PROBABILITY CONTENT OF REGIONS UNDER SPHERICAL NORMAL DISTRIBUTIONS, I[1]

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1. Introduction. The primary purpose of this series of papers is to attempt to lay the groundwork for a relatively well-rounded theory of the spherical normal distribution. Many distributional problems in mathematical statistics may be regarded as particular instances of one general problem, the determination of the probability content of geometrically well-defined regions in Euclidean N-space when the underlying distribution is centered spherical normal and has unit variance in any direction. Specifically then, we require for a definite region R

$$P(R) = (2\pi)^{-\frac{1}{2}N} \int_{\mathbf{x} \varepsilon R} e^{-\frac{1}{2}\mathbf{x}'\mathbf{x}}\, d\mathbf{x}, \tag{1.1}$$

in which $\mathbf{x}' = (x_1, \cdots, x_N)$. The class of problems represented by (1.1) is a very broad one and the literature on it is correspondingly quite enormous and well-diffused. In fact, all the distributional problems which occur in the theory of sampling from multivariate normal populations may in principle be brought under our general heading. Thus, let $\mathbf{y}_i$, $i = 1, 2, \cdots, n$, denote n mutually independent k-dimensional vectors each of which is governed by the elementary probability density

$$p(\mathbf{y}) = (2\pi)^{-\frac{1}{2}k} | \mathbf{V} |^{-\frac{1}{2}} e^{-\frac{1}{2}\mathbf{y}'\mathbf{V}^{-1}\mathbf{y}}. \tag{1.2}$$

The joint probability density function for the n vectors is $\prod_1^n p(\mathbf{y}_i)$ and integrals of the form

$$\begin{aligned}(2\pi)^{-\frac{1}{2}nk} | \mathbf{V} |^{-\frac{1}{2}n} \int_{\mathbf{z} \varepsilon T} \exp\left(-\tfrac{1}{2} \sum_1^n \mathbf{y}_i' \mathbf{V}^{-1} \mathbf{y}_i\right) \prod_1^n d\mathbf{y}_i \\ = (2\pi)^{-\frac{1}{2}N} | \mathbf{W} |^{-\frac{1}{2}} \int_{\mathbf{z} \varepsilon T} \exp\left(-\tfrac{1}{2}\mathbf{z}' \mathbf{W}^{-1} \mathbf{z}\right) d\mathbf{z},\end{aligned} \tag{1.3}$$

where $\mathbf{z}$ is a partitioned vector, $\mathbf{W}$ is a partitioned matrix,

$$\mathbf{z} = \begin{bmatrix} \mathbf{y}_1 \\ \cdots \\ \mathbf{y}_2 \\ \cdots \\ \vdots \\ \cdots \\ \mathbf{y}_n \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} \mathbf{V} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{V} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{V} \end{bmatrix}, \tag{1.4}$$

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$N = nk$ and T a specified region in Euclidean N-space, may be thrown in the form (1.1) by a linear orthogonal transformation chosen so as to orient the new axes along the axes of the ellipsoids of constant density of the distribution of $\mathbf{z}$, followed by a simple scaling transformation to convert the ellipsoids into spheres.

We recall a second and frequently more convenient method of reducing $\mathbf{z}'\mathbf{W}^{-1}\mathbf{z}$ to a sum of squares. By means of triangular resolution, $\mathbf{W}$ may be factored [1] in the form

$$\mathbf{W} = \mathbf{M}\mathbf{M}', \tag{1.5}$$

where the $N \times N$ matrix $\mathbf{M}$ is defined by

$$\mathbf{M} = \begin{bmatrix} \mathbf{L} & \vdots & \mathbf{0} \cdots\cdots \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & \mathbf{L} \cdots\cdots \mathbf{0} \\ \vdots & \vdots & \vdots \ddots \quad \vdots \\ \mathbf{0} & \vdots & \mathbf{0} \cdots\cdots \mathbf{L} \end{bmatrix}, \tag{1.6}$$

and $\mathbf{V} = \mathbf{L}\mathbf{L}'$, $\mathbf{L}$ denoting a $k \times k$ lower triangular matrix. On setting

$$\mathbf{z} = \mathbf{M}\mathbf{x}, \tag{1.7}$$

$$(2\pi)^{-\frac{1}{2}N} \mid \mathbf{W} \mid^{-\frac{1}{2}} \int_{\mathbf{z} \varepsilon T} \exp(-\tfrac{1}{2}\mathbf{z}'\mathbf{W}^{-1}\mathbf{z})\, d\mathbf{z} = (2\pi)^{-\frac{1}{2}N} \int_{\mathbf{x} \varepsilon R} \exp(-\tfrac{1}{2}\mathbf{x}'\mathbf{x})\, d\mathbf{x}, \tag{1.8}$$

where $R = \mathbf{M}^{-1}(T)$.

In view of the preceding discussion no loss of generality results in assuming, whenever necessary, that the distribution is given by (1.1).

We shall list, briefly review and discuss a number of important distributional problems, together with some applications, which are formally reduceable to integrals of the form (1.1). In the first few illustrations, the regions R constitute relatively simple geometrical entities, such as half-spaces, hyperspheres, hypercones and hypercylinders, for which the statistical applications are both classic and familiar, but in later illustrations more complex bodies, such as ellipsoids, simplices and polyhederal cones, are considered. In particular, the last named case of polyhederal cones, corresponding to the difficult and important problem of the multivariate normal integral, and more especially the bivariate normal integral (when the dimensionality of the polyhedral cone is 2), will be investigated in some detail.

Integrals of the form (1.1) are rarely capable of being expressed in closed form using well-known functions. Nevertheless, it is hoped that the current presentation will provide a unifying thread and thereby help to stimulate further research. In the sequel a quite powerful method, referred to as the "method of sections," will frequently be used to deal with the integrals. This consists in dividing up the region R by means of a series of parallel and adjoining $(N - 1)$-flats and in the exploitation of the following fundamental property of the spherical normal distribution of dimensionality N: The conditional probability distribution in any linear subspace of dimensionality $N - k$ $(k = 1, 2, \cdots, N - 1)$

is itself spherical normal with dimensionality $N - k$ and with variance in any direction equal to the variance of the original N-dimensional distribution. Let O be the center of distribution, P any point in R and M the foot of the perpendicular from O to the flat through P. Further, let $OP = r$, $OM = \xi$, $PM = \eta$, with $r^2 = \xi^2 + \eta^2$. Then the p.d.f. at P is

$$(2\pi)^{-\frac{1}{2}N}e^{-\frac{1}{2}r^2} = (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}\xi^2} \times (2\pi)^{-\frac{1}{2}(N-1)}e^{-\frac{1}{2}\eta^2},$$

and the distribution in the flat through P is spherical normal with dimensionality $N - 1$. It follows that the probability content of the infinitesimal region intercepted by R between two parallel flats distant ξ and $\xi + d\xi$ from O is of the form

$$(1.9) \qquad (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}\xi^2}\, d\xi Q(\xi;\, R),$$

where $Q(\xi,\, R)$ is itself obtained by evaluating an integral of the form (1.1), with N replaced by $N - 1$. Consequently,

$$(1.10) \qquad P(R) = \int_{\xi_0}^{\xi_1} (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}\xi^2}Q(\xi;\, R)\, d\xi,$$

where ξ_0 and ξ_1 are the distances of the bounding flats to R from O. If, further, the section of each cutting flat is a region of the same geometrical type as R (e.g. R an ellipsoid and the section an ellipsoid), with the center M of the $(N - 1)$-dimensional spherical distribution in the flat bearing the same geometrical relationship with respect to the $(N - 1)$-dimensional figure as does O with respect to R (e.g. both O and M are centers of ellipsoids), then (1.10) becomes an integral recurrence relationship (see Sections 7 and 8).

2. Probability content of a half-space. The probability content of the infinite parallel slab R defined by $p_1 \leqq \sum_1^N a_i x_i \leqq p_2$ is given directly by the method of sections as

$$(2.1) \qquad (2\pi)^{-\frac{1}{2}} \int_{p_1/(\Sigma a_i^2)^{\frac{1}{2}}}^{p_2/(\Sigma a_i^2)^{\frac{1}{2}}} e^{-\frac{1}{2}\xi^2}\, d\xi.$$

Here the flats dividing R are taken parallel to the bounding flats and $Q(\xi; R) = 1$, $\xi_0 = p_1/(\sum a_i^2)^{\frac{1}{2}}$, $\xi_1 = p_2/(\sum a_i^2)^{\frac{1}{2}}$. In particular, for the lower half-space $p_1 = -\infty$ and (2.1) becomes

$$(2.2) \qquad (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{p_2/(\Sigma a_i^2)^{\frac{1}{2}}} e^{-\frac{1}{2}\xi^2}\, d\xi.$$

These results are, of course, a reflection of the fact that $\sum a_i x_i$ is distributed normally with zero mean and variance $\sum a_i^2$.

3. Probability contents of centrally and non-centrally located hyperspheres. Historically, the central χ^2 distribution was one of the first directly entailing probability contents of regions in N-space when the density is spherical normal.[2]

[2] A geometrical derivation of the χ^2 distribution *for 3 degrees of freedom* is implicit in Maxwell's great work [2] concerning the energy distribution of gas molecules. Each of three orthogonal components of velocity have identical and independent normal distributions with zero mean, and the energy, suitably standardized, is a χ^2 with 3 degrees of freedom.

For the central χ^2 distribution the region in question is a sphere whose center coincides with the center of the distribution, while for the non-central distribution the region is a sphere whose center is non-coincident with the center of the distribution.

Let χ_N^2 and $\chi_{N;\kappa}^2$ refer generically to a variate distributed as χ^2 with N degrees of freedom and a non-central χ^2 variate with N degrees of freedom and non-centrality parameter κ, respectively ($\chi_{N;0}^2 \equiv \chi_N^2$). The latter variate is defined by $\chi_{N;\kappa}^2 = \sum_1^N (x_i - \kappa_i)^2$, where the x_i are independent normal variables with zero means and unit variances, and $\kappa^2 = \sum_1^N \kappa_i^2$. Further, denote the distribution functions of these two variates by $F_N(a^2)$ and $G_{N;\kappa}(a^2)$. Correspondingly, lower case letters shall denote the p.d.f.'s. Then

$$
\begin{aligned}
F_N(a^2) = P(\chi_N^2 \leqq a^2) &= \underset{R:\sum_1^N x_i^2 \leqq a^2}{\int \cdots \int} (2\pi)^{-\frac{1}{2}N} \exp\left(-\tfrac{1}{2}\sum_1^N x_i^2\right) dx_1 \cdots dx_N \\
&= \underset{R}{\int \cdots \int} (2\pi)^{-\frac{1}{2}N} \exp(-\tfrac{1}{2}r^2) r^{N-1}\, dr\, d\omega \\
&= S_N(1) \int_0^a (2\pi)^{-\frac{1}{2}N} \exp(-\tfrac{1}{2}r^2) r^{N-1}\, dr \\
&= \left(2^{\frac{1}{2}N}\Gamma\left(\frac{N}{2}\right)\right)^{-1} \int_0^{a^2} \exp(-\tfrac{1}{2}r^2)(r^2)^{\frac{1}{2}N-1}\, dr^2,
\end{aligned}
\tag{3.1}
$$

where $d\omega$ is the solid angle subtended at the center of the distribution by an infinitesimal volume element and $S_N(c)$ is the surface-content of a hypersphere of N dimensions with radius c,

$$
S_N(c) = 2\pi^{\frac{1}{2}N} c^{N-1}/\Gamma(N/2). \tag{3.2}
$$

This gives the usual Incomplete Gamma Function for the distribution function. On differentiating with respect to a^2,

$$
f_N(a^2) = [2^{\frac{1}{2}N}\Gamma(N/2)]^{-1} \exp(-\tfrac{1}{2}a^2)(a^2)^{\frac{1}{2}N-1}. \tag{3.3}
$$

Pedagogically, perhaps a more useful geometrical derivation is to "slice" up the sphere into infinitesimal thin slices by a set of parallel planes. This corresponds to a proof by induction (cf. [3], pp. 247–8). Let x denote the distance of a typical slice from the center of the sphere. Then

$$
\begin{aligned}
F_N(a^2) &= \underset{R:\sum_1^N x_i^2 \leqq a^2}{\int \cdots \int} (2\pi)^{-\frac{1}{2}N} \exp\left(-\frac{1}{2}\sum_1^N x_i^2\right) dx_1 \cdots dx_N \\
&= \int_{-a}^{a} (2\pi)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}x^2) \cdot F_{N-1}(a^2 - x^2)\, dx,
\end{aligned}
\tag{3.4}
$$

on noting that the density at a point on the "x-slice", which intersects the given sphere in a sphere of dimensionality $N - 1$ and radius $(a^2 - x^2)^{\frac{1}{2}}$, distant y

from the center of the latter sphere, is $(2\pi)^{-N/2}\exp[-\frac{1}{2}(x^2+y^2)]$. On differentiating with respect to a^2,

$$f_N(a^2) = \int_{-a}^{a}(2\pi)^{-\frac{1}{2}}\exp(-\tfrac{1}{2}x^2)f_{N-1}(a^2-x^2)\,dx$$

$$= \int_{-a}^{a}(2\pi)^{-\frac{1}{2}}\exp(-\tfrac{1}{2}x^2)$$

$$\cdot\left[2^{\frac{1}{2}(N-1)}\Gamma\left(\frac{N-1}{2}\right)\right]^{-1}\exp[-\tfrac{1}{2}(a^2-x^2)](a^2-x^2)^{\frac{1}{2}(N-3)}\,dx$$

$$= \exp(-\tfrac{1}{2}a^2)\left[(2\pi)^{\frac{1}{2}}2^{\frac{1}{2}(N-1)}\Gamma\left(\frac{N-1}{2}\right)\right]^{-1}\int_{-a}^{a}(a^2-x^2)^{\frac{1}{2}(N-3)}\,dx$$

$$= \left(2^{\frac{1}{2}N}\Gamma\left(\frac{N}{2}\right)\right)^{-1}\exp(-\tfrac{1}{2}a^2)(a^2)^{\frac{1}{2}N-1}.$$

For the non-central χ^2 distribution, we require the distribution of $\sum_1^N(x_i-\kappa_i)^2$, where the x_i are mutually independent standardized normal random variables. Let O be the center of the distribution which is taken as before to be the origin of coordinates and K the point $(\kappa_1, \kappa_2, \cdots, \kappa_N)$. Let P be any point with coordinates $(x_1, x_2, \cdots, x_N)$, let $OK=\kappa$, $\kappa=(\kappa_1^2+\kappa_2^2+\cdots+\kappa_N^2)^{\frac{1}{2}}$, $KP=\xi$, and let the angle between KP and the line OK, produced in the sense O to K, be θ. Then

$$\sum_1^N x_i^2 = OP^2 = \kappa^2+\xi^2+2\kappa\xi\cos\theta,$$

and

$$G_{N;\kappa}(a^2) = P\left\{\sum_1^N(x_i-\kappa_i)^2 \leqq a^2\right\}$$

$$= \int_{R:\sum_1^N(x_i-\kappa_i)^2\leqq a^2}\cdots\int(2\pi)^{-\frac{1}{2}N}\exp\left(-\tfrac{1}{2}\sum_1^N x_i^2\right)dx_1\cdots dx_N$$

$$= \int_0^a\int_0^\pi(2\pi)^{-\frac{1}{2}N}\exp[-\tfrac{1}{2}(\kappa^2+\xi^2+2\kappa\xi\cos\theta)]\xi^{N-1}\,d\xi\,d\omega$$

$$= (2\pi)^{-\frac{1}{2}N}\exp(-\tfrac{1}{2}\kappa^2)\int_0^a\int_0^\pi\exp(-\tfrac{1}{2}\xi^2-\kappa\xi\cos\theta)\xi^{N-1}\,d\xi\,d\omega.$$

Now

$$\int_0^\pi\exp(-\kappa\xi\cos\theta)\,d\omega = 2\pi^{\frac{1}{2}N}(\tfrac{1}{2}\kappa\xi)^{-(\frac{1}{2}N-1)}I_{\frac{1}{2}N-1}(\kappa\xi),$$

where $I_n(z) = i^{-n}J_n(iz)$ is the Bessel function of the first kind with purely imaginary argument. This follows directly by dividing up the surface of the hypersphere into annuli $d\theta$, the content of such an annulus being $S_{N-1}(\sin\theta)\,d\theta$.

Thus,

$$\int_0^\pi \exp(-\kappa\xi \cos\theta)\, d\omega = \int_0^\pi \exp(-\kappa\xi\cos\theta) S_{N-1}(\sin\theta)\, d\theta$$
$$= \left[2\pi^{\frac{1}{2}(N-1)} \Big/ \Gamma\left(\frac{N-1}{2}\right)\right] \int_0^\pi \exp(-\kappa\xi\cos\theta)\sin^{N-2}\theta\, d\theta,$$

and this integral is related to the Bessel function

$$I_n(z) = [\sqrt{\pi}\,\Gamma(n+\tfrac{1}{2})]^{-1}(\tfrac{1}{2}z)^n \int_{-1}^1 \exp(\pm zv)(1-v^2)^{n-\frac{1}{2}}\, dv \quad (R(n+\tfrac{1}{2})>0),$$

by setting $v = \cos\theta$. Alternatively, $\exp(-\kappa\xi\cos\theta)$ may be expanded as a power series in $\cos\theta$ after which integration is effected term by term. Hence, finally,

$$G_{N;\kappa}(a^2) = (2\pi)^{-\frac{1}{2}N}\exp(-\tfrac{1}{2}\kappa^2) \cdot \int_0^a \exp(-\tfrac{1}{2}\xi^2)\xi^{N-1}\cdot 2\pi^{\frac{1}{2}N}(\tfrac{1}{2}\kappa\xi)^{-(\frac{1}{2}N-1)} I_{\frac{1}{2}N-1}(\kappa\xi)\, d\xi \tag{3.5}$$

and

$$g_{N;\kappa}(a^2) = \tfrac{1}{2}\kappa^{-(\frac{1}{2}N-1)}\exp(-\tfrac{1}{2}\kappa^2)a^{\frac{1}{2}N-1}\exp(-\tfrac{1}{2}a^2)I_{\frac{1}{2}N-1}(\kappa a)$$
$$= 2^{-\frac{1}{2}N}\exp(-\tfrac{1}{2}\kappa^2)(a^2)^{\frac{1}{2}N-1}\exp(-\tfrac{1}{2}a^2)\sum_{r=0}^\infty [1/\Gamma(\tfrac{1}{2}N+r)] \cdot [(\tfrac{1}{2}\kappa a)^{2r}/r!]. \tag{3.6}$$

The above geometrical derivation seems to have been used first, in essence, by Patnaik [4].

An alternative and simpler geometrical method consists once again in dividing the sphere R by a set of parallel hyperplanes. Take these to be perpendicular to the line OK and let x be the distance of a typical plane from K. Then

$$G_{N;\kappa}(a^2) = \int\cdots\int_{R:\sum_1^N (x_i-\kappa_i)^2 \le a^2} (2\pi)^{-\frac{1}{2}N}\exp\left(-\frac{1}{2}\sum_1^N x_i^2\right) dx_1\cdots dx_N$$
$$= \int_{-a}^a (2\pi)^{-\frac{1}{2}}\exp[-\tfrac{1}{2}(\kappa+x)^2]F_{N-1}(a^2-x^2)\, dx. \tag{3.7}$$

Hence,

$$g_{N;\kappa}(a^2) = \exp(-\tfrac{1}{2}\kappa^2)\int_{-a}^a \exp(-\kappa x)\cdot(2\pi)^{-\frac{1}{2}}\exp(-\tfrac{1}{2}x^2)f_{N-1}(a^2-x^2)\, dx$$
$$= \exp(-\tfrac{1}{2}\kappa^2)\exp(-\tfrac{1}{2}a^2)\Big/\left[(2\pi)^{\frac{1}{2}}2^{\frac{1}{2}(N-1)}\Gamma\left(\frac{N-1}{2}\right)\right] \cdot \int_{-a}^a \exp(-\kappa x)(a^2-x^2)^{\frac{1}{2}(N-3)}\, dx$$
$$= \tfrac{1}{2}\kappa^{-(\frac{1}{2}N-1)}\exp(-\tfrac{1}{2}\kappa^2)a^{\frac{1}{2}N-1}\exp(-\tfrac{1}{2}a^2)I_{\frac{1}{2}N-1}(\kappa a),$$

after substitution for $f_{N-1}(a^2 - x^2)$ from (3.3) and using the above integral formula for $I_n(z)$.

Some exact values for the probability integral of the non-central χ^2 are available in [5] and [6]. A more extensive set of values is provided in Fix's tables [7] designed to yield the power function of χ^2. For studies in connexion with suitable approximations to the non-central χ^2 distribution, reference is made to [4], [8] and [9]. Finally, various tables of the non-central χ^2 distribution for the special case of two degrees of freedom have become available in recent years for application in ballistic problems ([10], [11], [12], [13]).

4. Probability content of a symmetrically and asymmetrically located hyperspherical cone. Consider a hyperspherical half-cone R with vertex at the center of the spherical normal distribution. Let the angle between the axis of the cone and a generator be θ. The probability content $P(R)$ of the cone is given by its relative solid angle, i.e., the ratio of the surface-content of the region, a cap, on a unit sphere with center at the vertex of the cone which is demarcated by th cone to the surface-content of the entire sphere. Hence, by division of the ʻ . into a set of annuli with radii sin θ',

$$
\begin{aligned}
P(R) &= \int_0^\theta S_{N-1}(\sin\theta')\, d\theta'/S_N(1) \\
&= \Gamma\left(\frac{N}{2}\right)\Big/\left[\sqrt{\pi}\,\Gamma\left(\frac{N-1}{2}\right)\right]\int_0^\theta \sin^{N-2}\theta'\, d\theta' \qquad (4.1)\\
&= \mathcal{I}_{\sin^2\theta}\left(\frac{N-1}{2}, \tfrac{1}{2}\right),
\end{aligned}
$$

where $\mathcal{I}$ denotes the Incomplete Beta Function *Ratio.*

Define

$$t_{N-1} = \xi/(\chi^2_{N-1}/(N-1))^{\frac{1}{2}}, \tag{4.2}$$

where ξ is a normal variate with zero mean and unit variance distributed independently of χ^2_{N-1}, a χ^2 variate with $N - 1$ degrees of freedom. The variate t_{N-1} may be expressed in the form

$$t_{N-1} = x_1\Big/\left(\sum_2^N x_i^2/(N-1)\right)^{\frac{1}{2}}, \tag{4.3}$$

where the x_i $(i = 1, 2, \cdots, N)$ are independent normal variates, each with zero mean and unit variance. The region $t_{N-1} \geqq c (c \geqq 0)$ defines a half-cone with vertex at the center of the distribution of the x_i and with axis oriented along the x_1-axis. The angle between the axis and a generator is $\theta = \text{arc cot}\,[c/(N-1)^{\frac{1}{2}}]$. The distribution function of t_{N-1} is given by (4.1), where here

$$\sin^2\theta = 1/[1 + c^2/(N-1)]. \tag{4.4}$$

The density function $-\partial P/\partial c$ is

$$g_{N-1}(c) = \left[(N-1)^{\frac{1}{2}} B\left(\frac{N-1}{2}, \tfrac{1}{2}\right)\right]^{-1}\left(1 + \frac{c^2}{N-1}\right)^{-\frac{1}{2}N}. \tag{4.5}$$

The simplest application of the t-distribution relates to the "Studentized" mean of a normal sample. The geometrical derivation of the latter quantity is well-known (see e.g. [3], pp. 239–40). The relevant hyperspherical cone in this instance has its axis along the line $x_1 = x_2 = \cdots = x_N$ equally inclined to the coordinate planes. The above argument implies, however, that the probability content of a hyperspherical cone of given angle and with vertex at the center of a spherical normal distribution is independent of its orientation.

Consider next a hyperspherical cone whose vertex does not coincide with the center of a given spherical normal distribution but whose axis passes through the latter point. As before, let the angle between the axis and a generator be θ. The probability content $P(R)$ of the cone may be obtained by considering sections perpendicular to the generator. Each such section is a hypersphere and the surfaces of equal density in the flat forming the section are hyperspheres. Hence[3],

$$P(R) = \int_{\lambda}^{\infty} (2\pi)^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}x^2\right) F_{N-1}[(x-\lambda)^2 \tan^2\theta]\, dx, \tag{4.6}$$

where λ is the distance of the vertex from the center of the distribution, and $F_{N-1}(\cdot)$ is defined in (3.1).

Define

$$t_{N-1;\lambda} = [\xi - \lambda]/[\chi^2_{N-1}/(N-1)]^{\frac{1}{2}}, \tag{4.7}$$

where ξ is a normal variate with zero mean and unit variance distributed independently of χ^2_{N-1}, a χ^2 variate with $N-1$ degrees of freedom ($t_{N-1} \equiv t_{N-1;0}$). The variate $t_{N-1;\lambda}$ may be expressed in the form

$$t_{N-1;\lambda} = [x_1 - \lambda] \Big/ \left[\sum_{2}^{N-1} x_i^2/(N-1)\right]^{\frac{1}{2}}, \tag{4.8}$$

where the $x_i (i = 1, 2, \cdots, N)$ are independent normal variates each with zero mean and unit variance. The region $t_{N-1;\lambda} \geqq c (c \geqq 0)$ defines a half-cone with vertex distant λ from the center of the distribution of the x_i. The axis of the cone is oriented along the x_1-axis and the angle between the latter and a generator is arc cot $[c/(N-1)^{\frac{1}{2}}]$. The distribution function of $t_{N-1;\lambda}$ is given by (4.6) with $\theta = \text{arc cot}\, [c/(N-1)^{\frac{1}{2}}]$. Setting $y = x - \lambda$ and differentiating with respect to y, the density $-\partial P/\partial c$ of $t_{N-1;\lambda}$ at c is obtained immediately as

$$\begin{aligned} g_{N-1;\lambda}(c) &= g_{N-1}(c) \exp\left(-\tfrac{1}{2}\lambda^2\right)\left[\Gamma\left(\frac{N}{2}\right)\right]^{-1} \\ &\quad \cdot \int_0^{\infty} z^{\frac{1}{2}N-1} \exp\left[-z - \lambda\sqrt{2z}\cos\theta\right] dz, \end{aligned} \tag{4.9}$$

where $y^2 \sec^2\theta/2$ has been replaced by z. This density function may be expressed in terms of the Hh function [14], defined by

$$Hh_m(y) = \int_0^{\infty} [x^m/m!] \exp\left[-\tfrac{1}{2}(x+y)^2\right] dx,$$

[3] This argument incidentally provides an alternative basis for the determination of the distribution and density functions of t_{N-1}, i.e., when $\lambda = 0$.

as follows:

$$g_{N-1;\lambda}(c) = g_{N-1}(c)\, \Gamma(N)\left[2^{\frac{1}{2}N-1}\Gamma\left(\frac{N}{2}\right)\right]^{-1} \cdot \exp\left(-\tfrac{1}{2}\lambda^2 \sec^2\theta\right) Hh_{N-1}(\lambda\cos\theta), \tag{4.10}$$

θ being given by (4.4). Alternatively, term by term integration yields

$$g_{N-1;\lambda}(c) = \frac{\exp\left(-\frac{1}{2}\lambda^2\right)}{\Gamma\left(\frac{N-1}{2}\right)((N-1)\pi)^{\frac{1}{2}}} \sum_{r=0}^{\infty} \frac{(-\lambda\sqrt{2})^r}{r!} \cdot \Gamma\left(\frac{N+r}{2}\right)\left(\frac{c^2}{N-1}\right)^{\frac{1}{2}r}\left(1+\frac{c^2}{N-1}\right)^{-\frac{1}{2}(N+r)} \tag{4.11}$$

The "tail-end area," obtained after term by term integration in (4.11), yields

$$P(t_{N-1;\lambda} \geqq c) = \frac{\exp\left(-\frac{1}{2}\lambda^2\right)}{2\sqrt{\pi}} \sum_{r=0}^{\infty} \Gamma\left[\tfrac{1}{2}(r+1)\right] \mathcal{G}_{\sin^2\theta}\left(\frac{N-1}{2}, \frac{r+1}{2}\right)\frac{(-\lambda\sqrt{2})^r}{r!} \qquad (c \geqq 0) \tag{4.12}$$

(cf. [15]). Tables of the non-central t-distribution have been provided by Neyman and Tokarska [16], Johnson and Welch [17] and, more recently, by Resnikoff and Lieberman [18] together with applications. The simplest application relates to the power of the test based on the Studentized mean-statistic from a normal sample. The axis of the relevant cone is then along the line $x_1 = x_2 = \cdots = x_N$. The above argument implies, however, that the probability content of a hyperspherical cone of given angle and with axis passing through the center of a spherical normal distribution is independent of its orientation provided that the distance between the vertex and the center of the distribution remains fixed.

5. Probability content of a region bounded by a variety of revolution of dimensionality $N - 1$ and of species p. Denote a hyperspherical surface (manifold) of dimensionality m by S_m. Then a variety of revolution S_{N-1} of dimensionality $N - 1$ and of species p is defined by the rotation of a S_{N-p-1}, imbedded in a $(N - p)$-flat A_{N-p} (linear space of dimensionality $N - p$), round a $(N - p - 1)$-flat Λ_{N-p-1} in A_{N-p} as axis (see e.g. Sommerville [19], pp. 137–8). The axial plane of revolution Λ_{N-p-1} may be regarded as defined by $N - p$ fixed points in a $(N - 1)$-flat A_{N-1} imbedded in N-space (Λ_{N-p-1} is a linear subspace of A_{N-1}) which has therefore p degrees of freedom and can rotate about Λ_{N-p-1} in such a way that each point of S_{N-p-1} generates the surface of a hypersphere with dimensionality $p + 1$, the latter surface itself being of dimensionality p. The center of the hypersphere is determined by the foot of the perpendicular to Λ_{N-p-1} from the given point.

If the equation of the generating surface S_{N-p-1}, referred to $N - p$ rectangular axes in A_{N-p} of which $N - p - 1$, designated the x_1-axis, $\cdots$, x_{N-p-1}-axis, are in Λ_{N-p-1}, is given by

$$x_{N-p}^2 = \phi(x_1, x_2, \cdots, x_{N-p-1}), \tag{5.1}$$

then the equation of the generated surface S_{N-1} is

$$x_{N-p}^2 + x_{N-p+1}^2 + \cdots + x_N^2 = \phi(x_1, x_2, \cdots, x_{N-p-1}), \tag{5.2}$$

since the expression on the left of equ. (5.2) represents the squared perpendicular distance of a point in S_{N-p-1}, when the latter is in a rotated position, from the axis Λ_{N-p-1}.

We shall now determine the probability content of the region R in N-space obtained by replacing the quality sign in equ. (5.2) by the sign $\leqq$, under the assumption that the distribution of the $x_i (i = 1, 2, \cdots, N)$ is governed by (1.1).

Let O be the center of the distribution, P the point $(x_1, x_2, \cdots, x_N)$ on S_{N-p-1} and P' the point $(x_1, \cdots, x_{N-p-1}, 0, \cdots, 0)$, P' being the foot of the perpendicular to Λ_{N-p-1} from P. The locus of P on rotation of S_{N-p-1} is a hypersphere with radius $\phi(x_1, x_2, \cdots, x_{N-p-1})$. Consider the infinitesimal element of R which projects into the element $d\Lambda_{N-p-1}$ located in Λ_{N-p-1} around P'. Since the density at any point is

$$(2\pi)^{-\frac{1}{2}N} \exp\left(-\tfrac{1}{2}\sum_1^N x_i^2\right) = (2\pi)^{-\frac{1}{2}(N-p-1)} \exp\left(-\tfrac{1}{2}\sum_1^{N-p-1} x_i^2\right) \cdot (2\pi)^{-\frac{1}{2}(p+1)} \exp\left(-\tfrac{1}{2}\sum_{N-p}^{N} x_i^2\right),$$

the distribution in the p-flat containing the locus of P is spherical normal with center P'. Hence, by (3.1), the probability content of the element is

$$(2\pi)^{-\frac{1}{2}(N-p-1)} \exp\left(-\tfrac{1}{2}\sum_1^{N-p-1} x_i^2\right) F_{p+1}(\phi(x_1, x_2, \cdots, x_{N-p-1}))\, d\Lambda_{N-p-1},$$

on recalling that $PP'^2 = \phi(x_1, x_2, \cdots, x_{N-p-1})$, where $F_{p+1}(\cdot)$ denotes the distribution function of a chi-square with $p + 1$ degrees of freedom (equ. (3.1)). The required probability content is then

$$\int \cdots \int (2\pi)^{-\frac{1}{2}(N-p-1)} \exp\left(-\tfrac{1}{2}\sum_1^{N-p-1} x_i^2\right) \cdot F_{p+1}(\phi(x_1, x_2, \cdots, x_{N-p-1}))\, d\Lambda_{N-p-1}. \tag{5.3}$$

Consider in particular the case

$$\phi(x_1, \cdots, x_{N-p-1}) = [(p + 1)/k(N - p - 1)](x_1^2 + \cdots + x_{N-p-1}^2) \tag{5.4}$$

when the region R becomes

$$[(x_1^2 + \cdots + x_{N-p-1}^2)/(N - p - 1)]/[(x_{N-p}^2 + \cdots + x_N^2)/(p + 1)] \geqq k. \tag{5.5}$$

Equation (5.1) now represents a hyperspherical conical surface in Λ_{N-p}, while equ. (5.2) represents the surface obtained by rotation round Λ_{N-p-1}. The generated surface is characterized by the property that the radius vector OP is inclined at a constant angle arc cot $(k(N - p - 1)/(p + 1))^{\frac{1}{2}}$ to the linear space Λ_{N-p-1}. Furthermore, in (5.3) $d\Lambda_{N-p-1}$ may conveniently be chosen as the annulus between two concentric hyperspherical surfaces in the Λ_{N-p-1}-subspace of radii ξ and $\xi + d\xi(\xi^2 = \sum_1^{N-p-1} x_i^2 = OP'^2)$, and the probability content of R is then

$$(5.6) \qquad \int_0^\infty (2\pi)^{-\frac{1}{2}(N-p-1)} \exp\left(-\tfrac{1}{2}\xi^2\right) \frac{2\pi^{\frac{1}{2}(N-p-1)}\xi^{N-p-2}}{\Gamma\left(\frac{N-p-1}{2}\right)} F_{p+1}\left[\frac{(p+1)\xi^2}{k(N-p-1)}\right] d\xi,$$

on using equ. (3.2) to substitute for $d\Lambda_{N-p-1}$.

The expression (5.6) represents $1 - P_{N-p-1,p+1}(k)$, where $P_{N-p-1,p+1}(\cdot)$ denotes the distribution function of an F-variate, $F_{N-p-1,p+1}$, with $N - p - 1$ and $p + 1$ degrees of freedom. The density function of the F-variate at the point k is

$$-\partial P_{N-p-1,p+1}/\partial k = \frac{2^{-\frac{1}{2}(N-2)}(p+1)^{\frac{1}{2}(p+1)}}{\Gamma\left(\frac{N-p-1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)(N-p-1)^{\frac{1}{2}(p+1)}}$$

$$(5.7) \qquad \frac{1}{k^{\frac{1}{2}(p+3)}} \int_0^\infty \xi^{N-1} \exp\left\{-\tfrac{1}{2}\left[1 + \frac{p+1}{(N-p-1)k}\right]\xi^2\right\} d\xi$$

$$= \frac{1}{B\left(\frac{N-p-1}{2}, \frac{p+1}{2}\right)} (N-p-1)^{\frac{1}{2}(N-p-1)}(p+1)^{\frac{1}{2}(p+1)} \cdot \frac{k^{\frac{1}{2}(N-p-1)-1}}{[(N-p-1)k + (p+1)]^{\frac{1}{2}N}},$$

after some reduction.

The case discussed in Section 4 corresponds to $p = N - 2$.

6. Probability contents of symmetrically and asymmetrically located hyperspherical cylinders. A hyperspherical cylinder in N-space is one such that the intersection with the cylinder of a $(N - 1)$-flat perpendicular to the axis of the cylinder is a hypersphere.

There are two distinct cases to consider:

(a) The axis of the cylinder passes through the center of the distribution.

(b) The axis of the cylinder does not pass through the center of the distribution.

The probability content may in both cases be readily evaluated by taking sections perpendicular to the axis. Let a be the radii of the cylinders in both (a) and (b), and let λ be the distance between the axis of the cylinder and the center of the distribution in (b). The probability contents of elements formed by adjoining parallel $(N - 1)$-flats distant x and $x + dx$ from the center of the distribution perpendicular to the axis of the cylinder is seen directly to be

$(2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}x^2)\, dxF_{N-1}(a^2)$ and $(2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}x^2)\, dxG_{N-1;\lambda}(a^2)$, respectively. Hence, by integration of x over $(-\infty, \infty)$, the probability content of the cylinder in case (a) is $F_{N-1}(a^2)$ and that of the cylinder in case (b) is $G_{N-1;\lambda}(a^2)$. A particularly simple application of (a) relates to the distribution of the sample variance in normal samples when the cylinder in question has its axis along the line $x_1 = x_2 = \cdots = x_N$ ([3], p. 238).

7. Probability content of a centrally situated ellipsoid. The problem treated in this section is equivalent to that of finding the distribution of the weighted sum of squares of mutually independent standardized normal variates. Formally, we require

$$(7.1) \qquad F_{N;a_1,\cdots,a_N}(t) = P\left(\sum_1^N a_i x_i^2 \leqq t\right), \qquad a_i \geqq 0, \quad \sum_1^N a_i = 1,$$

where the x_i are the variates referred to. The center of the ellipsoid coincides with the center of the distribution and the lengths of the semi-axes are $(t/a_i)^{\frac{1}{2}}(i = 1, 2, \cdots, N)$. The axes are oriented along the coordinate axes.

The distribution of $\sum a_i x_i^2$ has been discussed by Bhattacharya [20], Robbins [21], Robbins and Pitman [22], Hotelling [23], Gurland [24], [25], Pachares [26] and by Grad and Solomon [27]. The latter authors have tabulated $F_{N;a_1,\cdots,a_N}(t)$ for $N = 2, 3$, and for various selected sets of (a_1, a_2) and (a_1, a_2, a_3). We shall here obtain an integral recurrence relationship, based on the method of sections used previously in this paper, which should enable a systematic extension to be made of the available tables to values of $N > 3$, at least for moderate N[4], as well as of the tables for $N = 2, 3$. The following additional remarks are pertinent:

(i) There is no loss of generality in assuming $\sum a_i = 1$, since this can always be achieved by suitable standardization. However, the weights a_i are all non-negative.

(ii) The important statistical problem of the distribution of the weighted sum of independent χ^2 variates may be considered as a special case of our problem. Specifically, if $y = \sum_1^k c_i u_i$ where the u_i are independent χ^2 variates with n_i degrees of freedom, $\sum_1^k n_i = N$, then since N independent standardized normal variates, $x_1, x_2, \cdots, x_N$, may be introduced so that

$$(7.2) \qquad u_i = \sum_{j=1}^{n_i} x^2_{n_1+n_2+\cdots+n_{i-1}+j} \qquad (i = 1, 2, \cdots, k),$$

y may be expressed in terms of the x_i in the form

$$y = \sum_{i=1}^{k} \sum_{j=1}^{n_i} c_i x^2_{n_1+n_2+\cdots+n_{i-1}+j}$$

$$(7.3) \qquad = \sum_{\alpha=1}^{N} a_\alpha x_\alpha^2, \qquad a_\alpha = c_i \quad \text{for} \quad \alpha = n_1 + n_2 + \cdots + n_{i-1} + j$$

$$(j = 1, 2, \cdots, n_i).$$

[4] Extension of the Grad and Solomon tables for $N = 2$ and $N = 3$ has now been effected by Professor H. Solomon and the present author with the aid of (7.10). It is hoped to publish the extended tables shortly.

(iii) The problem of determining the distribution of a definite positive quadratic function of N variables when the latter are distributed as in a non-degenerate multivariate normal distribution reduces easily to our problem. Geometrically, the probability content of a given ellipsoid is required when the surfaces of constant density of the normal distribution are those of homothetic ellipsoids. A rotation of the coordinate axes and subsequent scaling converts the latter ellipsoids into spheres, while the given ellipsoid will in general remain an ellipsoid under these two transformations. Finally, a rotation of the new coordinate axes to bring them into coincidence with the axes of the given ellipsoid is effected.

Formally, one desires to evaluate the quantity

$$P(\mathbf{x}'\mathbf{A}\mathbf{x} \leqq c^2) = (2\pi)^{-\frac{1}{2}N} \,|\, \mathbf{V} \,|^{-\frac{1}{2}} \int_{\mathbf{x}'\mathbf{A}\mathbf{x} \leqq c^2} \exp(-\tfrac{1}{2}\mathbf{x}'\mathbf{V}^{-1}\mathbf{x}) \, d\mathbf{x}, \tag{7.4}$$

in which $\mathbf{A}$ and $\mathbf{V}$ are each of rank N. Set $\mathbf{x} = \mathbf{LRy}$, where $\mathbf{V}$ is decomposed by triangular resolution (as in the introductory section) in the form $\mathbf{V} = \mathbf{LL}'$, $\mathbf{L}$ being a lower $N \times N$ triangular matrix, while $\mathbf{R}$ is the orthogonal matrix of the characteristic vectors of the matrix $\mathbf{L}'\mathbf{AL}$. Then, after substitution,

$$P(\mathbf{x}'\mathbf{A}\mathbf{x} \leqq c^2) = (2\pi)^{-\frac{1}{2}N} \int_{\mathbf{y}'\mathbf{\Lambda}\mathbf{y} \leqq c^2} \exp(-\tfrac{1}{2}\mathbf{y}'\mathbf{y}) \, d\mathbf{y}, \tag{7.5}$$

where the diagonal matrix $\mathbf{\Lambda}$ is given by $\mathbf{\Lambda} = \mathbf{R}'\mathbf{L}'\mathbf{ALR}$.

If the diagonal elements of $\mathbf{\Lambda}$ are denoted by $\lambda_1, \lambda_2, \cdots, \lambda_N$, the characteristic numbers of $\mathbf{L}'\mathbf{AL}(\lambda_i > 0, i = 1, 2, \cdots, N)$, equation (7.5) is equivalent to

$$P(\mathbf{x}'\mathbf{A}\mathbf{x} \leqq c^2) = P\left(\sum_1^N \lambda_i y_i^2 \leqq c^2\right),$$

in which the y_i are mutually independent normal variates with zero means and unit variances. This establishes the equivalence of the problem dealt with in this subsection with that of equation (7.1).

(iv) A similar argument is applicable to the situations in which $\mathbf{A}$ is semi-definite positive. Here one wishes to evaluate the probability content of an elliptic cylinder under spherical normal distributions. The latter is clearly equal to the probability content of the ellipsoid, relating to an appropriately chosen subspace, obtained by projection into the latter subspace. The dimensionality of the subspace is equal to the rank of $\mathbf{A}$. A case in point is the mean square successive difference $\delta_{(1)}^2$, defined by

$$2(N-1)\delta_{(1)}^2 = \mathbf{x}'\mathbf{A}\mathbf{x} = \sum_{i=1}^{N-1} (x_{i+1} - x_i)^2 \tag{7.6}$$

([28], [29]), for which $\mathbf{A}$ is the continuant

$$\begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 1 \end{bmatrix},$$

empty spaces denoting zeros. The mean square successive difference has been proposed as a suitable estimator of variability when a secular trend in the mean is suspected. The inequality $\mathbf{x}'\mathbf{A}\mathbf{x} \leqq c^2$ defines the interior and boundary of an elliptic cylinder with axis $x_1 = x_2 = \cdots = x_N$ equally inclined to the coordinate axes. The secular equation $\mathbf{A} - \lambda\mathbf{I} = 0$ has one zero and $N - 1$ positive roots

$$\lambda_j = 4 \sin^2 (j\pi/2N) \qquad (j = 1, 2, \cdots, N - 1), \tag{7.7}$$

whence

$$P[2(N - 1)\delta_{(1)}^2 \leqq c^2] = P\left(\sum_1^{N-1} \lambda_j y_j^2 \leqq c^2\right),$$

in which the y_j are mutually independent standardized normal variates. Note that the sth cumulant of $2(N - 1)\delta_{(1)}^2$ is $(\sum_1^{N-1} \lambda_j^s)2^{s-1}(s - 1)!$, by the additive property of cumulants. The sums of powers of the characteristic numbers of $\mathbf{A}$ required for the specification of the cumulants of $\delta_{(1)}^2$, may be expressed in terms of the minors of $\mathbf{A}$, using well-known results relating to symmetric functions of the roots of a polynomial equation or, alternatively, by direct summation of the finite trigonometric series $\sum_1^{N-1} \sin^{2s} (j\pi/2N)$, after expressing the powers of the trigonometric ratios in terms of trigonometric ratios of multiples of the angles by standard formulae.

Similar results apply to higher order successive differences, useful in eliminating the inflationary effect of suspected given polynomial trends on estimates of variability [30], [31]. The mean square kth order advancing difference $\delta_{(k)}^2$ is defined as

$$\begin{aligned} (N - k)\binom{2k}{k}\delta_{(k)}^2 &= \sum_{i=1}^{N-k} (\Delta^k x_i)^2 \\ &= \sum_{i=1}^{N-k}\left[\sum_{\alpha=0}^{k} (-)^\alpha \binom{k}{\alpha} x_{i+\alpha}\right]^2 \end{aligned} \qquad (k = 1, 2, \cdots, N - 1). \tag{7.8}$$

The matrix of the quadratic form involved in the definition of $\delta_{(k)}^2$ is a continuant of order k in the sense that all the elements other than those in the leading diagonal and in the secondary upper and lower k diagonals are zero. The matrix is of rank $N - k$ and

$$P\left((N-k)\binom{2k}{k}\delta^2_{(k)} \leqq c^2\right) = P\left(\sum_{j=1}^{N-k} \lambda_j y_j^2 \leqq c^2\right),$$

where the λ_j are the $N - k$ non-zero characteristic numbers of the matrix and the y_j are mutually independent standardized normal variates. The distribution of $\delta^2_{(2)}$ has been considered in some detail by Kamat [32].

(v) One additional application is worth noting. Consider a dynamic programming or multifactorial design set-up in which the optimal course of action is represented by the N-dimensional vector $\mathbf{x}^*$. Suppose that $\mathbf{x}^*$ is not known exactly due either to a penumbra of vagueness surrounding the model from which $\mathbf{x}^*$ is deduced (i.e. faulty or imperfect theory), or to the fact that $\mathbf{x}^*$ is predicated on past experience (i.e. limited sampling), or for some other reason. Denote the estimate of $\mathbf{x}^*$ by $\hat{\mathbf{x}}^*$, and let the expectation vector and variance-covariance matrix of $\hat{\mathbf{x}}^*$ be $\mathbf{x}^*$ and $\mathbf{V}(\hat{\mathbf{x}}^*)(\mathbf{V}(\hat{\mathbf{x}}^*)$ of full rank). A course of action $\mathbf{x}$ will be adopted aiming to approach $\mathbf{x}$ as closely as possible to the *assumed* ideal course of action $\hat{\mathbf{x}}^*$ (not to $\mathbf{x}^*$ which is unknown). Due to imperfect control of the action variables exact coincidence is not possible. Assume that the expectation vector and variance-covariance matrix of $\mathbf{x}$ are $\hat{\mathbf{x}}^*$ and $\mathbf{V}(\mathbf{x})$, respectively ($\mathbf{V}(\mathbf{x})$ of full rank). Then $\mathbf{d} = \mathbf{x} - \mathbf{x}^* = (\mathbf{x} - \hat{\mathbf{x}}^*) + (\hat{\mathbf{x}}^* - \mathbf{x}^*)$ has zero expectation vector and variance-covariance martix $\mathbf{V}(\hat{\mathbf{x}}^*) + \mathbf{V}(\mathbf{x})$, provided the two kinds of errors are uncorrelated. Let the loss function due to imperfect matching of $\mathbf{x}$ with $\mathbf{x}^*$ be the quadratic $\mathbf{d}'\mathbf{A}\mathbf{d}$, $|\mathbf{A}| > 0$, and assume further that $\mathbf{x}$, $\hat{\mathbf{x}}^*$ and therefore $\mathbf{d}$ have multivariate normal distributions. In view of the discussion in (iii), it is clear that the probability of the loss not exceeding a given upper bound c^2 is equal to $P(\sum_1^N \lambda_j y_j^2 \leqq c^2)$, in which the y_j have the usual significance. (In particular, the expected loss is $\sum_1^N \lambda_j$.) The reader is referred to Grad and Solomon [27] who discuss an analogous ballistics problem for which $N = 3$.

(vi) There is one interesting case for which the distribution of the weighted sum of squares may be expressed in exact form. If the number of components N is even, $N = 2m$, and the weights c_j coincides in pairs, say $c_j = c_{N-j}$, then

$$z = \sum_{j=1}^{N} c_j x_j^2 = \sum_{j=1}^{m} c_j y_j$$

where the y_j are independent χ^2, each with 2 degrees of freedom. The characteristic function of $\sum c_j y_j$ is $\Pi(1 - 2c_j it)^{-1}$. By partial fraction decomposition the latter may be expressed in the form $\sum A_j/(1 - 2c_j it)$, which is obviously directly invertible to $\sum (A_j/2c_j) \exp(-2z/c_j)$, the density function of z. It follows that the complement of the distribution function of z, $P(z > z_0)$, is likewise expressible as a linear combination of exponentials.

The above remarks may easily be extended to the situation where the weights are repeated in groups of four (instead of groups of two), groups of six, etc., i.e., to the situation where $z = \sum c_j x_j^2$ may be identified as a weighted sum of independent χ^2 variates, each with the same *even* number of degrees of freedom.

More generally still, the degrees of freedom of the components, though still even, need not be the same. A partial fraction representation of the characteristic function enables the density function to be expressed as a linear combination of Gamma density functions with degrees of freedom 2, 4, $\cdots$, p, where p is the highest degree of freedom of the several components. It follows that the distribution function of the sum is a linear combination of Gamma distribution functions with degrees of freedom 2, 4, $\cdots$, p.

We now obtain the recurrence relationship referred to at the beginning of this section. Note first that the intersection of the flat $x_N = x$ with the N-dimensional ellipsoid $\sum_1^N a_i x_i^2 \leqq t$ is itself an elliposid but of dimensionality $N - 1$ and with semi-axes of lengths $((t - a_N x^2)/a_i)^{\frac{1}{2}}$, $i = 1, 2, \cdots, N - 1$. The amount of probability within the ellipsoid intercepted by two parallel and adjoining flats $x_N = x$ and $x_N = x + dx$ is therefore

$$(2\pi)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}x^2)\, dx \cdot F_{N-1;b_1,\cdots,b_{N-1}}\left[\frac{t - a_N x^2}{\sum_1^{N-1} a_i}\right],$$

where $b_i = a_i/\sum_1^{N-1} a_j$, $0 < a_N < 1 (i = 1, 2, \cdots, N - 1)$. Consequently, the probability content of the ellipsiod is

$$\begin{aligned} &F_{N;a_1,\cdots a_N}(t) \\ (7.9)\qquad &= 2\int_0^{(t/a_N)^{\frac{1}{2}}} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} F_{N-1;b_1,\cdots,b_{N-1}}\left[\frac{t - a_N x^2}{1 - a_N}\right] dx \quad (N = 2, 3, \cdots), \end{aligned}$$

or, on setting $y = x(a_N/t)^{\frac{1}{2}}$,

$$\begin{aligned} &F_{N;a_1,\cdots,a_N}(t) \\ (7.10)\qquad &= 2\left(\frac{t}{a_N}\right)^{\frac{1}{2}} \int_0^1 (2\pi)^{-\frac{1}{2}} \exp\left(\frac{-ty^2}{2a_N}\right) \\ &\qquad \cdot F_{N-1;b_1,\cdots,b_{N-1}}\left[\frac{t(1 - y^2)}{1 - a_N}\right] dy \quad (N = 2, 3, \cdots).^{4a} \end{aligned}$$

We may note that for the particular case of $N - 1$ equal components,

$$(7.11)\qquad F_{N;\alpha,\cdots,\alpha,\beta,}(t) = 2\left(\frac{t}{\beta}\right)^{\frac{1}{2}} \int_0^1 (2\pi)^{-\frac{1}{2}} \exp\left(\frac{-ty^2}{2\beta}\right) F_{N-1}\left[\frac{t(1 - y^2)}{1 - \beta}\right] dy,$$

in which $F_{N-1}(\cdot)$ denotes the distribution function of a χ^2 with $N - 1$ degrees of freedom, and $(N - 1)\alpha + \beta = 1$.

Finally, it will be convenient to record here an interesting relationship between the distribution of the weighted sum of squares of two independent standardized normal variables and that of the non-central χ^2 with two degrees of freedom. The relationship[5] in question is

[4a] (7.10) is, of course, just a convolution formula. It has been obtained here by a geometrical argument for consistency.

[5] I am indebted to one of the referees for having brought this useful result, for which an unpublished geometrical proof has been obtained by Dr. David C. Kleinecke, to my attention.

(7.12) $$F_{2;a_1,a_2}(t) = G_{2;\kappa}(u^2) - G_{2;u}(\kappa^2),$$

where

$$\kappa = \tfrac{1}{2} \mid (t/a_1)^{\frac{1}{2}} - (t/a_2)^{\frac{1}{2}} \mid, \qquad u = \tfrac{1}{2}[(t/a_1)^{\frac{1}{2}} + (t/a_2)^{\frac{1}{2}}],$$

and $G_{2;\kappa}(\cdot)$ is the distribution of $\chi^2_{2;\kappa}$, the non-central χ^2 with two degrees of freedom and non-centrality parameter κ, as defined in Section 3.

8. Probability content of a regular simplex. Denote the probability content of a regular N-dimensional simplex with sides of length a and centroid at the center of the spherical normal distribution with the same dimensionality by $K_N(a)$. To derive a formula for $K_N(a)$ it will be convenient to divide the simplex into $N + 1$ (non-regular) simplices, obtained by joining the centroid to the $N + 1$ vertices by straight lines.

Consider then one of these $N + 1$ derived simplices S. The probability content of this simplex may be obtained by first evaluating the amount of probability in a slab formed by two adjacent (N − 1)-flats parallel to the face opposite to that vertex of S which coincides with the centroid of the original simplex. Let x and $x + dx$ denote the distances of these flats from the latter vertex. The intersection of the first flat with S is a *regular* simplex of dimensionality $N - 1$ and with its edges of length y, where $y/a = x/d$, d denoting the distance of the centroid of the original simplex from one of its faces. Furthermore, the density at a point on the same flat distant ξ from the centroid of this regular simplex is

$$(2\pi)^{-\frac{1}{2}N} \exp(-\tfrac{1}{2}r^2) = (2\pi)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}x^2).(2\pi)^{-\frac{1}{2}(N-1)} \exp(-\tfrac{1}{2}\xi^2),$$

where $r^2 = x^2 + \xi^2$ is the square of the distance of the centroid of the original simplex from the point in question. It follows that the probability content of the slab is $(2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}x^2)\, dx K_{N-1}(ax/d)$, and the probability content of the original simplex is

(8.1) $$K_N(a) = (N + 1) \int_0^d (2\pi)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}x^2) K_{N-1}\left(\frac{ax}{d}\right) dx.$$

It is easily shown that

(8.2) $$d = a/(2N(N + 1))^{\frac{1}{2}}.$$

Substituting for d in (8.1), the desired integral recurrence relationship is obtained:

(8.3) $$K_N(a) = (N + 1) \int_0^{a/(2N(N+1))^{\frac{1}{2}}} (2\pi)^{-\frac{1}{2}} \exp(-\tfrac{1}{2}x^2) K_{N-1}[x(2N(N + 1))^{\frac{1}{2}}]\, dx \quad (N = 1, 2, \cdots),$$

or, equivalently [34][6],

[6] Godwin actually uses functions $G_m(\cdot)$ which are related to the K-functions by $G_m(x/\sqrt{2}) = (2\pi)^{m/2}(m + 1)^{-\frac{1}{2}}K_m(x)$. The K-function, from a theoretical point of view, seems to be more convenient and natural than the G-function, since it is (unlike the latter) a distribution function, in the usual statistical sense.

$$K_N(a) = \tfrac{1}{2}\left(\frac{N+1}{N\pi}\right)^{\frac{1}{2}} \int_0^a \cdot \exp\left(-\frac{u^2}{4N(N+1)}\right) K_{N-1}(u)\, du \quad (N = 1, 2, \cdots), \tag{8.4}$$

with $K_0(a) = 1$.

We shall give now one application involving a knowledge of $K_N(a)$. This relates to the problem of determining the distribution of the sum (or mean) of N independent observations from a half-normal distribution, or equivalently the sum (or mean) of the absolute values of observations from a normal population. The first four moments of this distribution have been obtained by Kamat [33] for $N \leqq 3$, but the actual distribution for general N does not appear to have been obtained previously.

The density function for each observation is $(2/\pi)^{\frac{1}{2}} \exp(-x^2/2)(0 \leqq x < \infty)$, and the joint density function of N independent observations is

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}N} \exp\left(-\tfrac{1}{2}\sum_{i=1}^N x_i^2\right) \qquad (x_i \geqq 0, \quad i = 1, 2, \cdots, N). \tag{8.5}$$

The determination of the density function of $u = \sum_1^N x_i$ thus reduces to the determination of the probability intercepted by the $(N-1)$-flats $\sum_1^N x_i = u$ and $\sum_1^N x_i = u + du$ in the positive orthant. To obtain this, note that $\sum_1^N x_i = u, x_i \geqq 0 (i = 1, 2, \cdots, N)$ defines a regular $(N-1)$-dimensional simplex with edges of length $u\sqrt{2}$. The distance of the flat $\sum_1^N x_i = u$ from the origin (i.e., the distance of the latter point from the centroid of the simplex) is $u/\sqrt{N}$. Further, the density at any point within the simplex distant η from the centroid may be expressed in the form

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}N} \exp\left(-\tfrac{1}{2}\sum_1^N x_i^2\right) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}N} \exp\left(-\tfrac{1}{2}\left[\frac{u^2}{N} + \eta^2\right]\right)$$
$$= \frac{1}{(2\pi)^{\frac{1}{2}(N-1)}} \exp(-\tfrac{1}{2}\eta^2)\, \frac{2^N}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2N}\right).$$

Consequently, the probability content of the element $u \leqq \sum x_i \leqq u + du$ is

$$h_N(u)\, du = \frac{2^N}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2N}\right) \frac{du}{\sqrt{N}} K_{N-1}(u\sqrt{2}), \tag{8.6}$$

after integration with respect to η over the simplex, where $h_N(u)$ denotes the p.d.f. of u. Observe that equation (8.6) reveals at the same time the intimate tie-up between the K_{N-1}-function and the N-fold convolution of the half-normal distribution. This tie-up was first demonstrated by Godwin [35], using an entirely different argument, in showing the equivalence of two expressions for the p.d.f. of the mean absolute deviation in normal samples, obtained respectively in [36] and [37].

We conclude with two further important applications of the K-function. The first relates (as already indicated) to the distribution of the mean absolute deviation in samples of size n from a normal population with zero mean and unit variance. The p.d.f., $p_n(t)$, of the latter variable is given by

$$p_n(t) = \left(\frac{n}{2}\right)^{\frac{3}{2}} \pi^{-\frac{1}{2}} \sum_{k=1}^{n-1} \binom{n}{k} [k(n-k)]^{-\frac{1}{2}} \exp\left[-\frac{n^3t^2}{8k(n-k)}\right] \cdot K_{k-1}\left(\frac{nt}{\sqrt{2}}\right) K_{n-k-1}\left(\frac{nt}{\sqrt{2}}\right), \tag{8.7}$$

(Godwin [36]). Tables of $\int_0^x p_n(t)\,dt$ for $n \leqq 10$ are available in [38], while percentage points of the distribution are given in [38] and [39] (Table 21, p. 165).

The second application relates to the distribution of the deviation of the largest observation from the mean in a sample of n independent observations from a normal population with zero mean and unit variance. The distribution function, $Q_n(t)$, of the latter variable is given by

$$Q_n(t) = K_{n-1}(nt\sqrt{2}) \tag{8.8}$$

(Nair [40]). Tables of $Q_n(t)$ are given in [40], while percentage points of the distribution are available in [40] and [39] (Table 25, p. 172).

It should be noted that the K-functions also find applications in connection with the distributions of a class of linear functions of normal order statistics [40][7].

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