A GENERALIZED PITMAN EFFICIENCY FOR NONPARAMETRIC TESTS

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0. Summary. Asymptotic expressions up to terms of order n^{-2} are given for the efficiency of the Wilcoxon two-sample test relative to the standard-normal test and t-test for nearby alternatives. The first term is the well-known Pitman efficiency; the remaining terms are corrections for finite sample sizes. Efficiency values are given for finite sample sizes in the case of normal and rectangular distributions, and comparisons of the asymptotic with the exact efficiency values are made. In general, the Wilcoxon test is shown to be nearly as good locally for moderate sample sizes as it is known to be asymptotically. A similar analysis is performed for the single-sample sign test.

1. The concept of efficiency. Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be independent and identically distributed according to the continuous distributions $F(x)$ and $G(x) = F(x - \theta)$, respectively. Let $p_{m,n}(\theta)$ and $\beta_{m,n}(\theta)$ denote the power of two tests for the hypothesis $\theta = 0$ against the alternative $\theta > 0$ at the same level of significance α. Then the efficiency of the first test relative to the second (for given values of θ, α, m and n), is

$$e(\theta, \alpha, m, n) = n^*/n \tag{1.1}$$

where n^* (not necessarily integer-valued) is defined by

$$p_{m,n}(\theta) = \beta_{m^*,n^*}(\theta), \qquad m/n = m^*/n^*. \tag{1.2}$$

Assume that the first derivatives of the power functions are continuous at $\theta = 0$, with values $\tilde{p}_{m,n} = p'_{m,n}(0)$ and $\tilde{\beta}_{m,n} = \beta'_{m,n}(0)$, respectively. Then condition (1.2) reduces in the vicinity of the hypothesis to

$$\tilde{p}_{m,n} = \tilde{\beta}_{m^*,n^*}, \qquad m/n = m^*/n^*, \tag{1.3}$$

which can often be easily expressed in the form of an asymptotic series, as for the sign test, which is done in Section 6.

In Sections 2, 3, and 4 we shall derive an approximation to the efficiency in terms of (1.3) for the one-sided Wilcoxon test, using the Edgeworth expansion up to terms of order $O(n^{-2})$. This expansion gives a good approximation to the null-distribution of the Wilcoxon test, [4] and [7], and its applicability in our problem seems also to be largely confirmed by comparison with the exact values,

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where such a comparison is possible. Whether it presents a valid asymptotic expression is an open question, which appears to be of secondary importance with regard to getting approximate values for the efficiency, but in any case we shall denote the error of the Edgeworth expansion by the O-symbol of the first neglected term to indicate in a simple manner what terms of this expansion are taken into account. In Section 5 we shall give the results of a corresponding analysis for the two-sided Wilcoxon test.

2. Edgeworth approximation of the Mann-Whitney statistic. The Mann-Whitney statistic U for the Wilcoxon test is

$$U = \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi(X_i, Y_j), \qquad \varphi(X, Y) = 1(0) \qquad \text{as } X > Y(X \leqq Y) \tag{2.1}$$

The first four moments of U under the null hypothesis $\theta = 0$ are [4]

$$(EU)^0 = \xi^0 = mn/2, \qquad (\text{Var } U)^0 = \mu_2^0 = mn(m + n + 1)/12, \qquad \mu_3^0 = 0,$$
$$\mu_4^0 = [mn(m + n + 1)/240][5(m^2n + mn^2) - 2(m^2 + n^2) + 3mn - 2(m + n)]. \tag{2.2}$$

General expressions for the first four moments are given by R. M. Sundrum [6], from which the following formulae for the derivatives at $\theta = 0$ can be obtained

$$\begin{aligned}
\tilde{\xi} &= -mnA \\
\tilde{\mu}_2 &= mn(m - n)(2B - A) \\
\tilde{\mu}_3 &= \tfrac{1}{2}Am^2n^2 + (-\tfrac{1}{2}A + 3B - 3C)(-4m^2n^2 + m^3n + mn^3 + m^2n + mn^2) \\
\tilde{\mu}_4 &= (-\tfrac{1}{2}A + B)(m^4n^2 - m^2n^4) + (2B - 6C + 4D)(m^4n - mn^4) \\
&\quad + (5A/2 - 27B + 66C - 44D)(m^3n^2 - m^2n^3) \\
&\quad + (-A + 12B - 30C + 20D)(m^3n - mn^3),
\end{aligned} \tag{2.3}$$

with the abbreviations

$$A = \int f^2\, dx, \qquad B = \int Ff^2\, dx, \qquad C = \int F^2f^2\, dx, \qquad D = \int F^3f^2\, dx. \tag{2.4}$$

In the particular case that the underlying distribution is symmetric, $F(x) + F(-x) = 1$, we have

$$2B = A, \qquad 4D = -A + 6C, \tag{2.5}$$

so that (2.3) simplifies to

$$\begin{aligned}
\tilde{\xi} &= -mnA, \qquad \tilde{\mu}_2 = 0, \qquad \tilde{\mu}_4 = 0, \\
\tilde{\mu}_3 &= \tfrac{1}{2}Am^2n^2 + (A - 3C)(-4m^2n^2 + m^3n + mn^3 + m^2n + mn^2).
\end{aligned} \tag{2.6}$$

Lehmann [5], among others, proved that U is asymptotically normally distributed. As mentioned above, the Edgeworth expansion with continuity correction up to terms $O(n^{-2})$ can be applied as an asymptotic expression, i.e., the power of this test can be approximated [2] by

$$p_\theta(U \leqq u \mid m, n) = \phi(x) + e_2\varphi^{(2)}(x) + e_3\varphi^{(3)}(x) + e_5\varphi^{(5)}(x) + \hat{e}\hat{\varphi}(x) + O(n^{-2}) \tag{2.7}$$

for a fixed value of x, where x is the normalized value of u with continuity correction,

$$x = (u + \tfrac{1}{2} - EU)/(\mu_2)^{1/2}. \tag{2.8}$$

The Edgeworth coefficients,

$$\begin{aligned} e_2 &= -\frac{1}{3!}\frac{\mu_3}{\mu_2^{3/2}} = \tilde{e}_2\,\theta + O(\theta^2), & e_2^0 &= 0, \\ e_3 &= \frac{1}{4!}\left(\frac{\mu_4}{\mu_2^2} - 3\right) = e_3^0 + \tilde{e}_3\theta + O(\theta^2), & e_3^0 &= \frac{1}{4!}\left(\frac{\mu_4^0}{(\mu_2^0)^2} - 3\right), \\ e_5 &= \frac{10}{6!}\frac{\mu_3^2}{\mu_2^3} = O(\theta^2), & e_5^0 &= \tilde{e}_5 = 0, \end{aligned} \tag{2.9}$$

are of order $n^{-1/2}$, n^{-1} and n^{-1}, respectively, and $\hat{e}\hat{\varphi}(x)$ symbolizes the terms of order $O(n^{-3/2})$.

In our problem, however, x is not a fixed constant, since the significance probability α is considered to be given, and the location parameter θ tends to zero. More precisely, on the one hand, u and α are connected by (2.7), for $\theta = 0$, by

$$P_0(U \leqq u \mid m, n) = \phi(x_0) + e_3^0\varphi^{(3)}(x_0) + O(n^{-2}) = \alpha, \tag{2.10}$$

where x_0 is the normalized value of u for $\theta = 0$, which can be determined from α by solving (2.10) asymptotically using $e_3^0 = O(n^{-1})$:

$$x_0 = \frac{u + \frac{1}{2} - mn/2}{(mn(m + n + 1)/12)^{1/2}} = \Phi^{-1}(\alpha) - e_3^0\,[3\Phi^{-1}(\alpha) - \{\Phi^{-1}(\alpha)\}^3] + 0(n^{-2}). \tag{2.11}$$

On the other hand, x and x_0 are connected by (2.8), which can be written in the form

$$x = x_0 + x_1\,\theta + O(\theta^2), \qquad x_1 = x'(0) = \frac{mnA}{(\mu_2^0)^{1/2}} - \frac{x_0}{2}\frac{\tilde{\mu}_2}{\mu_2^0}, \tag{2.12}$$

where $x_1 = O(n^{1/2})$.

Using the fact that both the normalized variable x and the Edgeworth coefficients depend on θ, and by differentiating (2.7) with respect to θ at $\theta = 0$, we get[3]

[3] Here the remainder is still $O(n^{-3/2})$, since $\hat{e}\hat{\varphi}(x)$ vanishes for $\theta = 0$.

$$(2.13)\quad \tilde{p} = x_1\varphi(x_0) + \overset{0}{e}_3 x_1\varphi^{(4)}(x_0) + \tilde{e}_2\varphi^{(2)}(x_0) + \tilde{e}_3\varphi^{(3)}(x_0) + O(n^{-3/2}),$$

or, by means of (2.2), (2.11) and (2.12),

$$(2.14)\quad \begin{aligned} \tilde{p} = \varphi(\Phi^{-1}(\alpha))\Bigg[&\left(\frac{12mn}{m+n+1}\right)^{1/2} A - \frac{\tilde{\mu}_2}{2\overset{0}{\mu}_2}\Phi^{-1}(\alpha) \\ &- \left(\tilde{e}_2 - 3\left(\frac{12mn}{m+n+1}\right)^{\frac{1}{2}} A\overset{0}{e}_3\right)(1 - \{\Phi^{-1}(\alpha)\}^2) \\ &+ \frac{\tilde{\mu}_2}{\overset{0}{\mu}_2}\overset{0}{e}_3\{\Phi^{-1}(\alpha)\}^3 + \tilde{e}_3(3\Phi^{-1}(\alpha) - \{\Phi^{-1}(\alpha)\}^3)\Bigg] + O(n^{-3/2}). \end{aligned}$$

In particular, for symmetric distributions $\tilde{\mu}_2 = \tilde{e}_3 = 0$ and (2.14) simplifies to

$$(2.15)\quad \tilde{p} = \varphi(\Phi^{-1}(\alpha))\left(\frac{12mn}{m+n+1}\right)^{1/2} A\,[1 + K_{mnAC}(1 - \{\Phi^{-1}(\alpha)\}^2) + O(n^{-2})]$$

with

$$(2.16)\quad K_{mnAC} = \frac{\begin{array}{r}[mn + 2(1 - 3(C/A))(-4mn + m^2 + n^2 + m + n) \\ - 0.15(m^2 + n^2 + mn + m + n)]\end{array}}{[mn(m+n+1)]}$$

3. Efficiency relative to the standard normal test. Let X_i and Y_j be normally distributed with known variance σ^2. Then $(\bar{Y} - \bar{X})/\sigma)\ (mn/m + n)^{1/2}$ is

$$N((\theta/\sigma)(mn/m + n)^{1/2}, 1)$$

and the power of the standard normal test, written $\bar{x}$-test for short, is

$$(3.1)\quad \begin{aligned} \beta(\theta) &= \Phi\left((\theta/\sigma)\left(\frac{mn}{m+n}\right)^{1/2} - \Phi^{-1}(1-\alpha)\right) \\ &= \alpha + (\theta/\sigma)\left(\frac{mn}{m+n}\right)^{1/2}\varphi(\Phi^{-1}(\alpha)) + O(\theta^2). \end{aligned}$$

Therefore, (1.3) can easily be solved for n^*/n in terms of m and n, and we get for the efficiency relative to the standard normal test

$$(3.2)\quad \begin{aligned} e_{\mathrm{AS}.\bar{x}} &= ((1/m) + (1/n))\,\sigma^2(\tilde{p}/\varphi(\Phi^{-1}(\alpha)))^2 = 12\sigma^2\left(\int f^2(x)\,dx\right)^2 \\ &\cdot\left[1 - (1/(m+n+1)) + 2K_{mnAC}(1 - \{\Phi^{-1}(\alpha)\}^2 + O(n^{-2})\right], \end{aligned}$$

where $A = 1/(2(\pi)^{1/2}) = 0.282095$, $1 - 3C/A = 0.087733$.

For small m and n, the exact values of $\tilde{p} = p'(0)$ can be derived from the integrals by which the power, $p(\theta)$, is represented. (Dixon [3] has used these integrals for computing the power numerically for some values of $\theta > 0$.) So a comparison is possible. The accuracy becomes worse, of course, for decreasing values of m and n. Some comparisons are shown in Table 3, Section 4.

TABLE 1

Comparison of the efficiency values $e_{As.\bar{x}}$ and $e_{Ex.\bar{x}}$ for underlying rectangular distribution

	$m = n = 20$	$m = 20, n = 10$	$m = n = 10$
α	0.0298	0.0245	0.0315
$e_{Ex.\bar{x}}$	0.9071	0.8780	0.8210
$e_{As.\bar{x}}$	0.9091	0.8838	0.8308
relative error	0.22%	0.66%	1.19%

TABLE 2

Efficiency values $e_{Ex.\bar{x}}$ for $\theta > 0$ compared with the corresponding limit values for $\theta \to 0$ for underlying rectangular distribution

$m = n = 20$		$m = 20, n = 10$		$m = n = 10$	
α	0.0298	α	0.0245	α	0.0315
$e(0)$	0.9071	$e(0)$	0.8780	$e(0)$	0.8210
$e(0.01)$	0.9031	$e(0.02)$	0.8684	$e(0.05)$	0.8139
$e(0.02)$	0.9022	$e(0.05)$	0.8553	$e(0.10)$	0.8053

When the X_i and Y_j are distributed according to any other distribution with finite fourth moment, μ_4, and known σ^2, then $((\bar{Y} - \bar{X})/\sigma)(mn/m + n)^{1/2}$ is normally distributed up to terms $O(n^{-2})$, and (3.2) can also be applied. For the rectangular distribution $R(0, 1)$, in particular, we have $A = 1, 1 - 3C/A = 0$. In this case the exact values of $\bar{p}$ can be shown to be

$$\begin{aligned}\bar{p} &= -(m + n)p_0(U \leqq u \mid m, n) + mp_0(U \leqq u \mid m - 1, n) \\ &\quad + np_0(U \leqq u \mid m, n - 1) = n[p_0(U \leqq u \mid m, n - 1) \\ &\qquad - p_0(U \leqq u - m \mid m, n - 1)].\end{aligned} \tag{3.4}$$

This comes from a private communication of Professor J. L. Hodges, Jr., which is based on the fact that only those X_i and Y_j which fall in the interval $\theta \leqq x \leqq 1$ contribute anything to (2.1), and they have there the same conditional distribution,

$$p_\theta(U \leqq u \mid m, n) = \sum_{k=0}^{m} \sum_{l=0}^{n} b(m, \theta, k)b(n, \theta, l)p_0(U \leqq u \mid m - k, n - l). \tag{3.5}$$

Expressions (3.4) and (3.5) can be evaluated numerically for $m, n \leqq 20$ by means of the tables of D. Auble [1] (Tables 1–4), on which the comparisons and statements are based.

The concept of efficiency for nearby alternatives is valid only in the special case of small values of θ, but here the choice of the appropriate test is of special importance. The zero-order approximation of (3.2) is the well-known Pitman efficiency, which is $3/\pi = 0.955$ in the case of normal alternatives and 1 for rectangular alternatives. The first-order approximation now indicates how the efficiency approximately changes with the significance probability α and the sample sizes m and n. Besides depending on the Pitman parameter $\sigma^2(\int f^2(x)\,dx)^2$, it depends on the special underlying distribution only through the parameter $C = \int F^2(x)f^2(x)\,dx$.

4. Efficiency relative to the t-test. A more realistic comparison than that made in the preceding section is one made with the t-test, which is appropriate for unknown, but common variance σ^2. Let us restrict ourselves to underlying normal distributions. Expanding the density $t_\delta(x)$ of the noncentral t-distribution with respect to the noncentrality parameter $\delta = (\theta/\sigma)(mn/m + n)^{1/2}$, one can verify that the derivative of the power function, $\beta(\theta) = \int_C^\infty t_\delta(x)\,dx$, at $\theta = 0$ is given by

$$\begin{aligned}\dot\beta &= \frac{1}{\sigma}\left(\frac{mn}{m+n}\right)^{1/2}\frac{1}{(2\pi)^{1/2}}\left(1+\frac{C^2}{m+n-2}\right)^{-(m+n-2)/2} \\ &= \frac{1}{\sigma}\left(\frac{mn}{m+n}\right)^{1/2}\varphi(C)\left(1+\frac{C^4}{4(m+n-2)}+O(n^{-2})\right).\end{aligned} \tag{4.1}$$

Here C can be determined asymptotically from α by expanding the central t-distribution tail integral with $f = (m + n - 2)$ degrees of freedom for large values of f,

$$\begin{aligned}\alpha = \int_C^\infty t_0(x)\,dx &= 1-\Phi\left[C\left(\frac{m+n-4}{m+n-2}\right)^{1/2}\right] \\ &\quad - \frac{1}{4(m+n-6)}\varphi^{(3)}\left[C\left(\frac{m+n-4}{m+n-2}\right)^{1/2}\right]+O(n^{-2}),\end{aligned} \tag{4.2}$$

and solving for C. One obtains

$$\begin{aligned}C = \left(\frac{m+n-2}{m+n-4}\right)^{1/2}&(\Phi^{-1}(1-\alpha)+(1/4(m+n-6))(3\Phi^{-1}(\alpha) \\ &- \{\Phi^{-1}(\alpha)\}^3)+O(n^{-2})).\end{aligned} \tag{4.3}$$

Therefore, asymptotically,

$$\begin{aligned}\dot\beta = \frac{1}{\sigma}\left(\frac{mn}{m+n}\right)^{1/2}&\varphi(\Phi^{-1}(\alpha))[1-(1/4(m+n-2))\{\Phi^{-1}(\alpha)\}^2 \\ &+ O(n^{-2})].\end{aligned} \tag{4.4}$$

Solving (1.3) asymptotically we get

$$e_{As.t} = e_{As.\bar x}\left(1+\frac{\{\Phi^{-1}(\alpha)\}^2}{24(m+n-2)\sigma^2(\int f^2(x)\,dx)^2}+O(n^{-2})\right). \tag{4.5}$$

For comparison of the asymptotic and the exact values, we give the following efficiency values; the values $e_{Ex.t}$ are obtained from (4.1) according to (1.3)

TABLE 3

Comparison of the efficiency values relative to the $\bar{x}$-test and the t-test for underlying normal distribution

	$m = n = 4$		$m = n = 5$		$m = 6, n = 4$	
α	0.0571	0.0286	0.0278	0.0159	0.0333	0.0190
$e_{Ex.\bar{x}}$	0.7840	0.7222	0.7697	0.7304	0.7787	0.7403
$e_{Ex.t}$	0.9772	0.9825	0.9774	0.9775	0.9705	0.9749
$e_{As.\bar{x}}$	0.7817	0.7311	0.7713	0.7369	0.7803	0.7455
$e_{As.t}$	0.9518	0.9620	0.9563	0.9593	0.9521	0.9553

TABLE 4

Efficiency values $e_{As.t}$ for different sample sizes m and n and reasonable values of α for underlying normal distribution

α	$m = n = 10$	$m = n = 20$	$m = n = 40$	$m = 10, n = 20$	$m = 10, n = 40$	$m = 20, n = 40$
0.100	0.9404	0.9466	0.9505	0.9437	0.9466	0.9488
0.050	0.9469	0.9498	0.9521	0.9471	0.9468	0.9505
0.010	0.9578	0.9566	0.9558	0.9531	0.9461	0.9542
0.005	0.9602	0.9591	0.9573	0.9546	0.9452	0.9556

by linear interpolation, after having taken C from Table 3, a table of the central t-distribution.

The following table of efficiency values, Table 4, indicates that the Wilcoxon test compared with the t-test is nearly as good locally for moderate sample sizes as it is known to be asymptotically.

5. Efficiency for the two-tail Wilcoxon test. Up to now we considered only the case of testing the hypothesis $\theta = 0$ against the one-sided alternatives $\theta > 0$ by one-tail tests. The analysis was based on the comparison of the first derivatives of the power functions at $\theta = 0$.

A similar analysis can be done in the case of two-tail tests for testing the hypothesis $\theta = 0$ against the two-sided alternatives $\theta \neq 0$. In the particular case that the first derivatives, $p'_{m,n}(\theta)$, $\beta'_{m,n}(\theta)$, both vanish at $\theta = 0$, condition (1.2) reduces to

$$\hat{p}_{m,n} = \hat{\beta}_{m^*,n^*}, \qquad m/n = m^*/n^*, \tag{5.1}$$

where $\hat{p}_{mn} = p''_{m,n}(0)$ and $\hat{\beta}_{m,n} = \beta''_{m,n}(0)$ are the second derivatives of the corresponding power functions at the hypothesis.

An analysis similar to that discussed in the preceding sections gives the following asymptotic expressions for the efficiency of the two-sided Wilcoxon test compared with the corresponding standard-normal and t-tests, respectively:

$$e_{As.\bar{x}} = 12\sigma^2 A^2 \Bigg[1 - \frac{1}{m+n+1} - \frac{m^2+n^2+mn+m+n}{mn(m+n+1)}$$

$$(5.2) \qquad \cdot \left(\frac{3}{5} - \frac{2}{5}\{\Phi^{-1}(\alpha/2)\}^2\right) + \frac{1}{2}\frac{m+n+1}{mn}\frac{\tilde{A}}{A^2} - \frac{m+n+1}{mn} + O(n^{-2})\Bigg]$$

$$(5.3) \qquad e_{As.t} = e_{As.\bar{x}}\left(1 + \frac{\{\Phi^{-1}(\alpha/2)\}^2}{24(m+n-2)\sigma^2(\int f^2(x)\,dx)^2} + O(n^{-2})\right).$$

In contrast to the known fact that the Pitman efficiency for the two-tail test is the same as for the one-tail test, the correction terms of order $O(n^{-1})$ differ from those of (3.2) and (4.5), respectively. On the other hand they depend on the special underlying distribution only through the Pitman parameter $12\sigma^2A^2$, and

$$(5.4) \qquad \tilde{A} = \int f^3(x)\,dx.$$

6. Efficiency of the sign test. Now, let $X_1, \cdots, X_n$ be independent and identically distributed according to the distribution $F(x - \theta)$, with $F(0) = \frac{1}{2}$ and with a density $f(x)$ continuous at $x = 0$. For testing the hypothesis $\theta = 0$ against the alternative $\theta > 0$ we consider the sign test with the power function

$$(6.1) \qquad p_n(\theta) = \sum_{\nu=k}^{n} \binom{n}{\nu} p^\nu q^{n-\nu}.$$

$$(6.2) \qquad p = \tfrac{1}{2} + \theta f(0) + O(\theta^2), \qquad q = 1 - p = \tfrac{1}{2} - \theta f(0) + O(\theta^2).$$

Here the Edgeworth series, integrated according to the Euler-Maclaurin summation formula, can be checked by a direct expansion of (6.1). For fixed x,

$$(6.3) \qquad \begin{aligned} p_n(\theta) &= 1 - \Phi(x) + \frac{1}{6}\frac{q-p}{(npq)^{1/2}}\varphi^{(2)}(x) + \frac{1}{24}\frac{1}{npq}\varphi'(x) \\ &- \frac{1}{24}\frac{1-6pq}{npq}\varphi^{(3)}(x) - \frac{1}{72}\frac{1-4pq}{npq}\varphi^{(5)}(x) + \hat{e}\hat{\varphi}(x) + O(n^{-2}), \end{aligned}$$

where x is the normalized value of k with continuity correction, and $\hat{e}\hat{\varphi}(x)$ symbolizes the terms of order $n^{-3/2}$, which vanish for $\theta = 0$. For small values of θ, (6.3) simplifies to

$$(6.4) \qquad \begin{aligned} p_n(\theta) &= 1 - \phi(x) - \frac{2}{3}\frac{\theta f(O)}{n^{1/2}}\varphi^{(2)}(x) + \frac{1}{6n}\varphi'(x) \\ &+ \frac{1}{12n}\varphi'''(x) + O(n^{-2}) + O(\theta^2), \end{aligned}$$

with

$$(6.5)\quad x = \frac{k - np - 1/2}{(npq)^{1/2}} = x_0 + x_1\theta + O(\theta^2), \qquad x_1 = -2n^{1/2}f(0).$$

x_0 and α are connected by $p_n(0) = \alpha$, which can be solved asymptotically for x_0 as follows:

$$(6.6)\quad x_0 = \phi^{-1}(1 - \alpha) - (1/12n)\phi^{-1}(\alpha) + (1/12n)\{\phi^{-1}(\alpha)^3\} + O(n^{-2}).$$

On the other hand, (6.4) gives, for fixed α,

$$(6.7)\quad \begin{aligned}\tilde{p}_n = & -x_1\varphi(x_0) - \frac{2}{3}\frac{f(0)}{n^{1/2}}\varphi^{(2)}(x_0) + \frac{x_1}{12n}\varphi^{(4)}(x_0) \\ & + \frac{x_1}{6n}\varphi''(x_0) + O(n^{-3/2}),\end{aligned}$$

or, by (6.5) and (6.6),

$$(6.8)\quad \tilde{p}_n = \varphi(\phi^{-1}(\alpha))2n^{1/2}f(0)(1 + (1/4n) - (1/12n)\{\phi^{-1}(\alpha)\}^2 + O(n^{-2})).$$

Let us first compare the sign test with the $\bar{x}$-test, the power function of which is

$$(6.9)\quad \begin{aligned}\beta_{n^*}(\theta) = \phi\left(\frac{(n^*)^{1/2}\theta}{\sigma} - \phi^{-1}(1 - \alpha)\right) = \alpha + \varphi(\phi^{-1}(\alpha))\frac{(n^*)^{1/2}\theta}{\sigma} \\ + O(\theta)^2).\end{aligned}$$

Expression (6.9) holds exactly for normally distributed random variables and up to (relative) errors of the order $O(n^{-2})$ for any other distribution with finite fourth moment. Therefore, in the vicinity of the hypothesis, the efficiency of the sign test compared with the $\bar{x}$-test is given by

$$(6.10)\quad \begin{aligned}e_{\mathrm{As}.\bar{x}} = \frac{n^*}{n} = \frac{\sigma^2}{n}\left(\frac{\tilde{p}_n}{\varphi(\phi^{-1}(\alpha))}\right)^2 = 4f^2(0)\sigma^2 \\ \cdot\left[1 + (1/2n) - \frac{\{\phi^{-1}(\alpha)\}^2}{6n} + O(n^{-2})\right],\end{aligned}$$

the first term of which is the well-known Pitman value for the efficiency. For the sign test with a value of $\alpha \approx \phi^{-1}(-3^{1/2})$, which is a reasonable choice, the first order correction is small and the approximation of the Pitman value is an especially good one.

Corresponding to (4.5) a comparison with the t-test gives

$$(6.11)\quad \begin{aligned}e_{\mathrm{As}.t} &= e_{\mathrm{As}.\bar{x}}\left(1 + \frac{\{\phi^{-1}(\alpha)\}^2}{8nf^2(0)\sigma^2} + O(n^{-2})\right) \\ &= 4f^2(0)\sigma^2(1 + (1/2n) + \{\phi^{-1}(\alpha)\}^2[(1/8nf^2(0)\sigma^2) \\ &\qquad - (1/6n)] + O(n^{-2})).\end{aligned}$$

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