MEASURABILITY OF EXTENSIONS OF CONTINUOUS RANDOM TRANSFORMS

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1. Summary. Extension theorems of Tietze and Hahn-Banach play an important part in functional analysis. It seems reasonable to deal with similar questions for random transforms. In the present paper some measurability problems arising in connection with this probabilistic generalization are solved.

2. Introduction. First of all we shall introduce some convenient notions, definitions of which follow those given in [1].

Let $(\Omega, \mathfrak{S})$ and $(Z, \mathfrak{Z})$ be two measurable spaces and U a mapping of the space Ω into the space Z so that the inclusion

$$\{\{\omega: U(\omega) \varepsilon B\}: B \varepsilon \mathfrak{Z}\} \subset \mathfrak{S}$$

holds. Then the mapping U will be called a generalized random variable, or, more precisely, a generalized random variable with values in the space $(Z, \mathfrak{Z})$.

If $(\Omega, \mathfrak{S})$ and $(Z, \mathfrak{Z})$ are two measurable spaces, X an arbitrary non-empty set and T a mapping of the Cartesian product $\Omega \times X$ into the space Z satisfying the condition

$$\{\{\omega: T(\omega, x) \varepsilon B\}: x \varepsilon X, B \varepsilon \mathfrak{Z}\} \subset \mathfrak{S},$$

then we shall speak about a random transform, or, more precisely, about a random transform of the Cartesian product $\Omega \times X$ into the space $(Z, \mathfrak{Z})$.

Let us remark that in case Z is a metric space, we usually choose the σ-algebra $\mathfrak{Z}$ as the class of all Borel subsets of the space Z. Under this additional agreement about the σ-algebra $\mathfrak{Z}$, a number of theorems and criteria have been stated in [1]. For the purposes of the present paper Criterion 6 is of most importance:

If Z is a separable Banach space then a mapping U is a generalized random variable if, and only if, for every bounded linear functional f from a subset Δ of the first adjoint Banach space Z^*, where the subset Δ is total on the whole Banach space Z, the compound mapping $f(U)$ is a real-valued random variable.

Some other definitions of a generalized random variable (or of a random element) have been given by other authors. Thus, for instance, Mourier [2] defines a random element only in the case Z is a Banach space in the following way: a mapping U is a random element if for every bounded linear functional f from the first adjoint space Z^* the compound mapping $f(U)$ is a real-valued random variable. Though for separable Banach spaces the definition of Mourier and the one of ours coincide, for arbitrary Banach spaces they differ. The definition of Mourier enables one to prove that the sum of random elements is again

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a random element, while for generalized random variables this statement need not hold as shown by Nedoma in [3]. On the other hand generalized random variables possess the important property that each compound mapping $\tau(U)$ formed by means of a Borel measurable mapping τ (of the measurable space $(Z, \mathfrak{Z})$ into another measurable space $(Y, \mathfrak{Y})$) and a generalized random variable U is a generalized random variable (with values in the measurable space $(Y, \mathfrak{Y})$).

Bharucha-Reid [4] follows essentially the definition of Mourier, provided his random elements have values in Orlicz spaces.

The conception of Kolmogoroff and Prochorow [5] is a generalization of the notion of a stochastic process, while Dubins [6] defines a generalized random variable as a homomorphism of some Boolean algebra into the measure ring induced by some probability space.

Let us remark that our definition does not depend on any probability measure defined on the measurable space $(\Omega, \mathfrak{S})$ and this is sometimes an advantage.

3. Probabilistic Tietze theorem. In what follows R denotes the space of all real numbers and $\mathfrak{R}$ the σ-algebra of all Borel subsets of the space R.

Theorem 1: *Let $(\Omega, \mathfrak{S})$ be a measurable space, X a separable metric space, M a closed subset of the space X and V a random transform of the Cartesian product $\Omega \times X$ into the space $(R, \mathfrak{R})$, which is for every fixed $\omega \in \Omega$ a continuous mapping $V(\omega, \cdot)$ of the set M into the space R, such that for every couple $(\omega, x) \in \Omega \times M$ the relation $|V(\omega, x)| \leqq s(\omega)$, where s is a real-valued random variable, holds.*

Then there exists a random transform T of the Cartesian product $\Omega \times X$ into the space $(R, \mathfrak{R})$ so that

(i) *for every couple $(\omega, x) \in \Omega \times M$ we have $T(\omega, x) = V(\omega, x)$;*

(ii) *for every $\omega \in \Omega$ the mapping $T(\omega, \cdot)$ is a continuous function from X into R;*

(iii) *for every couple $(\omega, x) \in \Omega \times X$ we have $|T(\omega, x)| \leqq s(\omega)$.*

Proof: We shall essentially follow the construction in the nonprobabilistic version of this theorem as given by Alexandroff (see pp. 182–183 in [7]), the only difference being in the definition of sets $A_n(\omega)$ and $B_n(\omega)$. For the sake of definiteness we shall briefly describe the construction of the random transform T.

We set $V_0(\omega, x) = V(\omega, x)$ for every couple $(\omega, x) \in \Omega \times M$, and for every $n = 0, 1, 2, \cdots$ we use the following recursive formulae: For every $\omega \in \Omega$ we define

$$A_n(\omega) = \{x: V_n(\omega, x) < -(2^n/3^{n+1})\cdot s(\omega)\}$$

and

$$B_n(\omega) = \{x: V_n(\omega, x) > (2^n/3^{n+1})\cdot s(\omega)\}.$$

Let $\rho(x, y)$ and $\rho(x, A)$ denote the distance from the point x to the point y or to the set A. Then for every couple $(\omega, x) \in \Omega \times X$ we put (the modification in case $A_n(\omega)$ or $B_n(\omega)$ is empty is omitted) $T_n(\omega, x) = (2/3)^{n+1}\cdot s(\omega)\cdot \rho(x, A_n(\omega))/(\rho(x, A_n(\omega)) + \rho(x, B_n(\omega))) - (2^n\cdot s(\omega)/3^{n+1})$ and for every couple $(\omega, x) \in \Omega \times M$

$$V_{n+1}(\omega, x) = V_n(\omega, x) - T_n(\omega, x).$$

Finally for every couple $(\omega, x) \varepsilon \Omega \times X$ we define

$$T(\omega, x) = \lim_{n\to\infty} \sum_{k=1}^{n} T_k(\omega, x).$$

It can be easily seen that the mapping T satisfies the three numbered requirements, hence we need only prove its measurability. First we shall prove that the mapping $\rho(x, A_n(\cdot))$ is for every $x \varepsilon X$ and for every $n = 0, 1, 2, \cdots$ a real-valued random variable. We have for every $c \geqq 0$ the equality

$$(1) \qquad \{\omega:\rho(x, A_n(\omega)) > c\} = \bigcup_{k=1}^{\infty} \bigcap_{y\varepsilon \tilde{O}(x,k)} \{\omega: V_n(\omega, y) \geqq -(2^n \cdot s(\omega)/3^{n+1})\},$$

where $\tilde{O}(x, k)$ is a countable set dense in the set $\{y:\rho(x, y) \leqq c + (1/k)\} \cap M$. Indeed, if ω_0 belongs to the set on the left hand side of (1), then there exists a positive integer k_0 (dependent in general on ω_0) so that

$$\inf_{y\varepsilon A_n(\omega_0)} \rho(x, y) > c + (1/k_0)$$

and hence the set $A_n(\omega_0)$ and the set $\{y:\rho(x, y) \leqq c + (1/k_0)\}$ are disjoint. Therefore for this k_0 and for every $y \varepsilon \tilde{O}(x, k_0)$ we have

$$(2) \qquad V_n(\omega_0, y) \geqq -(2^n \cdot s(\omega)/3^{n+1})$$

and this means that ω_0 belongs to the set on the right hand side of (1). Conversely, let ω_0 belong to the set on the right hand side of (1). Then there exists such a positive integer k_0, that for every $y \varepsilon \tilde{O}(x, k_0)$ the inequality (2) holds. Since the mapping $V_n(\omega_0, \cdot)$ is continuous, the inequality (2) holds for every element from the set $\{y:\rho(x, y) \leqq c + (1/k_0)\} \cap M$ and therefore the sets $A_n(\omega_0)$ and $\{y:\rho(x, y) \leqq c + (1/k_0)\}$ are disjoint. Hence

$$\inf_{y\varepsilon A_n(\omega_0)} \rho(x, y) \geqq c + (1/k_0) > c$$

and ω_0 belongs also to the set on the left hand side of (1). Thus, provided V_n is a random transform, we have that $\rho(x, A_n(\cdot))$ is a real-valued random variable for every $x \varepsilon X$ and quite a similar consideration holds for the mapping $\rho(x, B_n(\cdot))$. Therefore from the measurability of the mapping V_n it follows that both T_n and V_{n+1} are also random transforms. Since V_0 is a random transform, the same holds for T. The proof is complete.

4. Probabilistic Hahn-Banach theorem. The next theorem forms a probabilistic version of the well-known Hahn-Banach theorem for normed linear spaces.

THEOREM 2: *Let $(\Omega, \mathfrak{S})$ be a measurable space, X a separable real normed linear space, M a linear manifold in the space X, and V a random transform of the Cartesian product $\Omega \times M$ into the space $(R, \mathfrak{R})$, satisfying the following conditions:*

for every $\omega \varepsilon \Omega$, $\alpha \varepsilon R$, $\beta \varepsilon R$, $x \varepsilon M$ and $y \varepsilon M$

$$V(\omega, \alpha x + \beta y) = \alpha V(\omega, x) + \beta V(\omega, y);$$

for every couple $(\omega, x) \varepsilon \Omega \times M$ *we have* $| V(\omega, x) | \leqq s(\omega) \cdot \| x \|$, *provided the mapping s of the space* Ω *into the space R is for every* $\omega \varepsilon \Omega$ *defined by the formula* $s(\omega) = \sup_{x \varepsilon O \cap M} | V(\omega, x) |$, *where* $O = \{x\colon \| x \| = 1\}$.

Then there exists a random transform T of the Cartesian product $\Omega \times X$ *into the space* $(R, \mathfrak{R})$ *so that*

(iv) *for every couple* $(\omega, x) \varepsilon \Omega \times M$ *we have* $T(\omega, x) = V(\omega, x)$;

(v) *for every* $\omega \varepsilon \Omega$, $\alpha \varepsilon R$, $\beta \varepsilon R$, $x \varepsilon X$ *and* $y \varepsilon X$ *there holds* $T(\omega, \alpha x + \beta y) = \alpha T(\omega, x) + \beta T(w, y)$;

(vi) *for every couple* $(\omega, x) \varepsilon \Omega \times X$ *we have* $| T(\omega, x) | \leqq s(\omega) \cdot \| x \|$.

PROOF: First of all we shall describe the construction of the mapping T.

Since X is separable, there exists a countable set $\{x_1, x_2, \cdots\} \subset X - M$ dense in the set $X - M$. Let for every $n = 0, 1, 2, \cdots$ the symbol M_n denote the linear manifold generated by the set $M \cup \bigcup_{k=1}^{n} \{x_k\}$ and let $X_0 = \bigcup_{n=0}^{\infty} M_n$. We set for every couple $(\omega, x) \varepsilon \Omega \times M_0$,

$$V_0(\omega, x) = V(\omega, x)$$

and for every couple $(\omega, x) \in \Omega \times (X_0 - M_0)$,

$$V_0(\omega, x) = 0.$$

Then for every $n = 1, 2, \cdots$ we define recursively for every couple $(\omega, x) \varepsilon \Omega \times (X_0 - M_n)$

$$V_n(\omega, x) = V_{n-1}(\omega, x) = 0$$

and for every $\omega \varepsilon \Omega$, $x \varepsilon M_{n-1}$ and $t \varepsilon R$

$$V_n(\omega, x + tx_n) = V_{n-1}(\omega, x) + t \cdot \sup_{x \varepsilon M_{n-1}} (V_{n-1}(\omega, x) - s(\omega) \cdot \| x - x_n \|).$$

Further we put for every couple $(\omega, x) \in \Omega \times X_0$

$$T(\omega, x) = T_0(\omega, x) = \lim_{n \to \infty} V_n(\omega, x),$$

and finally for every $y \varepsilon X - X_0$ which can be written in the form $y = \lim_{n \to \infty} y_n$, where $y_n \varepsilon X_0$ for every $n = 1, 2, \cdots$, we set

$$T(\omega, y) = \lim_{n \to \infty} T_0(\omega, y_n).$$

It is well known that for every $\omega \varepsilon \Omega$ the mapping $T(\omega, \cdot)$ is a bounded linear functional which is an extension of the bounded linear functional $V(\omega, \cdot)$ from the linear manifold M to the whole space X with preservation of the norm. Thus, only measurability remains to be proved. However, we can write

$$\{\omega : s(\omega) \leqq c\} = \bigcap_{x \varepsilon \tilde{O}} \{\omega\colon | V_n(\omega, x) | \leqq c\},$$

where $\tilde{O}$ is a countable set dense in the set O. Since V is a random transform, the mapping V_0 is a random transform of the Cartesian product $\Omega \times X_0$ into the space $(R, \mathfrak{R})$. Further we have for every $c \in R$

$$\{\omega: \sup_{x \in M_{n-1}} (V_{n-1}(\omega, x) - s(\omega) \cdot \| x - x_n \|) \leqq c\} = \bigcap_{x \in \tilde{M}_{n-1}} \{\omega: V_{n-1}(\omega, x) - s(\omega) \cdot \| x - x_n \| \leqq c\},$$

where $\tilde{M}_{n-1} \subset M_{n-1}$ is a countable set dense in the set M_{n-1}. Thus, T_0 is a random transform of the Cartesian product $\Omega \times X_0$ into the space $(R, \mathfrak{R})$ and therefore the mapping T is a random transform of the Cartesian product $\Omega \times X$ into the space $(R, \mathfrak{R})$ and this proves Theorem 2.

Theorem 2 states that for separable Banach spaces a probabilistic version of the Hahn-Banach theorem is valid. Theorem 3 below shows that for (not necessarily separable) Hilbert spaces an equivalent statement is also true.

5. Conclusion. It would be of interest to know the extent to which Theorems 1 and 2 remain valid if the separability assumption is dropped. In this case the methods of proof used above obviously fail. Unfortunately, the author has not succeeded either in proving the non-separable versions or in constructing appropriate counterexamples. To get other positive results it seems necessary to lay further assumptions on the space X. Thus, a statement equivalent to Theorem 2 is true for not necessarily separable Hilbert spaces, owing to the possibility of defining an orthogonal complement to a given subspace.

THEOREM 3: *Theorem 2 remains valid provided X is a Hilbert space and M a Hilbert subspace of the space X.*

PROOF. Since every element $x \in X$ can be uniquely written in the form $x = z_1 + z_2$, where $z_1 \in M$ and $z_2 \perp M$, we can set for every $\omega \in \Omega$

$$T(\omega, x) = V(\omega, z_1)$$

and Theorem 3 follows immediately from this construction.

Finally, let us briefly sketch an application of our results.

The well-known Banach-Mazur Theorem asserts that every separable metric (Banach) space M can be imbedded in an isometric (isometric and isomorphic) way into the space C of all continuous functions defined on the closed interval $(0, 1)$. This theorem enables us sometimes to treat generalized random variables with values in the space $(M, \mathfrak{M})$ as generalized random variables with values in the space $(C, \mathfrak{C})$. This is the case in Theorem 16 in [1], where the measurability of the set $\{\omega: \bigcup_{n=1}^{\infty} \{V_n(\omega)\}$ is strongly compact$\}$ must be proved. Another example is Criterion 6 in [1] (for wording see Introduction of this paper). In both these cases the above mentioned treatment provides a simple and elegant proof of the statement in question.

Using Theorems 1 and 2 we are able to enlarge the number of problems in which not only generalized random variables with values in the space $(M, \mathfrak{M})$ are considered, but also random transforms of the Cartesian product $\Omega \times M$ into the space $(R, \mathfrak{R})$. In the present paper we shall mention only one problem of this kind, namely the Representation Theorem for random Schwartz distributions which, roughly speaking, reads: Every random Schwartz distribution can be represented on every compact interval with arbitrarily great probability as a derivative of a strictly continuous stochastic process. This theorem was proved by Ullrich [8] who applied our Theorem 2 with X replaced by C and M by K_r, where K_r stands for the space of rth derivatives of all continuous functions f defined on a closed interval $[a, b]$ that have derivatives of all orders, the functions f themselves and their derivatives taking the value 0 at both ends of the interval $[a, b]$.

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