ON THE MOMENTS OF THE TRACE OF A MATRIX AND APPROXIMATIONS TO ITS DISTRIBUTION

By K. C. S. Pillai[1] and Tito A. Mijares[2]

The Statistical Center, University of the Philippines

1. Summary. The first four moments of the sum of **s** non-null latent roots of a matrix occurring in multivariate analysis are studied. In particular, the first four moments of the sum of six roots are found, and are used to compare the upper percentage points obtained directly from the moment ratios with those from Pillai's approximate distribution.

2. Introduction. Distribution problems in multivariate analysis are often related to the distribution of the latent roots of a matrix derived from sample observations taken from multivariate normal populations. The form of the joint distribution of **s** non-null latent roots of a matrix in multivariate analysis as given by Roy [10], Hsu [3], and Fisher [2] is expressible as

$$f(\theta_1, \cdots, \theta_s) = C(s, m, n) \prod_{i=1}^{s} \theta_i^m (1 - \theta_i)^n \prod_{i>j} (\theta_i - \theta_j), \tag{2.1}$$
$$0 < \theta_1 \leqq \theta_2 \leqq \cdots \leqq \theta_s < 1,$$

where

$$C(s, m, n) = \frac{\pi^{s/2} \prod_{i=1}^{s} \Gamma\left(\frac{2m + 2n + s + i + 2}{2}\right)}{\prod_{i=1}^{s} \Gamma\left(\frac{2m + i + 1}{2}\right) \Gamma\left(\frac{2n + i + 1}{2}\right) \Gamma\left(\frac{i}{2}\right)} \tag{2.2}$$

and **m** and **n** are defined differently for various situations described in [7].

Pillai [6], [8] has studied the distribution of the proposed test criterion, $V^{(s)} = \sum_{i=1}^{s} \theta_i$, deriving the first three moments and obtaining the fourth moment for $s = 2$, 3 and 4. He has also suggested [6], [8] an incomplete Beta distribution approximation to the distribution of $V^{(s)}$, and tabulated approximate percentage points for various values of the supplementary parameters **m** and **n** [9]. In this paper, the work of Pillai has been extended to generalize the fourth moment.

3. The moment generating function of the sum of the roots. The joint distribution given in (2.1) can be thrown into a determinantal form of the Vander-

Received August 5, 1958; revised January 19, 1959.

[1]United Nations Senior Adviser in Mathematical Statistics and Visiting Professor of Statistics. Now with the United Nations, New York.

[2]Associate Professor of Statistics.

monde type and the moment generating function for the sum of s non-null roots, $V^{(s)}$, can be expressed in the determinantal form [6], [8]

$$E(e^{tV^{(s)}}) \tag{3.1}$$

$$= C(s, m, n) \begin{vmatrix} \int_0^1 \theta_s^{m+s-1}(1-\theta_s)^n e^{t\theta_s}\, d\theta_s \cdots \int_0^1 \theta_s^{m}(1-\theta_s)^n e^{t\theta_s}\, d\theta_s \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \int_0^{\theta_2} \theta_1^{m+s-1}(1-\theta_1)^n e^{t\theta_1}\, d\theta_1 \cdots \int_0^{\theta_2} \theta_1^{m}(1-\theta_1)^n e^{t\theta_1}\, d\theta_1 \end{vmatrix}.$$

We may denote the pseudo-determinant (P.D.) [6], [8] in (3.1) by

$$U(m + s - 1, m + s - 2, \cdots, m; n; t)$$

and more conveniently, when $t = 0$, by $U(s - 1, s - 2, \cdots, 1, 0)$.

Differentiating (3.1) successively [1], [4] with respect to t and setting $t = 0$ after each differentiation, we obtain

$$E(V^{(s)}) = \mu_1' = C(s, m, n)U(s, s-2, s-3, \cdots, 1, 0); \tag{3.2}$$

$$E[(V^{(s)})^2] = \mu_2' = C(s, m, n)[U(s+1, s-2, s-3, \cdots, 1, 0) + U(s, s-1, s-3, \cdots, 1, 0)]; \tag{3.3}$$

$$\begin{aligned} E[(V^{(s)})^3] = \mu_3' = C(s, m, n)[&U(s+2, s-2, s-3, \cdots, 1, 0) \\ &+ 2\,U(s+1, s-1, s-3, \cdots, 1, 0) \\ &+ U(s, s-1, s-2, s-4, \cdots, 1, 0)]; \end{aligned} \tag{3.4}$$

$$\begin{aligned} E[(V^{(s)})^4] = \mu_4' = C(s, m, n)[&U(s+3, s-2, s-3, \cdots, 1, 0) \\ &+ 3\,U(s+2, s-1, s-3, \cdots, 1, 0) \\ &+ 2\,U(s+1, s, s-3, s-4, \cdots, 1, 0) \\ &+ 3\,U(s+1, s-1, s-2, s-4, \cdots, 1, 0) \\ &+ U(s, s-1, s-2, s-3, s-5, \cdots, 1, 0)]. \end{aligned} \tag{3.5}$$

The method can be extended for any number of differentiations. It may be noted that the first relation (3.2) for μ_1' contains only one P.D., since the other five vanish because two columns in each are equal.

4. Values of the pseudo-determinants. In this section we give the values of the P.D.'s involved in the expressions for the first four raw moments following (3.2)–(3.5). These were evaluated using a reduction formula by Pillai [8]. For details the reader is referred to [4].

Let us set $(2g + a)(2g + b) \cdots = G(a, b, \cdots)$ and $(m + n) = p$. Then the P.D. for the *first raw moment* is

$$C(s, m, n)[U(s, s-2, s-3, \cdots, 1, 0)] = \frac{sM(s+1)}{P(2s+2)}. \tag{4.1}$$

For the *second raw moment,* we find

$$C(s, m, n)[U(s, s-1, s-3, s-4, \cdots, 1, 0)] = \frac{s(s-1)}{2!} \frac{M(s, s+1)}{P(2s+1, 2s+2)}, \tag{4.2}$$

$$C(s, m, n)[U(s+1, s-2, s-3, s-4, \cdots, 1, 0)] = \frac{s(s-1)}{2!} \frac{M(s, s+1)}{P(2s+1, 2s+4)} + \frac{sM(s+1, 2s+2)}{P(2s+2, 2s+4)}. \tag{4.3}$$

For the *third raw moment,* we find

$$C(s, m, n)[U(s, s-1, s-2, s-4, \cdots, 1, 0)] = \frac{s(s-1)(s-2)}{3!} \frac{M(s-1, s, s+1)}{P(2s, 2s+1, 2s+2)}, \tag{4.4}$$

$$\begin{aligned} C(s, m, n)&[U(s+1, s-1, s-3, \cdots, 1, 0)] \\ &= \frac{s(s-1)}{3!} \frac{M(s, s+1)}{P(2s, 2s+1, 2s+2, 2s+4)} \\ &\cdot \{4n[(2s-1)m + (s^2+2)] + 4(2s-1)m^2 + 12s^2m + (4s^3+2s^2+2s+4)\} \\ &+ \frac{m+s+1}{p+s+2} \cdot C(s, m, n)[U(s, s-1, s-3, s-4, \cdots, 1, 0)], \end{aligned} \tag{4.5}$$

$$\begin{aligned} C(s, m, n)&[U(s+2, s-2, s-3, \cdots, 1, 0)] \\ &= \frac{(s+1)s(s-1)}{3!} \frac{M(s, s+1, s+2)}{P(2s+1, 2s+2, 2s+6)} \\ &+ \frac{m+s+2}{p+s+3} \cdot C(s, m, n)[U(s+1, s-2, s-3, \cdots, 1, 0)] \end{aligned} \tag{4.6}$$

And for the *fourth raw moment,* we find

$$C(s, m, n)[U(s, s-1, s-2, s-3, s-5, \cdots, 1, 0)] = \frac{s(s-1)(s-2)(s-3)}{4!} \frac{M(s-2, s-1, s, s+1)}{P(2s-1, 2s, 2s+1, 2s+2)}, \tag{4.7}$$

$$\begin{aligned} C(s, m, n)&[U(s+1, s-1, s-2, s-4, s-5, \cdots, 1, 0)] \\ &= \frac{s(s-1)(s-2)}{3!2!} \frac{M(s-1, s, s+1)}{P(2s-1, 2s, 2s+1, 2s+2, 2s+4)} \\ &\cdot \{n[2(3s-1)m + (3s^2+s+10)] + 2(3s-1)m^2 \\ &+ (9s^2-s+2)m + (3s^3+s^2+2s+4)\} \\ &+ \frac{m+s+1}{p+s+2} \cdot C(s, m, n)[U(s, s-1, s-2, s-4, \cdots, 1, 0)], \end{aligned} \tag{4.8}$$

$$
\begin{aligned}
&C(s, m, n)[U(s+1, s, s-3, s-4, \cdots, 1, 0)] \\
&\quad = \frac{s(s-1)}{3!2!} \frac{M(s, s+1)}{P(2s-1, 2s, 2s+1, 2s+2, 2s+3, 2s+4)} \\
&\qquad \cdot \{n^2[16s(s+1)m^2 + 8s(2s^2+5s+9)m \\
&\qquad + 4(s^4+4s^3+11s^2+8s+12)] \\
&\qquad + n[32s(s+1)m^3 + 8s(8s^2+15s+13)m^2 \\
&\qquad + 4s(10s^3+30s^2+45s+19)m \\
&\qquad + 2(4s^5+17s^4+36s^3+31s^2+8s+12)] \\
&\qquad + s(s+1)M(s+1, s+2, 2s, 2s+1)\},
\end{aligned} \tag{4.9}
$$

$$
\begin{aligned}
&C(s, m, n)[U(s+2, s-1, s-3, s-4, \cdots, 1, 0)] \\
&\quad = \frac{(s+1)s(s-1)}{3!2!} \frac{M(s, s+1, s+2)}{P(2s, 2s+1, 2s+2, 2s+3, 2s+6)} \\
&\qquad \cdot \{n[2(3s-2)m + (3s^2-s+10)] + 2(3s-2)m^2 \\
&\qquad + (9s^2+s-2)m + (3s^3+2s^2+s+6)\} \\
&\qquad + \frac{m+s+2}{p+s+3} \cdot C(s, m, n)[U(s+1, s-1, s-3, \cdots, 1, 0)],
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
&C(s, m, n)[U(s+3, s-2, s-3, \cdots, 1, 0)] \\
&\quad = \frac{(s+2)(s+1)s(s-1)}{4!} \frac{M(s, s+1, s+2, s+3)}{P(2s+1, 2s+2, 2s+3, 2s+8)} \\
&\qquad + \frac{m+s+3}{p+s+4} \cdot C(s, m, n)[U(s+2, s-2, \cdots, 1, 0)].
\end{aligned} \tag{4.11}
$$

It may be noted that simplifications of relations (4.1) to (4.6) to obtain the first, second and third *central* moments yield exactly the results given by Pillai [6], [8]. The fourth central moment has not been obtained in general from (4.1) to (4.11); however, for particular cases, like that for $s = 6$ in the next section, these relations have been used to arrive at the expressions for $\beta_1 = \mu_3^2/\mu_2^3$ and $\beta_2 = \mu_4/\mu_2^2$, where μ's denote central moments [4].

5. Percentage points of $V^{(6)}$ using moment ratios and a Pearson curve approximation, and using Pillai's approximate beta distribution. Putting $s = 6$ in relations (4.1) to (4.11) the following expressions for the β's are obtained:

$$
\beta_1 = \frac{4(n-m)^2(p+1)^2(p+8)(2p+13)}{3(2m+7)(2n+7)(p+4)(p+6)^2(p+9)^2}, \tag{5.1}
$$

$$
\begin{aligned}
\beta_2 = &\frac{(p+8)(2p+13)}{(2m+7)(2n+7)(p+4)^2(p+6)(p+9)(p+10)(2p+11)(2p+15)} \\
&\cdot [4mn(6p^5 + 179p^4 + 2177p^3 + 13{,}176p^2 + 38{,}732p + 43{,}760) \\
&\qquad + (92p^6 + 2996p^5 + 40{,}869p^4 + 294{,}677p^3 \\
&\qquad + 1{,}169{,}444p^2 + 2{,}408{,}532p + 2{,}010{,}960)].
\end{aligned} \tag{5.2}
$$

Table 5A shows the values of β_1 and β_2 for several values of **m** and **n**. The upper 5% points of $V^{(6)}$ given in Table 5B for each given **m** and **n** were obtained by interpolating in Table 42 of [5], "Percentage points of Pearson Curves for given β_1, β_2, expressed in standardized measure."

The percentage points from Pillai's approximate distribution [9] were obtained by referring to Snedecor's F tables using the transformation

$$F = \frac{(2n + s + 1)}{(2m + s + 1)} \cdot \frac{V^{(s)}}{(s - V^{(s)})}, \tag{5.3}$$

with $f_1 = s(2m + s + 1)$ and $f_2 = s(2n + s + 1)$ degrees of freedom.

TABLE 5A

Values of $\begin{Bmatrix}\beta_1\\ \beta_2\end{Bmatrix}$ *of the exact distribution for* $s = 6$

n	m				
	0	5	10	20	30
10	0.03919 3.00925	0.00381 2.97815	0.00000 2.97594	0.00424 2.98561	0.00990 2.99674
30	0.10177 3.11486	0.01615 3.01774	0.00990 2.99674	0.00213 2.98724	0.00000 2.98732
60	0.13690 3.18341	0.04460 3.05136	0.02192 3.01996	0.00725 3.00025	0.00260 2.99453
100	0.15553 3.21669	0.05814 3.07233	0.03013 3.03597	0.01269 3.01125	0.00636 3.00253

TABLE 5B

Upper 5% points of $V^{(6)}$ *using* (1) *the* β*'s of the exact distribution and a Pearson curve approximation, and* (2) *Pillai's approximation*

n	m				
	0	5	10	20	30
10 (1)	1.601	2.705	3.362	4.112	4.529
(2)	1.647	2.728	3.394	4.133	4.542
30 (1)	0.762	1.460	1.983	2.729	3.239
(2)	0.771	1.469	2.002	2.747	3.247
60 (1)	0.427	0.863	1.226	1.811	2.264
(2)	0.431	0.870	1.241	1.830	2.280
100 (1)	0.269	0.559	0.813	1.249	1.615
(2)	0.271	0.563	0.822	1.259	1.628

It may be seen from Table 5B that the percentage points computed by the two methods practically agree in the first two places even for small values of m and n. Further, we may also study the difference in probabilities corresponding to the percentage points from the two approximations. This may be done by considering the percentage points for approximation (1) in Table 5B and evaluating the probability in each case from Pillai's beta distribution approximation. For example, taking $m = 0$, $n = 10$ and percentage point 1.601, exact integration of the incomplete beta function gives the probability 0.9298 as against 0.95. However, for $m = 0$ and $n = 60$, with percentage point 0.427, the probability obtained by the same procedure is 0.9462. Hence, for larger values of n, as in the case of percentage points, the difference in probabilities also tends to be small.

6. Acknowledgment. The authors wish to acknowledge the facilities offered by the Statistical Center, University of the Philippines, for preparing this paper.

REFERENCES

[1] A. C. AITKEN, *Determinants and Matrices*, Oliver & Boyd Ltd., Edinburgh, Great Britain, 1956.
[2] R. A. FISHER, "The sampling distribution of some statistics obtained from non-linear equations," *Ann. Eugenics*, Vol. 9 (1939), pp. 238–249.
[3] P. L. HSU, "On the distribution of the roots of certain determinantal equations," *Ann. Eugenics*, Vol. 9 (1939), pp. 250–258.
[4] T. A. MIJARES, "On the distribution of the sum of six roots of a matrix in multivariate analysis," unpublished M.S. thesis submitted in March, 1958 to the University of the Philippines Graduate School.
[5] E. S. PEARSON AND H. O. HARTLEY, *Biometrika Tables for Statisticians*, Vol. 1, Cambridge, 1954.
[6] K. C. S. PILLAI, "On some distribution problems in multivariate analysis," Mimeograph Series No. 88, Institute of Statistics, University of North Carolina, 1954.
[7] K. C. S. PILLAI, "Some new test criteria in multivariate analysis," *Ann. Math. Stat.*, Vol. 26 (1955), pp. 117–121.
[8] K. C. S. PILLAI, "Some results useful in multivariate analysis," *Ann. Math. Stat.*, Vol. 27 (1956), pp. 1106–1114.
[9] K. C. S. PILLAI, *Concise Tables for Statisticians*, The Statistical Center, University of the Philippines, Manila, 1957.
[10] S. N. ROY, "p-statistics or some generalizations in analysis of variance appropriate to multivariate problems," *Sankhya*, Vol. 4 (1939), pp. 381–396.