THE DUAL OF A BALANCED INCOMPLETE BLOCK DESIGN

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1. Summary. Shrikhande [9] and Roy [7] have shown that certain Balanced Incomplete Block Designs (BIBDs) can be dualised to give Partially Balanced Incomplete Block Designs (PBIBDs) with exactly two associate classes. Roy and Laha [8] have obtained a necessary and sufficient condition for the dual of a BIBD to be a PBIBD with two associate classes. In this paper, a general result regarding the dual of a BIBD is established and the results of Shrikhande and Roy are obtained as particular cases. An illustration to show the use of the result when the dual is not a 2-associate PBIBD is also given.

2. Two Lemmas connecting the parameters of a BIBD. For the definition of a BIBD the reader may refer to Kempthorne [4]. The following two lemmas will be stated without proof. Lemma 2.1 is due to Connor [2], while Lemma 2.2 is due to Hussain [3].

LEMMA 2.1: *If l_{ij} is the number of treatments in common with the* ith *and the* jth *blocks of a* BIBD *with parameters* v^*, b^*, r^*, k^*, λ^*; *the following inequalities hold*:

$$[2\lambda^* k^* + r^*(r^* - \lambda^* - k^*)]/r^* \geqq l_{ij} \geqq -(r^* - \lambda^* - k^*). \tag{2.1}$$

LEMMA 2.2: *If n_μ denotes the number of blocks having $u - 1$ treatments in common with a chosen initial block of a* BIBD *with parameters* v^*, b^*, r^*, k^*, λ^*, *and t is the largest integer contained in* $[2\lambda^*k^* + r^*(r^* - \lambda^* - k^*)]/r^*$, *such that $t < k + 1$, the following equalities hold*:

$$\sum_{u=1}^{t+1} n_u = b^* - 1, \tag{2.2}$$

$$\sum_{u=1}^{t+1} (u-1)n_u = k^*(r^* - 1), \tag{2.3}$$

$$\sum_{u=1}^{t+1} (u-1)(u-2)n_u = k^*(k^* - 1)(\lambda^* - 1). \tag{2.4}$$

Note that if (2.2), (2.3) and (2.4) admit a unique nonnegative integral solution, then, corresponding to each block of the design, the remaining $b^* - 1$ blocks may be divided into $t + 1 = m$ groups such that a block in the uth group has exactly $u - 1 = \lambda_u$ $(u = 1, 2, \cdots m)$ treatments in common with the chosen initial block, there being exactly n_u blocks in the uth group.

3. The definition of a PBIBD. An incomplete block design is said to be a PBIBD if it satisfies the following conditions:

(3.1) There are v treatments divided into b blocks of k plots each, different treatments being applied to the plots in the same block.

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(3.2) Each treatment occurs in exactly r blocks.

(3.3) There can be established an association relationship between any two treatments satisfying the following conditions:

(3.3a) Two treatments are either 1st, 2nd, $\cdots$ mth associates.

(3.3b) Each treatment has exactly n_u uth associates ($u = 1, \cdots m$).

(3.3c) Given any two treatments which are kth associates, the number of treatments which are the uth associates of the first and u'th associates of the second is $P^k_{uu'}$. Also, $P^k_{uu'} = P^k_{u'u}$.

(3.4) Two treatments which are uth associates will occur together in exactly $\lambda_u(u = 1, 2, \cdots, m)$ blocks.

For the necessary conditions satisfied by the parameters of a PBIBD the reader is referred to Bose and Nair [1] and Nair and Rao [6].

4. The dual of a design. Let $B_1, B_2, \cdots, B_{b^*}$ and $T_1, T_2, \cdots, T_{v^*}$ denote the blocks and treatments of a given design, D^*, in which $v^*(=b)$ treatments are arranged in $b^*(=v)$ blocks of $k^*(=r)$ plots each such that every treatment is replicated $r^*(=k)$ times. Let D be a new design with v treatments and b blocks constructed by placing the treatment numbered i in block numbered j of D, if in D^* the block B_i contains the treatment T_j. The designs D^* and D are said to be the duals of each other. Evidently, in D each block contains k plots and each treatment is replicated r times. Further, if $N^* = (n_{ij})$, $(i = 1, 2, \cdots, v^*; j = 1, 2, \cdots, b^*)$, where n_{ij} denotes the number of times the ith treatment occurs in the jth block, is the incidence matrix of D^*, the incidence matrix of D is $(N^*)'$, where $(N^*)'$ is the transpose of N^*. Also the element in the ith row and the jth column of the $v^* \times v^*$ matrix $(N^*)'N^*$ will be equal to the number of blocks in the dual design D in which the ith and the jth treatments occur together.

5. The dual of a BIBD. Consider a BIBD with parameters $v^*(=b)$, $b^*(=v)$, $r^*(=k)$, $k^*(=r)$, λ^*. Let $N^* = (n_{ij})$ be the incidence matrix. We have, by the well known properties of a BIBD,

$$N^*(N^*)' = \lambda^* E_{v^*} + (r^* - \lambda^*)I_{v^*}, \tag{5.1}$$

where E_{v^*} is a $v^* \times v^*$ matrix with all elements unity and I_{v^*} is a $v^* \times v^*$ identity matrix. Also,

$$(N^*)'N^* = \left(\sum_{i=1}^{v^*} n_{ij}n_{ij'}\right) = (\lambda_{jj'}), \tag{5.2}$$

where, as already observed in the previous section, $\lambda_{jj'}$ is the number of treatments common to the jth and the j'th blocks of the original BIBD, which is also equal to the number of blocks of the dual design in which the jth and the j'th treatments occur together. Thus, in the dual design, a pair of treatments can occur together in at most t blocks, where t is defined as in Lemma 2.2. Further, if the equations (2.2), (2.3) and (2.4) admit a unique integral non-negative solution, in the dual design, corresponding to each treatment, the remaining

$v - 1$ treatments can be divided into $t + 1 = m$ groups, such that a treatment in the uth group will occur in exactly $\lambda_u = u - 1$ ($u = 1, 2, \cdots, m$) blocks with the initial treatment, and, there will be exactly n_u treatments in the uth class. At this point, it may be noted that we do not exclude the possibility of some of the n_u's being zero, in which case the exact number of classes will be less than m. In fact, the total number of groups will be exactly equal to the total number of non-null n_u's.

We now proceed to investigate the conditions under which the dual will be a PBIBD. Evidently, if the equations (2.2), (2.3) and (2.4) admit a unique integral non-negative solution, then the conditions (3.1), (3.2), (3.3a), (3.3b) and (3.4) are satisfied by the dual design. Hence it remains to see when (3.3c) will also be satisfied.

Define m $v \times v$ matrices $B_u (u = 1, 2, \cdots, m)$ as

$$B_u = (b^u_{jj'}) \qquad j, j' = 1, 2, \cdots, v; \tag{5.3}$$

where $b^u_{jj} = 0$ for all j, and $b^u_{jj'} = 1$ if $\lambda_{jj'} = \lambda_u$ and 0 otherwise, for all $j \neq j'$.

The matrices B_u are symmetric, independent, and commutative with respect to multiplication. It is also clear that

$$\sum_{s=1}^{v} b^u_{is} b^{u'}_{sj} = \sum_{s=1}^{v} b^u_{si} b^{u'}_{js} = C^{uu'}_{ij}, \tag{5.4}$$

which is the number of treatments common to the uth and u'th groups of treatments with respect to the treatments numbered i and j in the dual design if $i \neq j$. It equals n_u if $i = j$ and $u = u'$, and it equals zero if $i = j$ and $u \neq u'$.

Now consider any block, B_i, of the original BIBD. There will be n_u blocks in the design that have exactly λ_u treatments in common with B_i. Of these n_u blocks, $C^{uu'}_{ij}$ blocks will have $\lambda_{u'}$ treatments in common with the block B_j Hence

(5.5) $\sum_{u'=1}^{m} C^{uu'}_{ij} = n_u$ if the blocks B_i and B_j do not have λ_u treatments in common,

$= n_u - 1$ otherwise.

Now using (5.2) and (5.3), and observing that $\lambda_{jj} = k^*$, we have,

$$(N^*)'N^* = k^* I_{b^*} + \sum_{u=1}^{m} \lambda_u B_u, \tag{5.6}$$

and hence,

$$\begin{aligned} [(N^*)'N^*][(N^*)'N^*] &= (N^*)'[N^*(N^*)']N^* \\ &= (N^*)'[\lambda^* E_{v^*} + (r^* - \lambda^*) I_{b^*}]N^* \\ &= \lambda^*(N^*)'E_{v^*}N^* + (r^* - \lambda^*)(N^*)'N^*. \end{aligned} \tag{5.7}$$

As N^* is the incidence matrix of a BIBD it is easy to verify that

$$(N^*)'E_{v^*}N^* = (k^*)^2 E_{b^*}, \tag{5.8}$$

and that the left hand side of (5.7) can also be expressed as

$$(N^*)'N^*\left[k^*I_{b^*} + \sum_{u=1}^{m} \lambda_u B_u\right]. \tag{5.9}$$

Hence, using (5.8) and (5.9) and noting that $\lambda_1 = 0$, we get from (5.7),

$$k^*(N^*)'N^* + (N^*)'N^*\left(\sum_{u=2}^{n} \lambda_u B_u\right) = \lambda^*(k^*)^2E_{b^*} + (r^* - \lambda^*)(N^*)'N^*.$$

Hence, from (5.6),

$$\lambda^*(k^*)^2E_{b^*} - k^*(k^* - r^* - \lambda^*)I_{b^*}$$
$$= (2k^* - r^* + \lambda^*)\sum_{u=2}^{m} \lambda_u B_u + \left[\sum_{u=2}^{m} \lambda_u B_u\right]^2.$$

Hence

$$\lambda^*(k^*)E_{b^*} - k^*(r^* - k^* - \lambda^*)I_b^*$$
$$= (2k^* - r^* + \lambda^*)\sum_{u} \lambda_u B_u + \sum_{u,u'}\sum \lambda_u\lambda_{u'}B_uB_{u'}. \tag{5.10}$$

Comparing the (ij)th non-diagonal terms on both sides of (5.10),

$$\sum_{u,u'}\sum \lambda_u\lambda_{u'}\sum_{s} b_{is}^{u}b_{sj}^{u'} = \lambda^*(k^*)^2 - (2k^* - r^* + \lambda^*)\sum_{u} \lambda_u b_{ij}^{u}.$$

Using the notation of (5.4),

$$\sum_{u,u'}\sum \lambda_u\lambda_{u'}C_{ij}^{uu'} = \lambda^*(k^*)^2 - (2k^* - r^* + \lambda^*)\sum \lambda_u b_{ij}^{u}. \tag{5.11}$$

We can divide the set of $(b^*)^2$ equations (5.11) into m mutually exclusive sets such that the qth set ($q = 1, 2, \cdots, m$) contains all the equations with $C_{ij}^{uu'}$ for $\lambda_{ij} = \lambda_q$ The coefficients in the left hand side, and the constant in the right hand side, are same for all the equations in a given set. In fact, the equations in the qth set will be obtained by giving all the values to i and j such that $\lambda_{ij} = \lambda_q$ in

$$\sum_{u,u'}\sum \lambda_u\lambda_{u'}C_{ij}^{uu'} = \lambda^*(k^*)^2 - (2k^* - r^* + \lambda^*)\lambda_q. \tag{5.12}$$

Thus it is clear that the values of $C_{ij}^{uu'}$ depend only on λ_u, $\lambda_{u'}$ and λ_{ij}. Hence, by writing $C_{ij}^{uu'} = P_{uu'}^{q}$ if $\lambda_{ij} = \lambda_q$, the equations (5.6) and (5.12) may be re-written as

$$\sum_{u'=1}^{m} P_{uu'}^{q} = n_u \qquad \text{if } u \neq q, \tag{5.13}$$
$$= n_u - 1 \qquad \text{if } u = q;$$

and

$$\sum_{u,u'}\sum \lambda_u\lambda_{u'}P_{uu'}^{q} = \lambda^*(k^*)^2 - (2k^* - r^* + \lambda^*)\lambda_q, \quad q = 1, 2, \cdots, m. \tag{5.14}$$

Hence, if (5.14) has a unique integral non-negative solution, it follows from (5.4) and (5.13) that the number of treatments common to the uth group and u'th group of two treatments is the same for all treatment pairs which belong to the qth group with respect to each other. This number is equal to $P^q_{uu'}$ with $P^q_{uu'} = P^q_{u'u}$. Thus we have proved Theorem 5.1.

THEOREM 5.1: *The dual of a* BIBD *with parameters* $v^*(=b)$, $b^*(=v)$, $r^*(=k)$, $k^*(=r)$, λ^* *is a* PBIBD *with parameters* $v, b, r, k; \lambda_1, \lambda_2, \cdots, \lambda_m; n_1, n_2, \cdots, n_m; P^q_{uu'}(u, u', q = 1, 2, \cdots, m)$, *where* $m = t + 1$ *is defined as in Lemma* 2.2, *provided the equations* (2.2), (2.3), (2.4) *and* (5.14) *admit unique integral non-negative solution subject to the conditions* (5.13).

6. Shrikhande's two theorems as particular cases of the Theorem 5.1.

(6.1) *The case* $\lambda^* = 1$. Consider a BIBD with parameters $v^*(=b)$, $b^*(=v)$, $r^*(=k)$, $k^*(=r)$, $\lambda^* = 1$. In this case we have $t = 1$ and the equations (2.2), (2.3) and (2.4) reduce to $n_1 + n_2 = b^* - 1$ and $n_2 = k^*(r^* - 1)$, giving the unique non-negative solution

$$n_1 = (v - 1) - r(k - 1),$$
$$n_2 = r(k - 1).$$

Noting that $\lambda_1 = 0$ and $\lambda_2 = 1$, we can solve the equations (5.14) uniquely to get the solution $P^1_{22} = r^2$, $P^2_{22} = r^2 - 2r + k - 1 = (r - 1)^2 + (k - 2)$. The other parameters can be easily obtained by using condition (5.13).

Thus we have proved Shrikhande's [9] Theorem 1 that the dual of a BIBD with parameters $v^* = rk - k + 1$, $b^* = k(rk - k + 1)/r$, $r^* = k$, $k^* = r$, $\lambda^* = 1$ is a PBIBD with parameters $v = k(rk - k + 1)/r$, $b = rk - k + 1$, $r = r$, $k = k$, $\lambda_1 = 0$, $\lambda_2 = 1$; $n_1 = r(k - 1)$, $n_2 = (k - r)(r - 1)(k - 1)/r$;

$$p^1_{uu'} = \begin{bmatrix} (k - r)^2 + 2(r - 1) - k(k - 1)/r & r(k - r - 1) \\ r(k - r - 1) & r^2 \end{bmatrix}$$

$$p^2_{uu'} = \begin{bmatrix} (r - 1)(k - r)(k - r - 1)/r & (r - 1)(k - r) \\ (r - 1)(k - r) & (k - 2) + (r - 1)^2 \end{bmatrix}.$$

(6.2) *The case* $\lambda^* = 2$. It can be easily seen that, if we exclude the solutions in which the same block is repeated, for all designs with $\lambda^* = 2$ and $r \leqq 10$, we must have $t = 2$. In this case the equations (2.2), (2.3) and (2.4) will have the unique solution given by

$$n_1 = (b^* - 1) - k^*(r^* - k^*) - k^*(k^* - 1)/2,$$
$$n_2 = k^*(r^* - k^*),$$
$$n_3 = k^*(k^* - 1)/2.$$

But, in general, equations (5.14) will not have a unique solution. However, if we consider the particular case $n_1 = 0$, i.e. when $r^* = k^* + 2$, the equations (5.14), when $q = 3$, reduce to $P^3_{22} + 4(P^3_{23} + P^3_{33}) = 2k^*(k^* - 1)$. Hence, using (5.13), we get, $P^3_{22} = 2k^*(k^* - 1) - 4(n_3 - 1) = 4$. Similarly, the other parameters may be found. Hence we have proved Theorem 3 of Shrikhande [6] that the dual of a BIBD with parameters

$$v^* = \binom{k-1}{2}, \quad b^* = \binom{k}{2}, \quad r^* = k, \quad k^* = k-2, \quad \lambda^* = 2,$$

is a PBIBD with parameters

$$v = \binom{k}{2}, \quad b = \binom{k-2}{2}, \quad r = k-2, \quad k = k;$$

$$\lambda_1 = 1, \quad \lambda_2 = 2; \quad n_1 = 2(k-2), \quad n_2 = \binom{k-2}{2};$$

$$P^1_{uu'} = \begin{bmatrix} k-2 & k-3 \\ k-3 & \binom{k-3}{2} \end{bmatrix}; \qquad P^2_{uu'} = \begin{bmatrix} 4 & 2(k-4) \\ 2(k-4) & \binom{k-4}{2} \end{bmatrix}.$$

Roy's [7] Theorem 3, regarding the dual of an affine resolvable BIBD, can be proved in a similar way by using Theorem 5.1 of this paper.

7. Application of Theorem 5.1 when the solution of the equations is not unique. When the solution of the equations (2.2), (2.3), (2.4) and (5.14) is not unique, Theorem 5.1 will not give complete information about the dual. However, if the structure of the original BIBD is known, Theorem 5.1 can be used to simplify the investigation about the properties of the dual. As an illustration, we consider the dual of a BIBD with parameters $v^* = 16$, $b^* = 24$, $r^* = 9$, $k^* = 6$, $\lambda^* = 3$. A plan of this design is given by Mann [5]. He constructed it by the process of residuation from the symmetric BIBD with parameters $v^* = b^* = 25$, $r^* = k^* = 9$, $\lambda^* = 3$. We shall denote Mann's design by D^*.

Since any two blocks of a symmetric BIBD must have λ^* treatments in common, any two blocks of the design D^* cannot have more than three treatments in common. Hence we must have $n_5 = n_6 = n_7 = 0$. Thus the equations (2.2), (2.3), and (2.4) can be written as

$$n_1 = 5 - n_4,$$

$$n_2 = 3\,(n_4 - 4),$$

$$n_3 = 3\,(10 - n_4).$$

From inspection of Mann's plan we can see that no two blocks of the design D^* have exactly one treatment in common. This gives the unique solution, $n_1 = 1$, $n_2 = 0$, $n_3 = 18$, $n_4 = 4$. For the sake of simplicity, we shall write n_1, n_2, n_3 instead of n_1, n_3, n_4 and make corresponding changes in $P^q_{uu'}$. Now, as $n_1 = 1$ and $P^1_{11} + P^1_{12} + P^1_{13} = n_1 - 1 = 0$, we must have $P^1_{11} = P^1_{12} = P^1_{13} = 0$. Therefore, the equations (5.14), when $q = 1$, may be solved uniquely to get the values of $P^1_{uu'}(u, u' = 2, 3.)$. Again, if the dual of the design D^* is a PBIBD, then P^2_{12} and P^3_{13} must both be unique and equal to $(n_1/n_2)P^1_{22}$ and $(n_1/n_3)P^1_{33}$ respectively. It can be verified that, for the design D^*, the values of P^2_{12} and P^3_{13} satisfy these conditions and are both equal to 1. Hence, as $n_1 = 1$, it follows from (5.13) that $P_{11}^2 = P_{13}^2 = P_{11}^3 = P_{12}^3 = 0$. It is now easy to see that the equations (5.14) will have the unique solution

$$P_{uu'}^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 4 \end{bmatrix},$$

$$P_{uu'}^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix},$$

$$P_{uu'}^3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 18 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Hence the dual of the design D^* is a PBIBD with the parameters $v = 24$, $b = 16$, $r = 6$, $k = 9$; $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 3$; $n_1 = 1$, $n_2 = 18$, $n_3 = 4$; $P_{uu'}^q(u, u', q = 1, 2, 3.)$.

Roy and Laha [8] have already pointed out that this PBIBD may be obtained as the dual of a BIBD. However, they have not stated how they arrived at this conclusion.

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