was from the first (second) sample, e.g., if the observed values in the first sample were $(-1.5, 2.6)$ and in the second sample $(3.4, -.9)$, then $z = (0101)$.

RULE II. To compute $P_{m,n}(z)$ add all $[(m + n + 1)$ in number$]$ of the $P_{m+1,n}(z^j)$ and divide by $(m + 1)$, where

$$z^j = (z_1, \cdots, 0, z_j, \cdots, z_{m+n}), j = 1, \cdots, (m + n + 1).$$

Note a. Several of the z^j will be the same.

Note b. The roles of m and n can be interchanged in the obvious manner.

Note c. The rule can be obtained using the analytic expression

$$P_{m,n}(z) = m!n! \int_{-\infty < w_1 \cdots w_{m+n} < \infty} \cdots \int \prod_{i=1}^{m+n} [f^{1-z_i}(w_i)g^{z_i}(w_i)\, dw_i],$$

where $f(w)[g(w)]$ is the density of the first [second] population. Another proof can be obtained by noting that, after the samples of size m and n have been obtained, an additional observation from the first population must either be between a pair of the observations of the original $m + n$ or before or after them.

EXAMPLE II. For the two-sample problem with $m = 3$ and $n = 2$,

$$P_{3,2}(00011) = [P_{3,3}(100011) + P_{3,3}(010011) + P_{3,3}(001011) + 3P_{3,3}(000111)]/3.$$

Teichroew [3] gives .0394 as the exact value, and .0410 as the Monte Carlo value (2000 samples) when the two populations are normal with means differing by $\frac{1}{2}$ of the common standard deviation. Using Teichroew's [3] Monte Carlo results for $m = 3$, $n = 3$ (4000 samples) in the above formula, one obtains $P_{3,2}(00011) = [.03250 + .01825 + .011875 + 3(.01675)]/3 = .03992$. Additional results for $m = 3$, $n = 2$ could be obtained from $m = 4$, $n = 2$ and from $m = 4$, $n = 3$ via $m = 3$, $n = 3$ [3].

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AN INEQUALITY FOR BALANCED INCOMPLETE BLOCK DESIGNS

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1. Summary. The inequality $b \geqq v + r - 1$ for a balanced incomplete block design was proved by Bose [1] under the assumption of resolvability. In this note

Received August 5, 1959; revised October 17, 1959.

the inequality is proved without that assumption, but with the weaker assumption that $v = nk$.

2. Introduction. A b.i.b. design is an arrangement of v treatments in b blocks of size $k < v$ such that (i) every block contains k distinct treatments, (ii) every treatment occurs in r blocks, and (iii) any two treatments occur together in λ blocks.

The parameters satisfy

$$vr = bk, \tag{2.1}$$

$$\lambda(v - 1) = r(k - 1), \tag{2.2}$$

$$b \geqq v, \qquad r \geqq k. \tag{2.3}$$

The last inequality is due to Fisher [2].

If the blocks can be partitioned into r sets of n blocks each so that in each set every treatment occurs exactly once, the design is called resolvable. Obviously then $v = nk$ and $b = nr$, but the converse need not hold. Bose [1] proved that if a resolvable design with parameters v, b, r, k, λ exists, then $b \geqq v + r - 1$.

3. Theorem. *If a* b.i.b. *design with parameters* $v = nk$, b, r, k, λ *exists, then*

$$b \geqq v + r - 1. \tag{3.1}$$

PROOF: Obviously $v > k$ implies $n \geqq 2$.

We first prove that $r > k$. Since $r \geqq k$, assume on the contrary that $r = k$. Then from (2.2), $\lambda(nk - 1) = k(k - 1)$. Hence $n\lambda = (k - 1) + \lambda/k$. Since $n\lambda$ is an integer, λ/k is an integer, which is a contradiction since, from (2.2), $\lambda < r = k$. Hence we have

$$r > k. \tag{3.2}$$

The inequality $b \geqq v + r - 1$, under the assumption that $v = nk$, is equivalent to

$$r \geqq (nk - 1)/(n - 1) \tag{3.3}$$

since $n - 1 \geqq 1$ is positive. Further, from (2.2), we have

$$r = \lambda(nk - 1)/(k - 1), \tag{3.4}$$

i.e.,

$$n = (r(k - 1) + \lambda)/\lambda k, \tag{3.5}$$

and

$$(k - 1)/(n - 1) = \lambda k/(r - \lambda). \tag{3.6}$$

From (3.3), (3.4), (3.6) we have

$$r - \lambda \geqq k. \tag{3.7}$$

It is therefore sufficient to prove (3.7).

Assume that the contrary is true, i.e., $k > r - \lambda$. Put

$$\lambda = r - k + i \tag{3.8}$$

where $1 \leqq i \leqq k - 1$, since $\lambda < r$. Substituting in (3.5), we get

$$n = (rk - k + i)/(rk - k^2 + ik).$$

From (3.2), we put $r = k + j$, where j is an integer > 1, and obtain

$$n = \frac{k}{j + i} + \frac{j - 1}{j + i} + \frac{i}{k(j + i)}. \tag{3.9}$$

Consider (3.9) and assume that $j + i$ divides k. Then, since $(j - 1)/(j + i)$ and $i/k(j + i)$ are both positive proper fractions and n is an integer, we must have $[(j - 1)/(j + i)] + (i/k(j + i)) = 1$, which implies that $i = -k/(k - 1) < 0$. This is a contradiction since $i \geqq 1$.

Now assume that $j + i$ does not divide k. Then, if $k < j + i$, all the terms on the right hand side of (3.9) are positive proper fractions and

$$\frac{j - 1}{j + i} + \frac{i}{k(j + i)} = \frac{kj + (i - k)}{kj + ki}, \tag{3.10}$$

which is <1 since k, j, i are all positive. Hence, since n is an integer, the only possibility is that $n = 1$, which is a contradiction.

Now assume that $k > j + i$ is not divisible by $j + i$. Then $k \equiv m \pmod{j + i}$, where $1 \leqq m \leqq j + i - 1$. In this case (3.9) gives

$$\frac{m}{j + i} + \frac{j - 1}{j + i} + \frac{i}{k(j + i)} = 1. \tag{3.11}$$

Since all the terms are positive proper fractions and the sum of the last two terms is also a positive proper fraction, (3.11) gives $i = m - 1 + [(m - 1)/(k - 1)]$, which is a contradiction since i is an integer and $k > j + i > m$ implies

$$m - 1 > k - 1.$$

Thus (3.8) is contradicted in all cases. Hence obviously $r - \lambda \geqq k$ in all cases. This completes the proof.

4. Acknowledgment. I am grateful to Dr. S. S. Shrikhande for kindly going through the proof and making suggestions.

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