CONFIDENCE BOUNDS CONNECTED WITH ANOVA AND MANOVA FOR BALANCED AND PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS[1]

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1. Introduction and summary. It is well known [3, 4] how, in the case of any general strongly testable [5] linear hypothesis for either ANOVA or MANOVA one can put simultaneous confidence bounds on a particular set of parametric functions, which might be regarded as measures of deviation from the "total" hypothesis and its various components. The parametric functions are such that, in each problem, one of these can be appropriately called the "total" and the rest "partials" of various orders. For each problem the "total" function, (i) in the univariate case, is related to, but not quite the same as, the noncentrality parameter of the usual F-test of the "total" hypothesis in ANOVA, and (ii) in the multivariate case, is the largest characteristic root of a certain parametric matrix which is related to, but not quite the same as, another parametric matrix whose nonzero characteristic roots occur as a set of noncentrality parameters in the power function for the test (no matter which of the standard tests we use) of the "total" hypothesis in MANOVA. The same remark applies to "partials" of various orders considered in the proper sense.

In this note, for both ANOVA and MANOVA, the hypothesis considered is that of equality of the treatment effects—vector equality in the case of MANOVA. Starting from such a hypothesis, explicit algebraic expressions are obtained for the total and partial parametric functions that go with the simultaneous confidence statements in the case of both ANOVA and MANOVA and for balanced and partially balanced designs. It is also indicated how to obtain, in a convenient form, the algebraic expression for the confidence bounds on each such parametric function, without a derivation of these expressions in an explicit form.

2. Notation and preliminaries.

(i) *Univariate case.* Let $\mathbf{x}$ denote a column-vector of n independent normal variables with a common variance σ^2 and the means given by

$$\mathcal{E}\mathbf{x} = \mathbf{A}_{n \times m}\boldsymbol{\theta}_{m \times 1}, \tag{1}$$

where $\mathbf{A}$ is a matrix of known constants and $\boldsymbol{\theta}$ is a vector of unknown parameters.

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The hypothesis

(2) $$\mathcal{H}_0: \quad \mathbf{B}_{s \times m}\boldsymbol{\theta} = \mathbf{0} \qquad [\text{Rank} \quad \mathbf{B} = s]$$

is said to be strongly testable if Rank $(\mathbf{A}', \mathbf{B}')$ is equal to Rank $(\mathbf{A})$. If we write

(3) $$\mathbf{B}\boldsymbol{\theta} = \boldsymbol{\phi}_{s \times 1},$$

then the "total" parametric function, Δ, associated with $\mathcal{H}_0$ is

(4) $$\Delta = \boldsymbol{\phi}'\mathbf{D}^{-1}\boldsymbol{\phi},$$

where $D\sigma^2$ is the variance-covariance matrix of the best unbiased linear estimates of $\boldsymbol{\phi}$. It may be observed that Δ/σ^2 is the noncentrality parameter of the F-test for $\mathcal{H}_0$. Confidence bounds on Δ, with a confidence coefficient greater than or equal to $(1 - \alpha)$, are then [3, 4] given by

(5) $$S_{H_0}^{\frac{1}{2}} - \left[\frac{s}{n-r} F_\alpha\right]^{\frac{1}{2}} S_E^{\frac{1}{2}} \leqq \Delta^{\frac{1}{2}} \leqq S_{H_0}^{\frac{1}{2}} + \left[\frac{s}{n-r} F_\alpha\right]^{\frac{1}{2}} S_E^{\frac{1}{2}},$$

where $r = $ Rank $\mathbf{A}$, F_α is the $100\alpha\%$ significance point of F with d.f. s and $n - r$ respectively, S_{H_0} is the sum of squares due to $\mathcal{H}_0$ and S_E is the sum of squares due to error. We also have the simultaneous confidence statements

(6) $$S_{(a)H_0}^{\frac{1}{2}} - \left[\frac{s}{n-r} F_\alpha\right]^{\frac{1}{2}} S_E^{\frac{1}{2}} \leqq \Delta_{(a)}^{\frac{1}{2}} \leqq S_{(a)H_0}^{\frac{1}{2}} + \left[\frac{s}{n-r} F_\alpha\right]^{\frac{1}{2}} S_E^{\frac{1}{2}},$$

where $\Delta_{(a)} = \boldsymbol{\phi}'_{(a)}\mathbf{D}_{(a)}^{-1}\boldsymbol{\phi}_{(a)}$, $\boldsymbol{\phi}_{(a)}$ is any subvector of $\boldsymbol{\phi}$, $\mathbf{D}_{(a)}$ is the corresponding submatrix of $\mathbf{D}$ and $S_{(a)H_0}$ is the corresponding sum of squares due to the partial hypothesis $\mathcal{H}_{(a)_0}$: $\boldsymbol{\phi}_{(a)} = \mathbf{0}$. (5) and (6) are implications of (13.2.21) on p. 90 in [3].

In the case of treatment-block designs, we have

(7) $$\mathcal{E}x_\alpha = t_i + b_j \qquad \begin{array}{l} i = 1, 2, \cdots, v, \\ j = 1, 2, \cdots, b, \end{array}$$

if the αth observation belongs to the ith treatment and jth block. The hypothesis of equality of treatment effects may be expressed as

(8) $$\mathcal{H}_0: \quad (\mathbf{I}_{v-1}, -\mathbf{J}_{v-1,1})\mathbf{t} = \mathbf{0},$$

where $\mathbf{t}' = (t_1, t_2, \cdots, t_v)$ and $\mathbf{J}_{r,s} = \{1\}_{r \times s}$. We shall write $\mathbf{J}_{r \times r}$ as $\mathbf{J}_r$. We assume that the design is connected. Let $n_{ij} = 1(0)$ if the ith treatment appears (does not appear) in the jth block. Then $\mathbf{N} = (n_{ij})_{v \times b}$ is the incidence-matrix of the design. Let r, k, $\mathbf{T}$ and $\mathbf{B}$ denote the number of replications of each treatment, the number of observations in each block, the vector of treatment totals and the vector of block-totals respectively. Then it is well-known [2] that the equations for $\mathbf{t}$ are

(9) $$\mathbf{C}\mathbf{t} = \mathbf{Q},$$

where $\mathbf{C} = r\mathbf{I} - (1/k)\mathbf{NN}'$ and $\mathbf{Q} = \mathbf{T} - (1/k)\mathbf{NB}$. Also

$$\text{Cov}(\mathbf{Q}) = \sigma^2\mathbf{C}. \tag{10}$$

Then, from (3) and (8), $\phi_i = t_i - t_v$, $i = 1, 2, \cdots, v - 1$. We may express Δ in a symmetrical form by taking $\boldsymbol{\phi} = (\mathbf{I}_{v-1}, -\mathbf{J}_{v-1,1})\boldsymbol{\xi}$, where

$$\xi_i = t_i - (1/v)(t_1 + t_2 + \cdots + t_v), \qquad i = 1, 2, \cdots, v.$$

From (4)

$$\Delta = \boldsymbol{\xi}'(\mathbf{I}_{v-1}, -\mathbf{J}_{v-1,1})'\mathbf{D}^{-1}(\mathbf{I}_{v-1}, -\mathbf{J}_{v-1,1})\boldsymbol{\xi}. \tag{11}$$

(ii) *Multivariate case.* Let $\mathbf{X}$ denote a matrix of n independent p-dimensional normal variables with a common variance-covariance matrix $\boldsymbol{\Sigma}$, p being the number of characters observed on each individual, and let the means be given by

$$\mathcal{E}\mathbf{X}_{n \times p} = \mathbf{A}_{n \times m}\boldsymbol{\Theta}_{m \times p}, \tag{12}$$

where $\boldsymbol{\Theta}$ is a matrix of unknown parameters. Suppose that

$$\mathcal{H}_0: \quad \mathbf{B\Theta U}_{p \times n} = \mathbf{0} \qquad [\text{Rank } \mathbf{U} = u \leqq p] \tag{13}$$

is the "strongly testable" hypothesis to be tested. If we write

$$\mathbf{B\Theta U} = \boldsymbol{\phi}_{s \times u}, \tag{14}$$

then the "total" parametric function, Δ, associated with $\mathcal{H}_0$ is [3, 4] given by

$$\Delta = C_{\max}[\boldsymbol{\phi}'\mathbf{D}^{-1}\boldsymbol{\phi}]. \tag{15}$$

It may be observed that the characteristic roots of $\boldsymbol{\phi}'\mathbf{D}^{-1}\boldsymbol{\phi}(\mathbf{U}'\boldsymbol{\Sigma}\mathbf{U})^{-1}$ are the noncentrality parameters in the power function of the test (no matter which of the standard tests we use) of the "total" hypothesis given by (13).

The confidence statement is [3] given by

$$\begin{aligned} &C^{\frac{1}{2}}_{\max}(\mathbf{S}_{H_0}) - \left[\frac{s}{n-r}C_\alpha\right]^{\frac{1}{2}} C^{\frac{1}{2}}_{\max}(\mathbf{S}_E) \leqq \Delta^{\frac{1}{2}} \leqq \\ &C^{\frac{1}{2}}_{\max}(\mathbf{S}_{H_0}) + \left[\frac{s}{n-r}C_\alpha\right]^{\frac{1}{2}} C^{\frac{1}{2}}_{\max}(\mathbf{S}_E), \end{aligned} \tag{16}$$

where $\mathbf{S}_{H_0}$ and $\mathbf{S}_E$ are the sum of products matrices due to the hypothesis and error respectively, and C_α is the $100\alpha\%$ significance point of the distribution of the largest characteristic root, with d.f. u, s, and $n - r$. In this case, we have simultaneous confidence statements, similar to (6), given by

$$\begin{aligned} C^{\frac{1}{2}}_{\max}[\mathbf{S}_{(a)H_0}] - \left[\frac{s}{n-r}C_\alpha\right]^{\frac{1}{2}} C^{\frac{1}{2}}_{\max}(\mathbf{S}_E) &\leqq \Delta^{\frac{1}{2}}_{(a)} \\ &\leqq C^{\frac{1}{2}}_{\max}[\mathbf{S}_{(a)H_0}] + \left[\frac{s}{n-r}C_\alpha\right]^{\frac{1}{2}} C^{\frac{1}{2}}_{\max}(\mathbf{S}_E), \end{aligned} \tag{17}$$

where $\Delta_{(a)} = C_{\max}[\boldsymbol{\phi}'_{(a)}\mathbf{D}^{-1}_{(a)}\boldsymbol{\phi}_{(a)}]$, $\boldsymbol{\phi}_{(a)}$ being a submatrix of $\boldsymbol{\phi}$ obtained by choosing some rows of ϕ. In addition, we have, by dropping some columns of ϕ, simultaneous confidence statements given by

$$(18)\quad \begin{aligned} C^{\frac{1}{2}}_{\max}[\mathbf{S}_{(b)H_0}] - \left[\frac{s}{n-r} C_\alpha\right]^{\frac{1}{2}} C^{\frac{1}{2}}_{\max}[\mathbf{S}_{(b)E}] &\leqq \Delta^{\frac{1}{2}}_{(b)} \\ &\leqq C^{\frac{1}{2}}_{\max}[\mathbf{S}_{(b)H_0}] + \left[\frac{s}{n-r} C_\alpha\right]^{\frac{1}{2}} C^{\frac{1}{2}}_{\max}[\mathbf{S}_{(b)E}], \end{aligned}$$

where $\Delta_{(b)} = C_{\max}[\phi'_{(b)}\mathbf{D}^{-1}\phi_{(b)}]$, $\phi_{(b)}$ being a submatrix of ϕ obtained by choosing some columns of ϕ, and $\mathbf{S}_{(b)H_0}$ and $\mathbf{S}_{(b)E}$ are the corresponding submatrices of $\mathbf{S}_{H_0}$ and $\mathbf{S}_E$. (16), (17) and (18) are implications of (14.6.3) on p. 101 in [.3]

In the case of treatment-block designs, we have

$$(19)\qquad \mathcal{E}_{x_\alpha}^{(k)} = t_i^{(k)} + b_j^{(k)}, \qquad \begin{aligned} i &= 1, 2, \cdots, v, \\ j &= 1, 2, \cdots, b, \\ k &= 1, 2, \cdots, p, \end{aligned}$$

where $x_\alpha^{(k)}$ denotes the kth character measured on the αth experimental unit or individual that turns up for the ith treatment and the jth block; and $t_i^{(k)}$, $b_j^{(k)}$ stand respectively for the contributions to the expectation of the kth variate made by the ith treatment and the jth block.

From (1) and (12) we have the same "structure matrix", $\mathbf{A}$, in the multivariate situation as in the univariate case. This "structure matrix" depends on the design as well as on what the experimental statisticians have called the model, e.g., (7) and (19).

In this set-up, so far as the hypothesis (13) is concerned, we shall take $\mathbf{U} = \mathbf{I}$ for simplicity.

3. Balanced incomplete block designs.

(i) *Univariate case.* Here

$$\mathbf{C} = r\mathbf{I}_v - \frac{1}{k}[(r-\lambda)\mathbf{I}_v + \lambda\mathbf{J}_v] = \frac{\lambda v}{k}\mathbf{I}_v - \frac{\lambda}{k}\mathbf{J}_v.$$

Imposing the usual condition, $\mathbf{J}_v\mathbf{t} = \mathbf{0}$, to get unique solutions, we have $\mathbf{t} = (k/\lambda v)\mathbf{Q}$. Therefore, $\phi = k/\lambda v(\mathbf{I}_{v-1}, -\mathbf{J}_{v-1,1})\mathbf{Q}$, and hence

$$(20)\qquad \mathbf{D} = \frac{k^2}{\lambda^2 v^2}(\mathbf{I}_{v-1}, -\mathbf{J}_{v-1,1})\mathbf{C}(\mathbf{I}_{v-1}, -J_{v-1,1})' = \frac{k}{\lambda v}(\mathbf{I}_{v-1} + \mathbf{J}_{v-1}),$$

whence

$$(21)\qquad \mathbf{D}^{-1} = \frac{\lambda v}{k}(\mathbf{I}_{v-1} - (1/v)\mathbf{J}_{v-1}).$$

Thus, using (11) and the relation $\mathbf{J}_v\xi = \mathbf{0}$,

$$\Delta = \frac{\lambda v}{k} \boldsymbol{\phi}' \left(\mathbf{I}_{v-1} - \frac{1}{v} \mathbf{J}_{v-1} \right) \boldsymbol{\phi}$$

(22)
$$= \frac{\lambda v}{k} \boldsymbol{\xi}' \left(\mathbf{I}_v - \frac{1}{v} \mathbf{J}_v \right) \boldsymbol{\xi}$$

$$= \frac{\lambda v}{k} \boldsymbol{\xi}' \boldsymbol{\xi}$$

$$= \frac{\lambda v}{k} \sum_{i=1}^{v} \xi_i^2 .$$

Then we can have a confidence statement of the form of (5) with $n = bk$, $r = b + v - 1$, $s = v - 1$ and Δ given by (22).

For the "partial" statements (6), if $\boldsymbol{\phi}'_{(a)} = (\phi_{i_1}, \phi_{i_2}, \cdots, \phi_{i_t})$, $(t < v - 1)$ then, from (20),

$$\mathbf{D}_{(a)} = \frac{k}{\lambda v} (\mathbf{I}_t + \mathbf{J}_t). \tag{23}$$

Hence

$$\mathbf{D}_{(a)}^{-1} = \frac{\lambda v}{k} \left(\mathbf{I}_t - \frac{1}{t+1} \mathbf{J}_t \right) \tag{24}$$

and

$$\Delta_{(a)} = \frac{\lambda v}{k} \boldsymbol{\phi}'_{(a)} \left(\mathbf{I}_t - \frac{1}{t+1} \mathbf{J}_t \right) \boldsymbol{\phi}_{(a)} .$$

For a symmetrical expression, we take $\boldsymbol{\phi}_{(a)} = (\mathbf{I}_t, -\mathbf{J}_{t,1})\boldsymbol{\xi}_{(a)}$, where

$$\xi_{i_j(a)} = t_{i_j} - (1/(t+1))[t_{i_1} + t_{i_2} + \cdots + t_{i_t} + t_v]$$

so that, using $\mathbf{J}_{t+1}\boldsymbol{\xi}_{(a)} = 0$,

$$\Delta_{(a)} = \frac{\lambda v}{k} \boldsymbol{\xi}'_{(a)} \left(\mathbf{I}_{t+1} - \frac{1}{t+1} \mathbf{J}_{t+1} \right) \boldsymbol{\xi}_{(a)}$$

(25)
$$= \frac{\lambda v}{k} \boldsymbol{\xi}'_{(a)} \boldsymbol{\xi}_{(a)}$$

$$= \frac{\lambda v}{k} \left[\sum_{j=1}^{t} \xi_{i_j(a)}^2 + \xi_{v(a)}^2 \right].$$

(ii) *Multivariate case.* We have the confidence bounds of the form of (16) with $n = bk$, $r = b + v - 1$, $s = v - 1$ and

$$\Delta = C_{\max} \left[\boldsymbol{\phi}' \frac{\lambda v}{k} \left(\mathbf{I}_{v-1} - \frac{1}{v} \mathbf{J}_{v-1} \right) \boldsymbol{\phi} \right].$$

Here again we may write $\boldsymbol{\phi} = (\mathbf{I}_{v-1}, -\mathbf{J}_{v-1,1})\boldsymbol{\xi}$, where $\boldsymbol{\xi} = (\boldsymbol{\xi}^{(1)}, \cdots, \boldsymbol{\xi}^{(p)})$ and

$\xi_i^{(j)} = t_i^{(j)} - v^{-1}\sum_{l=1}^{v} t_l^{(j)}$. Then $\Delta = (\lambda v/k)C_{\max}(\xi'\xi)$. We have, from (24), one set of "partial" statements of the form of (17) with

$$\Delta_{(a)} = \frac{\lambda v}{k} C_{\max}\left[\phi'_{(a)}\left(\mathbf{I}_t - \frac{1}{t+1}\mathbf{J}_t\right)\phi_{(a)}\right],$$

or, from (25), $\Delta_{(a)} = (\lambda v/k)C_{\max}[\xi'_{(a)}\xi_{(a)}]$, where $\xi_{(a)} = (\xi_{(a)}^{(1)}, \cdots, \xi_{(a)}^{(p)})$.

Similarly, we have, from (21), another set of "partial" statements of the form of (18) with

$$\Delta_{(b)} = C_{\max}\left[\phi'_{(b)}\frac{\lambda v}{k}\left(\mathbf{I}_{v-1} - \frac{1}{v}\mathbf{J}_{v-1}\right)\phi_{(b)}\right] = \frac{\lambda v}{k} C_{\max}(\xi'_{(b)}\xi_{(b)}).$$

4. Partially balanced incomplete block designs.

(i) *Univariate case.* Consider a PBIBD with m associate classes and association matrices $\mathbf{B}_i$ $(i = 0, 1, \cdots, m)$. Then it is well known [1] that $\mathbf{C} = \sum_{k=0}^{m}\alpha_k\mathbf{B}_k$, where $\mathbf{B}_0 = \mathbf{I}_v$, $\alpha_0 = r(k-1)/k$, $\alpha_i = -\lambda_i/k$, $i = 1, \cdots, m$; and, imposing the condition $\mathbf{J}_{1,v}\mathbf{t} = 0$ on (9), we have

$$\mathbf{t} = \left(\sum_{k=0}^{m} e_k\mathbf{B}_k\right)\mathbf{Q} = \mathbf{E}\mathbf{Q}, \quad \text{say.}$$

It is well known that, when the design is connected, Rank $\mathbf{C} = v - 1$, so that the condition $\mathbf{J}_{1,v}\mathbf{t} = 0$ is sufficient to give unique solutions. Further

$$\mathbf{J}_{1,v}\mathbf{C} = \mathbf{0} \quad \text{and} \quad \mathbf{J}_{1,v}\mathbf{Q} = 0. \tag{26}$$

Let

$$\underset{v\times v}{\mathbf{C}} = \begin{pmatrix}\mathbf{C}_1\\ \mathbf{c}'\end{pmatrix}\begin{matrix}v-1\\1\end{matrix} = (\mathbf{C}_1', \mathbf{c}) = \begin{pmatrix}\mathbf{C}_{11} & \mathbf{d}\\ \mathbf{d}' & c_0\end{pmatrix},$$

and

$$\mathbf{E} = \begin{pmatrix}\mathbf{E}_1\\ \mathbf{e}'\end{pmatrix}\begin{matrix}v-1\\1\end{matrix} = (\mathbf{E}_1', \mathbf{e}) = \begin{pmatrix}\mathbf{E}_{11} & \mathbf{f}\\ \mathbf{f}' & e_0\end{pmatrix}.$$

Then $\begin{pmatrix}\mathbf{C}_1\\ \mathbf{J}_{1,v}\end{pmatrix}\mathbf{t} = \begin{pmatrix}\mathbf{Q}_1\\ 0\end{pmatrix}$, where $\mathbf{Q}' = (\mathbf{Q}_1', Q_v)$. Hence

$$\mathbf{t} = \mathbf{E}\mathbf{Q} = (\mathbf{E}_1', \mathbf{e})\begin{pmatrix}\mathbf{Q}_1\\ Q_v\end{pmatrix} = \mathbf{E}_1'\mathbf{Q}_1 + \mathbf{e}Q_v.$$

Therefore, in view of (26),

$$\mathbf{t} = \mathbf{E}_1'\mathbf{Q}_1 - \mathbf{e}\mathbf{J}_{1,v-1}\mathbf{Q}_1 = (\mathbf{E}_1' - \mathbf{e}\mathbf{J}_{1,v-1})\mathbf{Q}_1 = (\mathbf{E}_1' - \mathbf{e}\mathbf{J}_{1,v-1} \,\vdots\, \mathbf{x})\begin{pmatrix}\mathbf{Q}_1\\ 0\end{pmatrix}.$$

Hence

$$\begin{pmatrix}\mathbf{C}_1\\ \mathbf{J}_{1,v}\end{pmatrix}^{-1} = (\mathbf{E}_1' - \mathbf{e}\mathbf{J}_{1,v-1} \,\vdots\, \mathbf{x}), \tag{27}$$

so that,

$$(\mathbf{E}_1' - \mathbf{e}\mathbf{J}_{1,v-1} \,\vdots\, \mathbf{x}) \begin{pmatrix} \mathbf{C}_1 \\ \mathbf{J}_{1,v} \end{pmatrix} = \mathbf{I}_v .$$

Thus $\mathbf{E}_1' \mathbf{C}_1 - \mathbf{e}\mathbf{J}_{1,v-1}\mathbf{C}_1 + \mathbf{x}\mathbf{J}_{1,v} = \mathbf{I}_v$. Hence, in view of (26),

$$\mathbf{E}_1'\mathbf{C}_1 + \mathbf{e}\mathbf{C}' = \mathbf{I}_v - \mathbf{x}\mathbf{J}_{1,v} ,$$

that is,

$$\mathbf{E}\mathbf{C} = \mathbf{I}_v - \mathbf{x}\mathbf{J}_{1,v} . \tag{28}$$

Also, from (27),

$$\begin{pmatrix} \mathbf{C}_1 \\ \mathbf{J}_{1,v} \end{pmatrix} (\mathbf{E}_1' - \mathbf{e}\mathbf{J}_{1,v-1} \,\vdots\, \mathbf{x}) = \mathbf{I}_v .$$

Hence $\mathbf{C}_1\mathbf{x} = \mathbf{0}$. But $\mathbf{C}_1\mathbf{J}_{v,1} = \mathbf{0}$ and Rank $\mathbf{C}_1 = v - 1$. Therefore, $\mathbf{x} = x\mathbf{J}_{v,1}$. Furthermore, $\mathbf{J}_{1,v}\mathbf{x} = 1$, whence $x\mathbf{J}_{1,v}\mathbf{J}_{v,1} = 1$, that is, $x = v^{-1}$. (28) thus reduces to

$$\mathbf{E}\mathbf{C} = \mathbf{I}_v - (1/v)\mathbf{J}_v . \tag{29}$$

Now $\hat{\boldsymbol{\phi}} = (\mathbf{I}_{v-1} , -\mathbf{J}_{v-1,1})\mathbf{t} = (\mathbf{I}_{v-1} , -\mathbf{J}_{v-1,1})\mathbf{E}\mathbf{Q}$, so that,

$$\mathbf{D} = (\mathbf{I}_{v-1} , -\mathbf{J}_{v-1,1})\mathbf{E}\mathbf{C}\mathbf{E} \begin{pmatrix} \mathbf{I}_{v-1} \\ -\mathbf{J}_{1,v-1} \end{pmatrix} .$$

Therefore, from (29) and (27),

$$\begin{aligned} \mathbf{D} &= (\mathbf{I}_{v-1} , -\mathbf{J}_{v-1,1}) \left(\mathbf{I}_v - \frac{1}{v}\mathbf{J}_v\right) (\mathbf{E}_1' - \mathbf{e}\mathbf{J}_{1,v-1}) \\ &= (\mathbf{I}_{v-1} , -\mathbf{J}_{v-1,1})(\mathbf{E}_1' - \mathbf{e}\mathbf{J}_{1,v-1}) \\ &= \mathbf{E}_{11} - \mathbf{f}\mathbf{J}_{1,v-1} - \mathbf{J}_{v-1,1}\mathbf{f}' + e_0 \mathbf{J}_{v-1} . \end{aligned} \tag{30}$$

Furthermore, premultiplying both sides of the equation,

$$\begin{bmatrix} \mathbf{E}_{11} - \mathbf{f}\mathbf{J}_{1,v-1} & \frac{1}{v}\mathbf{J}_{v-1,1} \\ \mathbf{f}' - e_0 \mathbf{J}_{1,v-1} & \frac{1}{v} \end{bmatrix} \begin{bmatrix} \mathbf{C}_{11} & \mathbf{d} \\ \mathbf{J}_{1,v-1} & 1 \end{bmatrix} = \mathbf{I}_v ,$$

by $(\mathbf{I}_{v-1} , -\mathbf{J}_{v-1,1})$, we have $\mathbf{D}\mathbf{C}_{11} = \mathbf{I}_{v-1}$ and, therefore,

$$\mathbf{D}^{-1} = \mathbf{C}_{11} . \tag{31}$$

Hence, from (11),

$$\Delta = \boldsymbol{\phi}'C_{11}\boldsymbol{\phi} = \xi'\mathbf{C}\xi. \tag{32}$$

Here, we may note that $c_{ii} = \alpha_0 = r(k - 1)/k$ and $c_{ij} = \alpha_l = -\lambda_l/k$ if ith and jth treatments are lth associates. Then we can have a confidence statement of the form of (5) with $n = bk$, $r = b + v - 1$, $s = v - 1$ and Δ given by (32).

The "partial" statements of the form of (6), however, cannot be made in a compact form, unless we know the association scheme. If we have $\boldsymbol{\phi}'_{(a)} = (\phi_1, \cdots, \phi_t)$, then $\mathbf{D}_{(a)}^{-1} = \mathbf{X} - \mathbf{Y}\mathbf{W}^{-1}\mathbf{Z}$, where

$$\mathbf{C}_{11} = \underset{t \quad v-t-1}{\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{W} \end{bmatrix}} \begin{matrix} t \\ v-t-1 \end{matrix}$$

and thus $\Delta_{(a)} = \boldsymbol{\phi}'_{(a)}[\mathbf{X} - \mathbf{Y}\mathbf{W}^{-1}\mathbf{Z}]\boldsymbol{\phi}_{(a)}$.

(ii) *Multivariate case.* We have the confidence bounds of the form of (16) with $n = bk$, $r = b + v - 1$, $s = v - 1$ and

$$\Delta = C_{\max}[\boldsymbol{\phi}'\mathbf{C}_{11}\boldsymbol{\phi}] = C_{\max}[\boldsymbol{\xi}^1\mathbf{C}\boldsymbol{\xi}].$$

We have, as before, one set of "partial" statements of the form of (17) with

$$\Delta_{(a)} = C_{\max}[\boldsymbol{\phi}'_{(a)}(\mathbf{X} - \mathbf{Y}\mathbf{W}^{-1}\mathbf{Z})\boldsymbol{\phi}_{(a)}].$$

The other set of "partial" statements is of the form of (18) with

$$\Delta_{(b)} = C_{\max}[\boldsymbol{\phi}'_{(b)}\mathbf{C}_{11}\boldsymbol{\phi}_{(b)}] = C_{\max}[\boldsymbol{\xi}'_{(b)}\mathbf{C}\boldsymbol{\xi}_{(b)}].$$

5. General "connected" incomplete block designs. It is well known [2] that, in general, $\mathbf{C}\mathbf{t} = \mathbf{Q}$, which, on imposing the condition $\mathbf{J}_{1,v}\mathbf{t} = 0$, yields $\mathbf{t} = \mathbf{E}\mathbf{Q}$. Then, arguing as before, from (26) to (32), we have

$$\Delta = \boldsymbol{\phi}'\mathbf{C}_{11}\boldsymbol{\phi} = \boldsymbol{\xi}'\mathbf{C}\boldsymbol{\xi}. \tag{33}$$

Then we can have a confidence statement of the form of (5) with

$$n = \sum_{i=1}^{v} r_i = \sum_{j=1}^{b} k_j, \qquad r = b + v - 1, \quad s = v - 1$$

and Δ given by (33). We can have "partial" statements and confidence bounds in the multivariate situation analogous to those for PBIBD.

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