[7] S. S. SHRIKHANDE, "On a characterization of the triangular association scheme," *Ann. Math. Stat.*, Vol. 30 (1959), pp. 39–48.
[8] S. S. SHRIKHANDE, "Relation between certain types of incomplete block designs," Institute of Statistics, University of North Carolina, *Mimeograph Series No. 207* (1958).
[9] S. S. SHRIKHANDE, "The uniqueness of the L_2 association scheme," *Ann. Math. Stat.* Vol. 30 (1959), pp. 781–798.

OPTIMALITY CRITERIA FOR INCOMPLETE BLOCK DESIGNS[1]

BY K. R. SHAH[2]

Forest Research Institute, Dehra Dun, India

1. Introduction and Summary. Several optimality criteria have been suggested for the efficiency of incomplete block designs. This note surveys these criteria, extends certain results and puts forward a new and simpler criterion.

2. Existing Criteria. Important aims in experimental design are to estimate the effects of treatment comparisons with maximum precision for a given total number of experimental units, or total cost, and to perform a test of the null hypothesis. These two considerations lead us to different criteria for choosing from among the designs.

Consider the class of incomplete block designs, D_{vkb}, for fixed values of v, k and $b(v > k)$, where v treatments are arranged in b blocks of k plots each, and each treatment is replicated r times. In the usual notation, (see for example, Kempthorne [2]) intra-block estimates of treatment effects are given by

$$\mathbf{C}\hat{\mathbf{t}} = \mathbf{Q}, \tag{2.1}$$

where $\mathbf{C} = r\mathbf{I} - \mathbf{NN}'/k$, $\mathbf{N}$ being the incidence matrix of the design. We consider only connected designs, so that the rank of $\mathbf{C}$ is $v - 1$. Let $\lambda_1, \lambda_2, \cdots, \lambda_v$, be the $v - 1$ non-zero latent roots of $\mathbf{C}$. It is proved in [2] that the average variance of all elementary treatment contrasts is proportional to $\sum \lambda_i^{-1}$. Let $\mathbf{P}_i'\mathbf{t}(i = 1, 2, \cdots, v - 1)$ be any complete set of $v - 1$ orthogonal normalised contrasts. Set

$$\mathbf{P} = [\mathbf{P}_1, \mathbf{P}_2, \cdots, \mathbf{P}_{v-1}], \qquad \mathbf{P}'\mathbf{t} = \boldsymbol{\varrho}, \qquad \boldsymbol{\varrho} = \{\rho_1, \cdots, \rho_{v-1}\}.$$

It can be shown that $\mathbf{P}'\mathbf{CP}$ is a non-singular matrix with latent roots $\lambda_1, \cdots, \lambda_{v-1}$, and that (2.1) leads to

$$\mathbf{P}'\mathbf{CP}\hat{\boldsymbol{\varrho}} = \mathbf{P}'\mathbf{Q} \quad \text{or} \quad \hat{\boldsymbol{\varrho}} = (\mathbf{P}'\mathbf{CP})^{-1}\mathbf{P}'\mathbf{Q}. \tag{2.2}$$

Let us denote the dispersion matrix of $\mathbf{x}$ by $V(\mathbf{x})$. Now $V(\mathbf{Q}) = \mathbf{C}\cdot\sigma^2$, which

Received June 8, 1959.

[1] This work was supported by a senior research training scholarship of the Government of India.

[2] Present address: Indian Statistical Institute, Calcutta, India.

gives $V(\hat{\varrho}) = (\mathbf{P}'\mathbf{C}\mathbf{P})^{-1} \cdot \sigma^2$. Hence the generalised variance of $\hat{\varrho}$ is given by

$$(2.3) \qquad | V(\hat{\varrho}) | = | (\mathbf{P}'\mathbf{C}\mathbf{P})^{-1} | \cdot \sigma^2 = \sigma^2 \prod_{i=1}^{v-1} \lambda_i^{-1}.$$

The usual null hypothesis H_0 is $t_1 = t_2 = \cdot = t_v$, which is equivalent to $\rho_1 = \rho_2 = \cdot = \rho_{v-1} = 0$. The sum of squares for testing H_0 is $\mathbf{t}'\mathbf{Q}$, which can be shown to be equal to $\varrho'\mathbf{P}'\mathbf{C}\mathbf{P}\varrho$. Hence the power of the F test is a monotonically increasing function of $\beta = \varrho'\mathbf{P}'\mathbf{C}\mathbf{P}\varrho/\sigma^2$.

The efficiency criteria considered so far by various authors are as follows:

(A) If we wish to minimise the average variance of all elementary treatment contrasts, we should minimise $\sum \lambda_i^{-1}$, [2], [4].

(B) Wald [6] argues that it is not possible to maximise power for all values of ϱ. Hence we should maximise β for fixed values of $\varrho'\varrho/\sigma^2$. It is reasonable to maximise the minimum of β subject to $\varrho'\varrho/\sigma^2 =$ constant. This leads to maximising $\lambda_{\min}$, [1], [6].

(C) Wald [6] further argues that from certain mathematical considerations it would be simpler to minimise $\prod_{i=1}^{v-1} \lambda_i^{-1}$. This minimises the generalised variance. Also, as Nandi [5] has pointed out, this has the desirable effect of minimising the volume of equi-power ellipsoid given by $\varrho'\mathbf{P}'\mathbf{C}\mathbf{P}\varrho/\sigma^2 =$ const. In a sense this minimises the range of ϱ subject to constant power. It should also be noted that the design which minimises $\prod \lambda_i^{-1}$ gives certain optimum properties for the usual F test associated with it, [3].

It is easy to see that the optima for all the criteria are reached when the λ's are all equal. Hence, when a balanced incomplete block design (BIB) exists in the class D_{vbk}, it is the most efficient design in that class [4].

In [2] and [4] only the equi-replicate designs are considered. But the results follow from the roots of $\mathbf{C}$, and the only condition used in [4] is $\sum \lambda_i =$ constant. Hence the results in [2] and Section 2 of [4] are valid also for the case of unequal number of replications. The extension of these results is not of mere academic interest; there are important classes of designs, such as inter and intra-group block designs and reinforced incomplete block designs, where the number of replications are usually unequal.

Since efficiency should relate to the manner of utilization of the resources, in framing an efficiency criterion, it seems natural to take into account the amount of experimental material used. This would enable us to compare designs with different sizes. Hence, we consider the class of designs, D_{vk}, for fixed values of v and $k (v > k)$, where v treatments are arranged in blocks of k plots each. Denote by r_i and R, the number of replications for the ith treatment and the average number of replications respectively. Since $\sum \lambda_i = \text{Trace } \mathbf{C} = (k-1)\sum r_i/k = (k-1)vR/k$; it is linearly related to the total number of plots.

The efficiency criteria, analogous to those in (A), (B) and (C) would be

$$(2.4) \qquad E_1 = (v-1)/R\sum \lambda_i^{-1}, \quad E_2 = \lambda_{\min}/R, \quad E_3 = 1/R(\prod \lambda_i)^{1/v-1}.$$

Now for fixed R, the theoretical maxima of E_1, E_2, E_3 are attained when $\lambda_1 = \lambda_2 = \cdots = \lambda_{v-1}$. Since this maximising solution is independent of R, it is also the unconditional maximising solution. Now in the class of designs D_{vk}, a BIB design always exists. Hence, judged by any of the three critreia, within the class of designs D_{vk} any of the BIB designs is the most efficient. It can be easily seen that for the BIB design $E_1 = E_2 = E_3 = (1 - 1/k)/(1 - 1/v)$. And as is to be expected, each one of them increases with k. In the limit when $k = v$, i.e., for randomised complete block designs, $E_1 = E_2 = E_3 = 1$.

3. A New Criterion. The above three criteria are based on different considerations and need not necessarily agree in comparing two given designs. Which criterion should be adopted depends upon our aim in conducting the experiment. But most often we shall be interested in both the interval estimation of treatment effects and in the test of the null hypothesis.

It should be noted that, in the limit when optimality is reached, all the three criteria lead to the same result, *viz.*, the λ's should be all equal. In fact for the first and the third certeria, we are concerned with the geometric and the harmonic means subject to the arithmetic mean being constant. When the experiment is symmetrical, i.e., the λ's are all equal, the three means coincide. This suggests the use of $\sum (\lambda_i - \bar{\lambda})^2/(v - 1)$ with $\sum \lambda_i =$ const., as a criterion for optimality, i.e. among designs of given size, we should make $\sum \lambda_i^2$ as small as possible, subject to existence of a design. To eliminate the effect of the size of the design we define

$$(3.1) \quad E_4 = \bar{\lambda}^2/[R\,(\textstyle\sum \lambda_i^2/(v - 1))^{\frac{1}{2}}] = (v - 1)^{-\frac{3}{2}}\,(\textstyle\sum \lambda_i)^2/[R\,(\textstyle\sum \lambda_i^2)^{\frac{1}{2}}].$$

When the design is balanced, $E_4 = (1 - 1/k)/(1 - 1/v)$, and hence the efficiency of a BIB increases with k increasing, reaching unity when $k = v$. Nevertheless, the criterion is suggested only for comparisons of different designs within the class D_{vk}, with v and k fixed.

Though this criterion does not agree exactly with any of the three criteria given above, it will tend to be as good as any of them. In any case, we are not able to satisfy all the three criteria simultaneously. Smaller values of $\sum \lambda_i^2$ will tend to give smaller values of $\sum \lambda_i^{-1}$ and $\prod \lambda_i^{-1}$, though this does not hold exactly in all cases. Though the contours of equal efficiency (in the space of the λ's) are not identical with those for the other three criteria (which themselves are not identical), our criterion will be quite useful. For the points on the line given by $\lambda_1 = \lambda_2 = \cdots = \lambda_{v-1}$ all give the same result and for the class of designs with higher efficiency, i.e., for λ's not too widely spread, they will be more or less equal. This is the region where our criterion will be quite effective. As shown below this criterion has the advantages of simplicity and practical usefulness.

We can express $\mathbf{C}$ as $\sum \lambda_i \mathbf{L}_i \mathbf{L}_i'$, where $\mathbf{L}_i$ is the canonical vector corresponding to λ_i. This immediately gives $\mathbf{C}^2 = (\sum \lambda_i \mathbf{L}_i \mathbf{L}_i')(\sum \lambda_i \mathbf{L}_i \mathbf{L}_i') = \sum \lambda_i^2 \mathbf{L}_i \mathbf{L}_i^2$. Hence Trace $\mathbf{C}^2 = \sum \lambda_i^2$, but Trace $\mathbf{C}^2 = \sum_i \sum_j c_{ij}^2$, and therefore $\sum \lambda_i^2 = \sum_i \sum_j c_{ij}^2$. Hence,

$$E_4 = (v - 1)^{-\frac{3}{2}}((k - 1)/k)^2 v^2 R/\textstyle\sum_i \sum_j c_{ij}^2 .$$

A further simplification can be had for PBIB and circulant designs, where $\sum_j c_{ij}^2$ is the same for all i.

For the other three criteria, elegant expressions are seldom available. Since E_4 follows directly from the **C** matrix it is easiest to compute; we do not have to solve the normal equations or evaluate the λ's.

Acknowledgment. My sincere thanks are due to Dr. K. R. Nair for his guidance in writing this paper.

REFERENCES

[1] SYLVAIN EHRENFELD, "On the efficiency of experimental designs," *Ann. Math. Stat.*, Vol. 26 (1955), pp. 247–255.
[2[OSCAR KEMPTHORNE, "The efficiency factor of an incomplete block design," *Ann. Math. Stat.*, Vol. 27 (1956), pp. 846–849.
[3] J. KIEFER, "On the non-randomised optimality and randomised non-optimality of symmetrix design," *Ann. Math. Stat.*, Vol. 29 (1958), pp. 675–699.
[4] A. M. KSHIRSAGAR, "A note on incomplete block designs," *Ann. Math. Stat.*, Vol. 29 (1958), pp. 907–910.
[5] H. K. NANDI, "On the efficiency of experimental designs," *Calcutta Stat. Assoc. Bull.*, Vol. 3 (1950), pp. 167–171.
[6] ABRAHAM WALD, "On the efficient design of statistical investigations," *Ann. Math. Stat.*, Vol. 14 (1943), pp. 134-140.

ON THE COMPLETENESS OF ORDER STATISTICS[1]

BY C. B. BELL[2]

University of California, La Jolla, and San Diego State College

AND

DAVID BLACKWELL AND LEO BREIMAN

University of California, Berkeley

1. Introduction and summary. Let $X_1, X_2, \cdots, X_n$ be a sample of a one-dimensional random variable X; let the order statistic $T(X_1, X_2, \cdots, X_n)$ be defined in such a manner that $T(x_1, x_2, \cdots, x_n) = (x^{(1)}, x^{(2)}, \cdots, x^{(n)})$ where $x^{(1)} \leqq x^{(2)} \leqq \cdots \leqq x^{(n)}$ denote the ordered x's; and let Ω be a class of one-dimensional *cpf*'s, i.e., cumulative probability functions.

The order statistic, T, is said to be a complete statistic with respect to the class, $\{P^{(n)} \mid P \varepsilon \Omega\}$, of n-fold power probability distributions if

$$E_{P^{(n)}}\{h[T(X_1, \cdots, X_n)]\} = 0$$

Received October 7, 1959; revised April 18, 1960.

[1] This paper was prepared with the partial support of the Office of Naval Research (Nonr-222-53). This paper in whole or in part may be reproduced for any purpose of the United States Government.

[2] The research was supported in part by a National Science Foundation Grant. A conversation with Professor Herman Rubin was helpful.