A ONE-SIDED ANALOG OF KOLMOGOROV'S INEQUALITY[1]

By Albert W. Marshall

Stanford University

1. Introduction and summary. It is well known (see e.g. [4] p. 198) that for every positive ϵ and every square integrable random variable X with zero expectation, $P\{X \geqq \epsilon\} \leqq E(X^2)/[\epsilon^2 + E(X^2)]$. In this paper an inequality is obtained that generalizes this in the same way that Kolmogorov's inequality generalizes Chebyshev's inequality. The inequality is proved in Section 2 and an example is given to show that equality can be achieved. In Section 3 an extension to continuous parameter martingales is obtained, and a condition under which equality can be achieved is given.

2. The inequality.

Theorem 2.1. *Let* $X_1, X_2, \cdots, X_n$ *be random variables with* $E(X_1) = 0$, $E(X_i \mid X_1, X_2, \cdots, X_{i-1}) = 0$ a.e. $(i = 2, 3, \cdots, n)$, *and* $E(X_i^2) = \sigma_i^2 < \infty$, $(i = 1, 2, \cdots, n)$. *Then, for every positive* ϵ,

$$(1) \quad P\{\max_{1 \leqq i \leqq n} (X_1 + X_2 + \cdots + X_i) \geqq \epsilon\} \leqq s_n/(\epsilon^2 + s_n), \quad \textit{where } s_n = \sum_{i=1}^n \sigma_i^2 .$$

Note that, if $Y_i = \sum_{k=1}^i X_k$, $i = 1, 2, \cdots, n$, then $\{Y_i, 1 \leqq i \leqq n\}$ is a martingale and $E(Y_n^2) = s_n$.

Proof. Let $F(x) = F(x_1, x_2, \cdots, x_n) = (\epsilon \sum_{i=1}^n x_i + s_n)^2/(\epsilon^2 + s_n)^2$, and let

$$B_k = \{X_1 + X_2 + \cdots + X_i < \epsilon, i = 1, 2, \cdots, k-1,$$
$$X_1 + X_2 + \cdots + X_k \geqq \epsilon\}, \qquad k = 1, 2, \cdots, n.$$

Then

$$\int F(X)\, dP \geqq \sum_{k=1}^n \int_{B_k} F(X)\, dP \geqq \frac{1}{(\epsilon^2 + s_n)^2} \sum_{k=1}^n \int_{B_k} \left(\epsilon \sum_{i=1}^k X_i + s_n\right)^2 dP$$
$$\geqq \sum_{k=1}^n P(B_k) = P\{\max_{1 \leqq i \leqq n} (X_1 + \cdots + X_i) \geqq \epsilon\}.$$

Since $\int F(X)\, dP = s_n/(\epsilon^2 + s_n)$, the proof is complete. Note the similarity of this proof to the standard proof of Kolmogorov's inequality (see e.g. [1] p. 105, 314 or [3] p. 235, 386).

To show that equality can be achieved in (1), let $s_k = \sum_{i=1}^k \sigma_i^2$, $k = 1, 2, \cdots, n$, and let $Z = (Z_1, Z_2, \cdots, Z_n)$ be a random variable having the following distribution:

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$$P\{Z = (\epsilon, 0, \cdots, 0)\} = \sigma_1^2/(\epsilon^2 + s_1) = p_1,$$

$$P\left\{ Z = \epsilon^{-1}(-\sigma_1^2, -\sigma_2^2, \cdots, -\sigma_{k-1}^2, \epsilon^2 + s_{k-1}, 0, \cdots, 0)\right\}$$
$$= \frac{\epsilon^2 \sigma_k^2}{(\epsilon^2 + s_{k-1})(\epsilon^2 + s_k)} = p_k, \qquad k = 2, 3, \cdots, n,$$

$$P\{Z = \epsilon^{-1}(-\sigma_1^2, -\sigma_2^2, \cdots, -\sigma_n^2)\} = \epsilon^2/(\epsilon^2 + s_n).$$

It is easily verified by induction on j that

$$\sum_{k=1}^{j} p_k = 1 - \epsilon^2/(\epsilon^2 + s_j), \qquad j = 1, 2, \cdots, n, \tag{2}$$

so that this is a valid probability distribution. Clearly, $E(Z_1) = 0$. It can be shown that $E(Z_j \mid Z_1, \cdots, Z_{j-1}) = 0$ a.e. (by first computing

$$E(Z_j \mid Z_{j-1} \neq \sigma_{j-1}^2/\epsilon) \qquad \text{and} \qquad E(Z_j \mid Z_{j-1} = -\sigma_{j-1}^2/\epsilon))$$

and that $E(Z_j^2) = \sigma_j^2$, $j = 1, 2, \cdots, n$. Thus the random variable Z satisfies the conditions of Theorem 2.1; furthermore, equality holds in (1) whenever $(X_1, \cdots, X_n) = Z$ a.e.

Kolmogorov's inequality has been extended under certain conditions by Hájek and Rényi [2] to provide a bound for

$$P\{\max_i \epsilon_i^{-1} \mid X_1 + \cdots + X_i \mid \geqq 1\} \qquad (\epsilon_i > 0, i = 1, 2, \cdots, n),$$

and it is natural now to ask what the best upper bound is for

$$P\{\max_i \epsilon_i^{-1}(X_1 + \cdots + X_i) \geqq 1\}$$

under the conditions of Theorem 2.1. Unfortunately this bound has no simple expression even for small n, and is not easily obtained. It is given here only for $n = 2$.

THEOERM 2.2. *Let X_1 and X_2 be random variables with $E(X_1) = 0$, $E(X_2 \mid X_1) =$ 0 a.e., and $E(X_i^2) = \sigma_i^2 < \infty$, $i = 1, 2$. Then if $\epsilon_1 > 0$ and $\epsilon_2 > 0$,*

$$P\{X_1 \geqq \epsilon_1 \text{ or } X_1 + X_2 \geqq \epsilon_2\} \leqq \frac{\sigma_2^2 + \sigma_1^2(\alpha_2/\alpha_1)^2}{\sigma_2^2 + \alpha_2^2/\alpha_1}, \tag{3}$$

where $\alpha_i = \sigma_1^2 + \eta_1\eta_i$, $i = 1, 2$, and $\eta_1 = \min(\epsilon_1, \epsilon_2)$, $\eta_2 = \epsilon_2$.

PROOF. Following the method of Hájek and Rényi [2], we let $F(x_1, x_2) = c_1F_1^2(x_1) + c_2F_2^2(x_1 + x_2)$, where

$$c_1 = \frac{\eta_1^2}{\alpha_1^2} - \frac{\eta_1^2(\alpha_2 + \sigma_2^2)^2}{(\alpha_2^2 + \sigma_2^2\alpha_1)^2}, \qquad c_2 = \frac{\eta_1^2\alpha_2^2}{(\alpha_2^2 + \sigma_2^2\alpha_1)^2},$$

$$F_1(x) = \left(x + \frac{\sigma_1^2}{\eta_1}\right), \qquad F_2(x) = \left(x + \frac{\sigma_1^2}{\eta_1} + \frac{\sigma_2^2\alpha_1}{\eta_1\alpha_2}\right),$$

and we let $B_1 = \{X_1 \geqq \eta_1\}$, $B_2 = \{X_1 < \eta_1, X_1 + X_2 \geqq \eta_2\}$. Since $\alpha_2 \geqq \alpha_1 > 0$, it follows that $c_1 \geqq 0$, and, as in the proof of Theorem 2.1,

$$\int_{B_1} F(X_1, X_2)\, dP \geqq P(B_1),$$

$$\int_{B_2} F(X_1, X_2)\, dP \geqq \int_{B_2} c_2 F_2^2(X_1 + X_2)\, dP = P(B_2).$$

Thus $\int F(X_1, X_2)\, dP \geqq P(B_1) + P(B_2) = P\{X_1 \geqq \eta_1 \text{ or } X_1 + X_2 \geqq \eta_2\} \geqq P\{X_1 \geqq \epsilon_1 \text{ or } X_1 + X_2 \geqq \epsilon_2\}$. It is straightforward to verify that, upon integrating the function $F(X_1, X_2)$, one obtains the bound given in (3), and this completes the proof.

Equality is achieved in (3) whenever (X_1, X_2) has the following distribution:

$$P\{(X_1, X_2) = (\eta_1, 0)\} = \sigma_1^2/\alpha_1, \qquad P\left\{(X_1, X_2) = \left(-\frac{\sigma_1^2}{\eta_1}, \frac{\alpha_2}{\eta_1}\right)\right\}$$

$$= \frac{\eta_1^2\, \sigma_2^2}{\alpha_1\, \sigma_2^2 + \alpha_2^2}, \qquad P\left\{(X_1, X_2) = \left(-\frac{\sigma_1^2}{\eta_1}, -\frac{\sigma_2^2\, \alpha_1}{\eta_1\, \alpha_2}\right)\right\} = \frac{\eta_1^2\, \alpha_2^2}{\alpha_1(\alpha_1\, \sigma_2^2 + \alpha_2^2)}.$$

In this case, $P\{X_1 \geqq \eta_1 \text{ or } X_1 + X_2 \geqq \eta_2\} = P\{X_1 \geqq \epsilon_1 \text{ or } X_1 + X_2 \geqq \epsilon_2\}$.

Several inequalities follow from (3) simply by a change of variables. The corollaries below are given to illustrate the possibilities.

COROLLARY 2.3. *Let X_1 and X_2 be random variables with $E(X_1) = a$, $E(X_2 \mid X_1) = bX_1 + c$ a.e. (where $b \neq -1$), and* $\operatorname{Var}(X_i) = \sigma_i^2 < \infty$, $i = 1, 2$. *Then if $\epsilon_1 - a > 0$ and $[\epsilon_2 - \delta(a + ab + c)]/|b + 1| > 0$ where* $\delta = \operatorname{sign}(b + 1)$,

$$(4) \qquad P\{X_1 \geqq \epsilon_1 \text{ or } \delta(X_1 + X_2) \geqq \epsilon_2\} \leqq \frac{\sigma_2^2 - b^2\sigma_1^2 + \sigma_1^2[(b + 1)\alpha_2/\alpha_1]^2}{\sigma_2^2 - b^2\sigma_1^2 + [(b + 1)^2\alpha_2^2/\alpha_1]}$$

where $\alpha_i = \sigma_i^2 + \eta_1\eta_i$, $i = 1, 2$, and $\eta_2 = [\epsilon_2 - \delta(a + ab + c)]/|b + 1|$, $\eta_1 = \min(\epsilon_1 - a, \eta_2)$.

PROOF. This follows from Theorem 2.2 by making the change of variables

$$X_1' = X_1 + a, \qquad X_2' = bX_1 + (b + 1)X_2 + ab + c,$$

$$\epsilon_1' = \epsilon_1 + a, \qquad \epsilon_2' = \epsilon_2 |b + 1| + \delta(a + ab + c)$$

and dropping the primes.

Note that by taking $a = b = c = 0$ in this corollary, one obtains Theorem 2.2.

COROLLARY 2.4. *Let X_1 and X_2 be random variables such that $E(X_i) = \mu_i$,* $\operatorname{Var}(X_i) = \sigma_i^2 < \infty$, $i = 1, 2$, $\operatorname{Cov}(X_1, X_2) = \sigma_{12} \neq 0$, *and suppose that the regression of X_2 on X_1 is linear. Then, if $\epsilon_1 - \mu_1 > 0$ and $(\delta\epsilon_2 - \sigma_1^2\mu_2)/\sigma_{12} > 0$, where $\delta = \operatorname{sign} \sigma_{12}$,*

$$(5) \qquad P\{X_1 \geqq \epsilon_1 \text{ or } \delta X_2 \geqq \epsilon_2\} \leqq \frac{\sigma_1^2(\sigma_1^2\, \sigma_2^2 - \sigma_{12}^2) + \sigma_1^2(\alpha_2\, \sigma_{12}/\alpha_1)^2}{\sigma_1^2(\sigma_1^2\, \sigma_2^2 - \sigma_{12}^2) + (\alpha_2^2\, \sigma_{12}^2/\alpha_1)},$$

where $\alpha_i = \sigma_i^2 + \eta_1\eta_i$, $i = 1, 2$, and $\eta_2 = (\delta\epsilon_2 - \sigma_1^2\mu_2)/\sigma_{12}$, $\eta_1 = \min(\epsilon_1 - \mu_1, \eta_2)$.

PROOF. To obtain (5) from (3), make the change of variables $X_1' = X_1 + \mu_1$,

$X_2' = [\sigma_{12}'(X_1 + X_2)/\sigma_1'^2] + \mu_2$, $\epsilon_1' = \epsilon_1 + \mu_1$ and $\epsilon_2' = \delta(\epsilon_2\sigma_{12}' + \sigma_1'^2\mu_2)$ in (3), and then remove the primes.

3. An extension to continuous parameter martingales. We begin by assuming that the underlying probability space is such that P is complete. Then we have the following:

THEOREM 3.1. *If* $\{Y_t, t \geqq 0\}$ *is a separable martingale with* $E(Y_t) = 0$ *and* $E(Y_t^2) = \sigma^2(t) < \infty$ *for all* $t \geqq 0$, *then, for every positive* ϵ *and* τ,

$$P\left\{ \sup_{t\epsilon[0,\tau]} Y_t \geqq \epsilon \right\} \leqq \frac{\sigma^2(\tau)}{\epsilon^2 + \sigma^2(\tau)}. \tag{6}$$

PROOF. Let $0 = t_1 \leqq t_2 \leqq \cdots \leqq t_n = \tau$. Since $X_1 = Y_{t_1}$ and $X_i = Y_{t_i} - Y_{t_{i-1}}$, $i = 2, 3, \cdots, n$ satisfy the conditions of Theorem 2.1,

$$P\{\max_{1\leqq i\leqq n} Y_{t_i} \geqq \epsilon\} \leqq \sigma^2(\tau)/[\epsilon^2 + \sigma^2(\tau)]. \tag{7}$$

Let S be a countable set satisfying the definition of separability and containing the points 0 and τ. Taking the supremum of the left side of (7) over all finite subsets of $S \cap [0, \tau]$, we obtain

$$P\{\sup_{t\epsilon S\cap[0,\tau]} Y_t \geqq \epsilon\} \leqq \sigma^2(\tau)/[\epsilon^2 + \sigma^2(\tau)].$$

But

$$P\{\sup_{t\epsilon S\cap[0,\tau]} Y_t \geqq \epsilon\} = P\{\sup_{t\epsilon[0,\tau]} Y_t \geqq \epsilon\},$$

and the proof is complete.

THEOREM 3.2. *Equality can be achieved in* (6) *if* $\sigma^2(\cdot)$ *is right continuous.*

PROOF. In order to define a martingale that achieves equality in (6), let $\Omega = \{-1\} \cup [0, \infty)$, $\mathcal{B}$ be the Borel subsets of Ω, and let P be the probability measure defined on $\mathcal{B}$ by

$$P(B) = \{\epsilon^2/[\epsilon^2 + \lim_{x\to\infty} \sigma^2(x)]\}\chi_{B\cap\{-1\}} + \mu(B \cap [0, \infty)),$$

where χ_E is the characteristic function of the set E and μ is the measure induced on the Borel subsets of $[0, \infty]$ by the right continuous distribution function $\sigma^2(\cdot)/[\epsilon^2 + \sigma^2(\cdot)]$. Let $\{Z_t, t \geqq 0\}$ be defined on $(\Omega, \mathcal{B}, P)$ by

$$Z_t(\omega) = \begin{cases} -\sigma^2(t)/\epsilon & 0 \leqq t < \omega \\ \epsilon & 0 \leqq \omega \leqq t \\ \text{or } \omega = -1 & \end{cases}$$

Then

$$P\{\sup_{t\epsilon[0,\tau]} Z_t \geqq \epsilon\} = P\{0 \leqq \omega \leqq \tau\} = \sigma^2(\tau)/[\epsilon^2 + \sigma^2(\tau)],$$

and it remains only to verify that the process $\{Z_t, t \geqq 0\}$ satisfies the conditions of Theorem 3.1. We compute

$$E(Z_t) = [-\sigma^2(t)P\{\omega > t \text{ or } \omega = -1\}/\epsilon] + \epsilon P\{0 \leqq \omega \leqq t\} = 0,$$

and similarly obtain $E(Z_t^2) = \sigma^2(t)$, $t \geqq 0$. Clearly $E\{Z_t \mid Z_s = \epsilon\} = Z_s$ where $0 \leqq s < t$ are fixed. Let $\theta = E\{Z_t \mid Z_s = -\sigma^2(s)/\epsilon\}$; using the relation

$$0 = E(Z_t) = E[E(Z_t \mid Z_s)] = \epsilon P\{Z_s = \epsilon\} + \theta P\{Z_s = -\sigma^2(s)/\epsilon\},$$

we obtain $\theta = -\sigma^2(s)/\epsilon$. Hence the process $\{Z_t, t \geqq 0\}$ is a martingale satisfying the conditions of Theorem 3.1 and achieving equality in (3.1).

REFERENCES

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