AN EXAMPLE OF WIDE DISCREPANCY BETWEEN FIDUCIAL AND CONFIDENCE INTERVALS

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1. Introduction. Fisher [1], [2] has emphasized that when he chooses a set in the parameter space on the basis of certain observations and attributes to it a certain fiducial probability α, he does not intend that, for fixed values of the parameter the probability that this random set contains the parameter point should be α. Examples of this distinction for the Behrens-Fisher problem have been given by Fisher [1], [2] and Neyman [3], [4]. In these cases the numerical differences are not extremely large. In order to bring out more clearly the contrast between fiducial probability and confidence sets I shall give, for each α and ϵ in the interval $(0, 1)$, an example where a fiducial interval for a parameter with fiducial probability equal to α has probability less than ϵ of covering the true parameter for a large range of parameter values. This means that although a large fiducial probability is claimed, it is practically certain that the interval will not cover the true parameter value. Of course this cannot happen when the fiducial sets are obtained by Pitman's methods [6], [7].

2. The example. Let $X_1, \cdots, X_n$ be independently normally distributed real random variables with unknown means $\xi_1, \cdots, \xi_n$ and variance 1. Suppose we are interested in fiducial or confidence sets for $\sum \xi_i^2$ of the form

$$[f(X_1, \cdots, X_n), \infty].$$

We consider the one-sided case only in order to avoid irrelevant computational details. The fiducial distribution of $\xi_1, \cdots, \xi_n$ is that they are independently normally distributed with means $X_1, \cdots, X_n$ and variance 1 (see Fisher [1], p. 132, where the case $n = 2$ is given, but see also Tukey [5] for a different fiducial distribution). Thus the fiducial distribution of $\sum_1^n \xi_i^2$ is a non-central χ^2 distribution with n degrees of freedom and non-centrality parameter $\sum_1^n X_i^2$. On the basis of this we determine a fiducial interval

$$[\Phi_{\alpha,n}(\textstyle\sum X_i^2), \infty) \tag{1}$$

with fiducial probability α for the unknown parameter $\sum \xi_i^2$. Here $\Phi_{\alpha,n}(\sum X_i^2)$ is the value which will be exceeded with probability α by a non-central χ^2 variate with n degrees of freedom and non-centrality parameter $\sum X_i^2$. But this non-central χ^2 distribution is, for large n, approximately a normal distribution with mean $n + \sum X_i^2$ and variance $2n + 4\sum X_i^2$, the approximation being uniform

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in $\sum X_i^2$. Thus for sufficiently large n

$$\Phi_{\alpha,n}(\sum X_i^2) > n + \sum X_i^2 - t'_\alpha \sqrt{2n + 4\sum X_i^2}, \tag{2}$$

for all values of $\sum X_i^2$ where t'_α is a fixed number independent of n satisfying

$$t'_\alpha > t_\alpha \tag{3}$$

where

$$1 - \alpha = \frac{1}{\sqrt{2\pi}} \int_{t_\alpha}^{\infty} e^{-\frac{1}{2}u^2}\, du. \tag{4}$$

Thus for fixed $\xi_1, \cdots, \xi_n$ the probability that the fiducial interval will cover the true value of $\sum \xi_i^2$ is

$$\begin{aligned} &P_{\xi_1 \cdots \xi_n}\{\sum \xi_i^2 \geqq \Phi_{\alpha,n}(\sum X_i^2)\} \\ &\qquad \leqq P_{\xi_1 \cdots \xi_n}\{\sum \xi_i^2 \geqq n + \sum X_i^2 - t'_\alpha \sqrt{2n + 4\sum X_i^2}\} \\ &\qquad = P_{\xi_1 \cdots \xi_n}\{\sum X_i^2 \leqq \sum \xi_i^2 - n + t'_\alpha \sqrt{2n + 4\sum X_i^2}\} \end{aligned} \tag{5}$$

for sufficiently large n. Now let $n \to \infty$ with

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_1 \xi_i^2 = 0. \tag{6}$$

From Chebyshev's inequality it follows that, for any $\epsilon > 0$, we have, for sufficiently large n,

$$P\{2n + 4\sum X_i^2 > \epsilon n^2\} < \epsilon. \tag{7}$$

Thus

$$\begin{aligned} &P_{\xi_1 \cdots \xi_n}\{\sum X_i^2 \leqq \sum \xi_i^2 - n + t'_\alpha \sqrt{2n + 4\sum X_i^2}\} \\ &\qquad \leqq P_{\xi_1 \cdots \xi_n}\{\sum X_i^2 \leqq \sum \xi_i^2\} + \epsilon, \end{aligned} \tag{8}$$

Again applying Chebyshev's inequality, or the limiting distribution of $\sum X_i^2$. it follows from (6) and (8) that

$$\lim_{n\to\infty} P_{\xi_1 \cdots \xi_n}\{\sum \xi_i^2 \geqq \Phi_{\alpha,n}(\sum X_i^2)\} = 0. \tag{9}$$

Let us compare these results with the natural confidence sets. Since $\sum X_i^2$ has a non-central χ^2 distribution with n degrees of freedom and non-centrality parameter $\sum \xi_i^2$, the confidence sets of the desired form are

$$[\Phi_{1-\alpha,n}^{-1}(\sum X_i^2), \infty), \tag{10}$$

or, approximately for large n,

$$\sum X_i^2 \leqq \sum \xi_i^2 + n + t_\alpha \sqrt{2n + 4\sum \xi_i^2} \tag{11}$$

(which must be inverted to obtain an explicit lower confidence bound for $\sum \xi_i^2$) as compared with the fiducial interval

$$[\Phi_{\alpha,n}(\sum X_i^2), \infty) \tag{12}$$

which is approximately, for large n

$$\sum \xi_i^2 \geqq \sum X_i^2 + n - t_\alpha \sqrt{2n + 4 \sum X_i^2}\,. \tag{13}$$

However a different argument leads to a more reasonable fiducial distribution. Because of the rotational symmetry of the problem, it seems reasonable to base our procedure only on $Z = \sum X_i^2$, ignoring the individual observations. Then the fiducial argument leads to the intervals based on (10), i.e. confidence intervals.

At first I intended to write this paper without extended comments, letting the example speak for itself. However some remarks of the editor and referees and the fact that I have since read the discussion of fiducial inference in Chapter 6 of Quenouille [8] lead me to believe that some discussion may be useful. Two questions may be asked in connection with the above example. Is the argument used a fiducial argument as this is understood by the advocates of fiducial inference, and is the resulting fiducial distribution of $\sum \xi_i^2$ absurd? The fiducial argument has two parts. First the joint fiducial distribution of $\xi_1 \ldots \xi_n$ is given on the authority of [1]. Then the distribution of $\sum \xi_i^2$ is calculated from this joint distribution, the joint fiducial distribution being treated as an ordinary probability distribution. The first step seems to be in agreement with the practice of the advocates of fiducial inference. For example, Quenouille on p. 139 of [8] argues against the different fiducial distribution given by Tukey [5]. Anyone who argues that the second step is not justified seems to be saying that fiducial distributions cannot be treated as ordinary probability distributions. In [8] on pp. 114–119, Quenouille imposes restrictions on the way some fiducial distributions can be used, but (at least to me) it is not clear whether these restrictions are meant to apply to cases as simple as the one discussed in this paper, nor is it clear whether my derivation meets his requirements if they are applicable to the present case.

Finally it may be contended that the fiducial interval (1) is the correct one and should be used. Because of the conflict with the argument immediately below (13), I do not think many people will take this attitude. Apart from this, there is an important question of principle here. If n is large and $\sum \xi_i^2$ is small compared with n^2 (which is commonly the case if the ξ_i are coordinates of a high order interaction), then the probability that the fiducial interval (1) will cover the true value of $\sum \xi_i^2$ has been shown to be small if α is moderate. This has the practical interpretation that, when the fiducial interval (1) is applied in such situations, it will not cover the true value in the vast majority of cases that actually arise. For this reason I cannot understand the contention that the probability of covering a fixed parameter point is irrelevant to inferences of this type.

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