SOME REMARKS ON HERBACH'S PAPER, "OPTIMUM NATURE OF THE F-TEST FOR MODEL II IN THE BALANCED CASE"[1]

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1. Summary. The purpose of this note is to present a lemma which will settle a question of completeness left open in Section 6 of the above mentioned paper [5]. We give two applications of the lemma,

(i) by proving that, in addition to Herbach's results, also the standard F-test for $\sigma_{ab}^2 = 0$ is a uniformly most powerful similar test,

(ii) by pointing out that the standard form introduced in [5] together with our lemma provide convenient tools to prove that in a balanced model II design (with the usual normality assumptions) *the standard estimates of variance components are minimum variance unbiased*. This result is well known ([2], [3]) and it has in fact been pointed out by Graybill and Wortham [3] that a completeness argument may be used to demonstrate the minimum variance property of the usual estimators for the variance components. The present lemma shows that the estimators do indeed have the necessary completeness property. We will follow Herbach's notation throughout.

2. A completeness lemma. The following lemma guarantees completeness for a certain class of probability densities to which the results of Lehmann and Scheffé do not apply directly. It takes care of a difficulty mentioned in [5], Section 6, which is caused when $g(\theta)$ does not equal one of the θ_i $(i = 2, \cdots, r)$. If $g(\theta)$ does, the product-densities could immediately be reduced to the exponential form considered by Lehmann and Scheffé in [7], Theorem 7.3. Our lemma is more general than the Lehmann and Scheffé Theorem 7.1 [7] in the sense that we allow instead of their $g''_{\theta''}(x'')$ to have $g''_{\theta',\theta''}(x'')$ which, however, we assume to factor into $h'_{\theta'}(x'')h''_{\theta''}(x'')$ with $h'_{\theta'}(x'') > 0$ and $\{h_{\theta''}(x'')\, d\mu^{x''}\}$ strongly complete. It is of course more special in that we take both $\mu^{x''}$ and $\mu^{x'|x''}$ as Lebesgue measure and for $g'_{\theta'}(x')$, $g''_{\theta',\theta''}(x'')$ specific functions. Our proof is modelled along the same lines as the one given by Lehmann and Scheffé in [7] p. 221.

LEMMA: *Let*

$$\mathfrak{P}^t = \{P_\theta^t\,;\, \theta\, \varepsilon\, \mathfrak{D}\}, \qquad t = (t_2, \cdots, t_r), \theta = (\theta_2, \cdots, \theta_r)$$

$$\mathfrak{P}^{t_1} = \{P_{\theta_1,\theta}^{t_1}\,;\, (\theta_1, \theta)\, \varepsilon\, \mathfrak{D}_1 \times \mathfrak{D}\}, \qquad \theta_1 \text{ real}$$

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[1] This is a cut-down version of a paper in which the author independently considered standard forms for model II designs. He acknowledges, however, the priority of Dr. Herbach's approach (see [4] as compared to [1]) and restricts himself to giving some results supplementing those of Herbach.

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be two families of probability measures on the Borel sets of the Euclidean space E_{r-1} and the real line E_1 respectively, having the densities

$$p_\theta(t) = c(\theta)h(t_2, \cdots, t_r)e^{\theta_2 t_2 + \cdots + \theta_r t_r} \tag{1}$$

$$p_{\theta_1,\theta}(t_1) = c(\theta_1, \theta)e^{g(\theta)t_1^2 + \theta_1 t_1} \tag{2}$$

with respect to Lebesgue measure. If $\mathfrak{D}_1$ is the real line and $\mathfrak{D}$ a Borel set in E_{r-1} containing a non-degenerate $(r - 1)$-dimensional interval then the family of product measures $\mathfrak{P} = \{P_{\theta_1,\theta}^{t_1} \times P_\theta^t ; (\theta_1, \theta) \varepsilon \mathfrak{D}_1 \times \mathfrak{D}\}$ is strongly complete (in the sense of Lehmann and Scheffé [7]).

PROOF: Suppose

$$I = \iint f(t_1, t)p_{\theta_1,\theta}(t_1)p_\theta(t)\, dt_1\, dt = 0 \qquad (\text{a.e. } L^{\theta_1 \times \theta}).^2 \tag{3}$$

Let N be the set of parameter points (θ_1, θ) for which $I \neq 0$. If N_θ denotes the θ-section of N, i.e. $N_\theta = \{\theta_1 ; (\theta_1, \theta) \varepsilon N\}$, then $L^{\theta_1}(N_\theta) = 0$ except possibly for $\theta \varepsilon N_0$, where $L^\theta(N_0) = 0$.

According to Fubini's theorem we may write

$$I = \int p_{\theta_1,\theta}(t_1)\Phi(t_1, \theta)\, dt_1,$$

where $\Phi(t_1, \theta) = \int f(t_1, t)p_\theta(t)\, dt$. Since $p_{\theta_1,\theta}(t_1) > 0$, for fixed $\theta \not\varepsilon N_0$ the exceptional set of points t_1 for which the integral defining $\Phi(t_1, \theta)$ does not exist has L^{t_1}-measure zero. Furthermore, if $\theta \not\varepsilon N_0$, we can, in virtue of (2), rewrite (3) as

$$\int e^{\theta_1 t_1}\left[e^{g(\theta)t_1^2}\,\Phi(t_1, \theta)\right] dt_1 = 0 \qquad (\text{a.e. } L^{\theta_1}), \theta \not\varepsilon N_0 .$$

From the unicity property of the bilateral Laplace transform (see, for instance, [8], Ch. VI, Theorem 6b) it follows that

$$\Phi(t_1, \theta) = 0 \qquad (\text{a.e. } L^{t_1}), \theta \not\varepsilon N_0 .$$

Thus, if S denotes the (measurable) set of points (t_1, θ) for which Φ is either not defined or $\neq 0$, almost every θ-section of S has L^{t_1}-measure zero, hence $L^{t_1 \times \theta}(S) = 0$.

This in turn implies that almost all t_1-sections of S have L^θ-measure zero, i.e.

$$\Phi(t_1, \theta) = \int f(t_1, t)p_\theta(t)\, dt = 0 \qquad (\text{a.e. } L^\theta) \quad \text{if} \quad t_1 \not\varepsilon N_1,$$

where $L^{t_1}(N_1) = 0$. Since the family of probability densities $p_\theta(t)$ is strongly complete (Lehmann and Scheffé [7], Theorem 7.3) we conclude

[2] L with a superscript denotes Lebesgue measure. The superscript indicates the space on which the measure is taken.

$$f(t_1, t) = 0 \qquad (\text{a.e. } \mathfrak{P}^t),\ t_1 \not\in N_1,$$

from which $f(t_1, t) = 0$ (a.e. $\mathfrak{P}$) follows immediately.

3. Applications. (a) *Tests of hypotheses in balanced model* II *designs*. Consider the balanced two-way classification ([5], Section 6) and the hypothesis $\omega: \sigma_{ab}^2 = 0$. The statistic

$$T_1 = Z_{111}, \qquad T_2 = S_2, \qquad T_3 = S_3, \qquad T_4 = S_4 + S_5$$

is not only sufficient under ω but also complete on ω. In fact, if we let

$$\theta_1 = \frac{\sqrt{N}\mu}{\lambda_2 + \lambda_3 - \lambda_4}, \qquad \theta_2 = -\frac{1}{2\lambda_2}, \qquad \theta_3 = -\frac{1}{2\lambda_3}, \qquad \theta_4 = -\frac{1}{2\lambda_4},$$

the densities of T_1 and $T = (T_2, T_3, T_4)$ are easily recognized to have the form given in our lemma. Proceeding therefore in the same fashion as in [5], Section 6, we would find that *also the standard F-test of the hypothesis* $\omega: \sigma_{ab}^2 = 0$ *is a uniformly most powerful similar test.* The same situation prevails in higher order classifications. As is well known, in a complete n-way classification F-tests exist for the non-existence of anyone of the $(n - 1)$st or $(n - 2)$nd order interactions. All these tests are uniformly most powerful similar tests.

(b) *Point estimation in balanced model* II *designs.* To fix the ideas consider the standard form for the balanced two-way classification. A sufficient statistic for the parameters involved is

$$T_1 = Z_{111}, \qquad T_2 = S_2, \cdots, \qquad T_5 = S_5. \tag{4}$$

If we let

$$\theta_1 = \frac{\sqrt{N}\mu}{\lambda_2 + \lambda_3 - \lambda_4}, \qquad \theta_2 = -\frac{1}{2\lambda_2}, \cdots, \qquad \theta_5 = -\frac{1}{2\lambda_5},$$

the densities of T_1 and $T = (T_2, \cdots, T_5)$ are again of the form given in our lemma and thus the statistic (4) is complete on Ω. Unbiased estimates for the variance components, in terms of (4), are

$$\hat{\sigma}_e^2 = \frac{T_5}{\nu_e}, \quad \hat{\sigma}_{ab}^2 = \frac{1}{K}\left[\frac{T_4}{\nu_{ab}} - \frac{T_5}{\nu_e}\right], \quad \hat{\sigma}_b^2 = \frac{1}{IK}\left[\frac{T_3}{\nu_b} - \frac{T_4}{\nu_{ab}}\right], \quad \hat{\sigma}_a^2 = \frac{1}{JK}\left[\frac{T_2}{\nu_a} - \frac{T_4}{\nu_{ab}}\right], \tag{5}$$

where $\nu_a = I - 1$, $\nu_b = J - 1$, $\nu_{ab} = (I - 1)(J - 1)$, $\nu_e = IJ(K - 1)$ and are therefore minimum variance unbiased estimates ([6], Theorem 5.1). On the other hand the standard estimates in terms of the various mean squares have the same distribution as those in (5) and must consequently be of minimum variance among all unbiased estimates based on the original observation vector X.

Higher order layouts could be treated in a similar manner.

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