elements of the vectors $\boldsymbol{\beta}_i$ and $\boldsymbol{\eta}_{i+1}$ may be regarded

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as unknown parameters, and hence, when Y_i is fixed, the hypothesis $H_i: \beta_i = 0$ is a linear hypothesis in univariate analysis with the linear model given by (1).

We observe that rank $Y_i = i$, a.e. and rank $(A \vdots Y_i) = r + i$, a.e. Hence β_i is estimable and the hypothesis H_i is testable. Let $\hat{\beta}_i$ be the Gauss-Markov estimator of β_i in the conditional set-up. Denote the variance-covariance matrix of $\hat{\beta}_i$ by $\sigma_{i+1}^2 C_i$ where $C_i: i \times i$ is a positive-definite matrix. Let $s_i^2/n - r - i$ denote the usual error mean square giving an unbiased estimator of σ_{i+1}^2. Then, as in [1],

$$(4) \qquad F_i \equiv \frac{(\hat{\beta}_i - \beta_i)' C_i^{-1} (\hat{\beta}_i - \beta_i)/i}{s_i^2/(n - r - i)}, \qquad i = 1, \cdots, p - 1,$$

has the F distribution with i and $n - r - i$ degrees of freedom.

Thus the conditional distribution of F_i, given Y_i, does not involve Y_i and hence does not involve $F_1, \cdots, F_{i-1}$. Therefore, the statistics $F_1, \cdots, F_{p-1}$ have independent F distributions with degrees of freedom i and $n - r - i$, $i = 1, \cdots, p - 1$ respectively.

For a preassigned constant α_i, $0 < \alpha_i < 1$, let f_i denote the upper 100 α_i percent point of the F distribution with i and $n - r - i$ degrees of freedom. Then the probability that simultaneously

$$(5) \qquad F_i \leqq f_i, \qquad i = 1, \cdots, p - 1,$$

is equal to $\prod_{i=1}^{p-1} (1 - \alpha_i)$.

Since $H_o \leftrightarrow H_i: \beta_i = 0$, $i = 1, \cdots, p - 1$, we utilize (4) and propose the following test procedure for H_o:
Accept H_o, if

$$(6) \qquad u_i = \frac{\hat{\beta}_i C_i^{-1} \hat{\beta}/i}{s_i^2/n - r - i} \leqq f_i \qquad \text{for all } i = 1, \cdots, p - 1;$$

otherwise reject H_o.

To carry out the test one should first compute u_1. If $u_1 > f_1$, H_o is rejected. If $u_1 \leqq f_1$, u_2 is computed. If $u_2 > f_2$, H_o is rejected. If $u_2 \leqq f_2$, u_3 is computed and so on. The level of significance for this test is obviously $1 - \prod_{i=1}^{p-1} (1 - \alpha_i)$. One possibility is to choose $\alpha_1 = \cdots = \alpha_{p-1}$. We prefer choosing α's so that $f_1 = \cdots = f_{p-1}$, for reasons discussed in [3].

3. Confidence bounds associated with the test.

Now from (4), $F_i \leqq f_i \Rightarrow (\hat{\beta}_i - \beta_i)'(\hat{\beta}_i - \beta_i) \leqq \lambda_{\max}(C_i) l_i^2 s_i^2$ where $l_i^2 = i f_i/(n - r - i)$ and $\lambda_{\max}(C_i)$ is the maximum characteristic root of C_i. Hence, in view of (5), with a probability greater than $\prod_{i=1}^{p-1} (1 - \alpha_i)$,

$$(7) \qquad (\hat{\beta}_i - \beta_i)'(\hat{\beta}_i - \beta_i) \leqq \lambda_{\max}(C_i) l_i^2 s_i^2, \qquad i = 1, \cdots, p - 1.$$

Now (7) implies

$$(8) \qquad \mathbf{a}_i' \hat{\beta}_i - l_i s_i \lambda_{\max}^{1/2}(C_i) \leqq \mathbf{a}_i' \beta_i \leqq \mathbf{a}_i' \hat{\beta}_i + l_i s_i \lambda_{\max}^{1/2}(C_i)$$

for all non-null $\mathbf{a}_i : i \times 1$ such that $\mathbf{a}_i'\mathbf{a}_i = 1$, $(i = 1, \cdots, p-1)$. This again implies

$$(9)\quad \begin{aligned} (\hat{\boldsymbol{\beta}}_i'\hat{\boldsymbol{\beta}}_i)^{1/2} - l_i s_i \lambda_{\max}^{1/2}(C_i) &\leqq (\boldsymbol{\beta}_i'\boldsymbol{\beta}_i)^{1/2} \\ &\leqq (\hat{\boldsymbol{\beta}}_i'\hat{\boldsymbol{\beta}}_i)^{1/2} + l_i s_i \lambda_{\max}^{1/2}(C_i), \quad i = 1, \cdots, p-1. \end{aligned}$$

Thus (9) holds with probability greater than $\prod_{i=1}^{p-1}(1-\alpha_i)$. We may obtain partial statements by choosing some elements of $\mathbf{a}_i$ in (8) to be zero. Thus we have the simultaneous confidence bounds given by (9) for all possible subsets of $\boldsymbol{\beta}_i$ for all $i = 1, \cdots, p-1$ with the confidence coefficient greater than $\prod_{i=1}^{p-1}(1-\alpha_i)$.

4. Remarks.

(a) It will be easily seen that when Y represents a random sample of size n from $N(\boldsymbol{\mu}, \Sigma)$, (1) takes the form

$$E(y_{i+1,k} \mid Y_i) = \mu_{i+1} + \sum_{j=1}^{i} \beta_{ij}(y_{jk} - \mu_j),$$

where $\mathbf{y}_i' = (y_{i1}, \cdots, y_{in})$ and $\boldsymbol{\beta}_i' = (\beta_{i1}, \cdots, \beta_{ii})$, $i = 1, \cdots, p-1$.

If we write $s_{ij} = \sum_{k=1}^{n}(y_{ik} - \bar{y}_i)(y_{jk} - \bar{y}_j)$ and $S_i = (s_{jk})$, $j, k = 1, \cdots, i$, then it is well-known that

$$\hat{\boldsymbol{\beta}}_i = S_i^{-1}\begin{pmatrix} s_{i+1,1} \\ \vdots \\ s_{i+1,i} \end{pmatrix} = \mathbf{b}_i, \qquad C_i = S_i^{-1}$$

and

$$s_i^2 = s_{i+1,i+1} - (s_{i+1,1}; \cdots; s_{i+1,i})S_i^{-1}(s_{i+1,1}; \cdots; s_{i+1,i})',$$

so that

$$u_i = \frac{\mathbf{b}_i'S_i\,\mathbf{b}_i/i}{s_i^2/n-1-i} = \frac{r_{i+1.1,\cdots,i}^2}{1-r_{i+1.1,\cdots,i}^2}\,\frac{n-1-i}{i},$$

where $r_{i+1.1,\cdots,i}$ denotes the multiple correlation coefficient of $(i+1)$ with $(1, \cdots, i)$, thus giving as a special case the test procedure already obtained in [3]. This is, of course, as it should be.

(b) In this model, as in (iii), it is of interest to investigate whether the test of the usual multivariate linear hypothesis of the type

$$(10)\qquad H_0' : \Phi = B\theta = 0,$$

where Φ is estimable, and the above test of independence are quasi-independent (see e.g. Roy [2]). As shown in [1], the step-down test procedure for (10) gives, when Y_i is fixed,

$$(11)\qquad F_i' \equiv \frac{(\hat{\boldsymbol{\phi}}_{i+1} - \boldsymbol{\phi}_{i+1})'D_{i+1}^{-1}(\hat{\boldsymbol{\phi}}_{i+1} - \boldsymbol{\phi}_{i+1})/t}{s_i^2/n-r-1}, \quad i = 0, 1, \cdots, p-1$$

where $\mathbf{\Phi}_{i+1} = B\mathbf{n}_{i+1}$ and the variance-covariance matrix of $\hat{\mathbf{\phi}}_{i+1}$ is $D_{i+1}\sigma_{i+1}^2$.

F_i given by (4) and F_i' given by (11) are, for fixed Y_i, quasi-independent if the numerators, which are marginally distributed as $\chi_i^2\sigma_{i+1}^2/i$ and $\chi_t^2\sigma_{i+1}^2/t$ respectively, are independent. It can be easily verified that χ_i^2 and χ_t^2 are not independent and hence the tests for H_o and H_o' are not quasi-independent. It may be noted that, when Y_i is fixed, the test of $\mathbf{\beta}_i = 0$ is like testing significance of regression, as seen from (1), while the test of $\mathbf{\Phi}_{i+1} = 0$ is like covariance-analysis.

(c) We may consider extension of (vi) to blocks, as in (ii), and test

$$H_o: \Sigma_{ij} = 0,$$

as pointed out by the referee. It is easy to check that a similar step-down procedure with respect to blocks will result in $k - 1$ independent tests in multivariate analysis of variance of the same general structure as in [1] and [3].

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